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QUANTUM CLASSIFYING SPACES AND UNIVERSAL QUANTUM CHARACTERISTIC CLASSES

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Abstract. A construction of the noncommutative-geometric counterparts of classical classifying spaces is presented, for general compact matrix quantum structure groups. A quantum analogue of the classical concept of the classifying map is introduced and analyzed. Interrelations with the abstract algebraic theory of quantum characteristic classes are discussed. Various non-equivalent approaches to defining universal characteristic classes are outlined.

1. Introduction. Let us begin by recalling basic concepts of the classical classification theory of principal bundles. From the point of view of this theory, a given compact Lie group G is represented by the corresponding *classifying space* B_G. This space can be equipped with a (infinite-dimensional) smooth manifold structure. It classifies principal G-bundles in the following sense. For every compact smooth manifold M, isomorphism classes of principal G-bundles P over M are in a natural correspondence with homotopy classes of maps $\Phi: M \to \mathrm{B}_G$. For a given Φ, the bundle P is reconstructed as the pull-back of a natural contractible principal G-bundle E_G over B_G, via Φ. This scheme allows us to associate intrinsically cohomological invariants of M, to principal bundles P. Such *characteristic classes* are defined as pull-backs, via Φ, of the cohomology classes of B_G. The elements of the algebra $H(\mathrm{B}_G)$ are called *universal characteristic classes*.

The algebra $H(\mathrm{B}_G)$ can be also described in a completely different way, without appealing to the construction of the classifying space B_G. Let $\mathfrak{g}$ be the Lie algebra of G, and $P(\mathfrak{g})$ the algebra of polynomial functions on $\mathfrak{g}$. Let $\Sigma(\mathfrak{g}) \subseteq P(\mathfrak{g})$ be the subalgebra consisting of elements invariant under the coadjoint action of G on $P(\mathfrak{g})$. Then it can be shown that

$$H(\mathrm{B}_G) = \Sigma(\mathfrak{g}),$$

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in a natural manner (assuming the complex coefficients and the standard cohomology theory).

In terms of the above identification, the association of characteristic classes is given via the classical Weil homomorphism $W\colon \Sigma(\mathfrak{g}) \to H(M)$, obtained by formally replacing generators of $P(\mathfrak{g})$ by the curvature of an arbitrary connection on the bundle P.

In this letter we are going to discuss how to incorporate the classical theory of classifying spaces into the conceptual framework of noncommutative differential geometry [1]. All considerations are logically based on a general theory of quantum principal bundles [3,4], in which the bundle and the base are quantum objects and quantum groups play the role of structure groups. A detailed exposition of the theory of quantum classifying spaces will be given in [7].

The first part of the paper is devoted to the construction of the quantum classifying spaces, and universal quantum principal bundles over them. This will be done for general compact [10] matrix quantum groups G. Then we shall explain in which sense such spaces are 'classifying'. The essential point consists in establishing a variant of the cross-product structuralization [2] of the algebra describing an arbitrary quantum G-bundle P over a quantum space M. Geometrically, introducing such a cross-product is equivalent to the specification of a 'classifying map'.

The second part of the paper is devoted to the study of the relations between cohomology classes associated to a quantum classifying space in the framework of the general theory of quantum principal bundles [4], and 'purely algebraic' universal characteristic classes analyzed in [4–6] generalizing the classical Weil approach. Finally, in the last section some concluding remarks are made.

2. The construction of quantum classifying spaces. Let G be an arbitrary compact matrix quantum group [10]. Let $\mathcal{A}$ be the Hopf *-algebra representing polynomial functions on the quantum space G. We shall denote by $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\epsilon\colon \mathcal{A} \to \mathbb{C}$ and $\kappa\colon \mathcal{A} \to \mathcal{A}$ the coproduct, counit and the antipode respectively.

Let us fix a unitary fundamental representation $u \in \mathrm{M}_n(\mathcal{A})$ of G such that the conjugate representation $\bar{u}$ is realizable as a direct summand in u^m, for some m. Equivalently, we can say that the matrix elements u_{ij} generate the whole algebra $\mathcal{A}$. This is a generalization of the classical unimodularity condition for u.

For each $d \in \mathbb{N}$, let us denote by $\mathcal{O}_d$ the unital *-algebra generated by elements ψ_{ki} where $k \in \{1, ..., d\}$ and $i \in \{1, ..., n\}$, and the following relations

$$\sum_k \psi_{ki}^* \psi_{kj} = \delta_{ij} 1. \tag{1}$$

By definition, there exists a natural action map $\delta\colon \mathcal{O}_d \to \mathcal{O}_d \otimes \mathcal{A}$ defined on the generators by

$$\delta(\psi_{ki}) = \sum_j \psi_{kj} \otimes u_{ji}, \tag{2}$$

and extended to the whole $\mathcal{O}_d$ by requiring that δ is a unital *-homomorphism. Let us denote by $\mathcal{H}_d$ a unital subalgebra of $\mathcal{O}_d$ generated by the elements ψ_{ki}. By definition,

$$\delta(\mathcal{H}_d) \subseteq \mathcal{H}_d \otimes \mathcal{A}.$$

We shall denote by $\mathcal{O}_{d,G} \subseteq \mathcal{O}_d$ and $\mathcal{H}_{d,G} \subseteq \mathcal{H}_d$ the corresponding subalgebras of G-invariant elements.

It is possible to find the m-th order elements $\widehat{\psi}_{\alpha i} \in \mathcal{H}_d$ forming the irreducible $\bar{u}$-multiplets, with

$$\widehat{\psi}_{\alpha i} = \sum_{\omega} a_{i\omega} \prod_{\substack{l\in\alpha\\ j\in\omega}} \psi_{lj}$$

where α and ω are appropriate multiindexes and $a_{i\omega} \in \mathbb{C}$, so that we have

$$\delta(\widehat{\psi}_{\alpha i}) = \sum_{j} \widehat{\psi}_{\alpha j} \otimes u^*_{ji} \tag{3}$$

$$\sum_{\alpha} \widehat{\psi}^*_{\alpha i} \widehat{\psi}_{\alpha j} = C_{ji} 1. \tag{4}$$

Here $C \in \mathrm{M}_n(\mathbb{C})$ is the canonical intertwiner [10] between u and the second contragradient representation u^{cc}.

The formula

$$S_{\alpha k} = \sum_{ij} \widehat{\psi}_{\alpha i} C^{-1}_{ji} \psi_{kj} \tag{5}$$

defines a matrix $S_{\alpha k}$. It is easy to see that the introduced matrix elements are actually from $\mathcal{H}_{d,G}$.

Let us now consider the maps $\tau_{kl} \colon \mathcal{O}_d \to \mathcal{O}_d$, defined by

$$\tau_{kl}(a) = \sum_{i} \psi_{ki} a \psi^*_{li}. \tag{6}$$

We have then

$$\sum_{r} \tau_{kr}(a)\tau_{rl}(b) = \tau_{kl}(ab) \qquad \tau_{kl}(a)^* = \tau_{lk}(a^*), \tag{7}$$

in other words the associated matrix map $\tau \colon \mathcal{O}_d \to \mathrm{M}_d(\mathcal{O}_d)$ is a *-homomorphism. From the definition of τ it follows that the algebra $\mathcal{O}_{d,G}$ is τ-invariant. In what follows, the domain of τ will be restricted to $\mathcal{O}_{d,G}$.

The algebra $\mathcal{O}_d$ can be naturally decomposed as

$$\mathcal{O}_d = \sum_{r\in\mathcal{T}}^{\oplus} \mathcal{O}^r_d$$

where $\mathcal{T}$ is a complete set of mutually inequivalent irreducible representations of G, and $\mathcal{O}^r_d \subseteq \mathcal{O}_d$ are the multiple irreducible $\mathcal{O}_{d,G}$-submodules corresponding to the decomposition of δ. We can similarly decompose the δ-invariant subalgebra $\mathcal{H}_d$.

Let us denote by $\mathcal{Q}_d \subseteq \mathcal{O}_{d,G}$ the minimal unital τ-invariant subalgebra containing all the elements $S^*_{\alpha k}$.

LEMMA 1. *(i) The following commutation relations hold*

$$\psi_{ki} a = \sum_{l} \tau_{kl}(a) \psi_{li} \tag{8}$$

$$\psi^*_{ki} a = \sum_{\alpha\beta} S^*_{\alpha k} \tau^m_{\alpha\beta}(a) \widehat{\psi}_{\beta i}. \tag{9}$$

(ii) Every element ψ of a multiple irreducible module $\mathcal{O}_d^r$ can be written in the form

$$\psi = \sum_{f\varphi} f\varphi, \tag{10}$$

where $\varphi \in \mathcal{H}_d^r$ and $f \in \mathcal{Q}_d$.

Proof. The statement (i) follows from definitions of $S_{\alpha k}$, τ and $\widehat{\psi}_{\alpha i}$. Decomposition (10) follows from commutation relations (8) and (9). ∎

3. Reconstruction and classifying maps. Let us consider a unital *-algebra $\mathcal{V}$, interpreted as consisting of 'smooth functions' on a quantum space M. Let $P = (\mathcal{B}, i, F)$ be a quantum principal G-bundle [4] over M. Here $\mathcal{B}$ is a *-algebra representing P as a quantum space, while $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ and $i\colon \mathcal{V} \to \mathcal{B}$ are *-homomorphisms corresponding to the right action of G on P, and to the projection of P onto M respectively.

Explicitly, the following properties hold

$$(\mathrm{id} \otimes \phi)F = (F \otimes \mathrm{id})F \qquad (\mathrm{id} \otimes \epsilon)F = \mathrm{id}$$
$$i(\mathcal{V}) = \Big\{ b \in \mathcal{B} \mid F(b) = b \otimes 1 \Big\},$$

and for each $a \in \mathcal{A}$ there exist elements $q_\alpha, b_\alpha \in \mathcal{B}$ satisfying

$$\sum_\alpha b_\alpha F(q_\alpha) = 1 \otimes a.$$

The last condition corresponds to the classical requirement that the structure group is acting freely on the bundle.

It is possible to prove [7] that under relatively weak 'regularity' conditions for the bundle P, there exist a natural number d and a $\mathcal{B}$-matrix $B = \{b_{ki}\}$, where $k \in \{1, \dots, d\}$ and $i \in \{1, \dots, n\}$ such that the following relations hold:

$$F(b_{ki}) = \sum_j b_{kj} \otimes u_{ji}, \tag{11}$$

$$\sum_k b_{ki}^* b_{kj} = \delta_{ij} 1. \tag{12}$$

DEFINITION 1. We say that a quantum principal bundle P admitting a matrix B of the above described type has a complexity level d.

Let us observe that the number d figuring in the above definition is not fixed uniquely- we can always increase this number as necessary. Having a matrix B we can introduce a homomorphism $\rho\colon \mathcal{V} \to \mathrm{M}_d(\mathcal{V})$, by the formula

$$\rho_{kl}(f) = \sum_i b_{ki} f b_{li}^*. \tag{13}$$

It follows immediately that the following commutation relations hold

$$b_{ki} f = \sum_l \rho_{kl}(f) b_{li}, \tag{14}$$

between the elements of B and $\mathcal{V}$.

Let QB_G^d be the quantum space corresponding to $\mathcal{O}_{d,G}$, and let $\iota\colon \mathcal{O}_{d,G} \to \mathcal{O}_d$ be the canonical inclusion map.

LEMMA 2. *The triplet* $\mathrm{QE}_G^d = (\mathcal{O}_d, \iota, \delta)$ *is a quantum principal bundle over* QB_G^d, *having the complexity level* d. ∎

Furthermore, there exists the unique unital *-homomorphism $\gamma\colon \mathcal{O}_d \to \mathcal{B}$, specified by

$$\gamma(\psi_{ki}) = b_{ki}, \tag{15}$$

and we have $\gamma(\mathcal{O}_{d,G}) \subseteq \mathcal{V}$, because γ intertwines δ and F.

LEMMA 3. *The following natural decomposition holds*

$$\mathcal{B} \leftrightarrow \mathcal{V} \otimes_* \mathcal{O}_d \leftrightarrow \mathcal{O}_d \otimes_* \mathcal{V}, \tag{16}$$

where $\otimes_*$ *denotes the tensor product over* $\mathcal{O}_{d,G}$. ∎

We are going to reverse the above construction. Let $\rho\colon \mathcal{V} \to \mathrm{M}_d(\mathcal{V})$ be a (non-trivial) *-homomorphism and let $\gamma\colon \mathcal{O}_{d,G} \to \mathcal{V}$ be a unital *-homomorphism such that

$$\sum_\beta \rho_{\alpha\beta}(f)\gamma(o_{\beta j}) = \gamma(o_{\alpha j})f \tag{17}$$

$$\gamma\tau_{kl} = \rho_{kl}\gamma, \tag{18}$$

for every $f \in \mathcal{V}$ and every multi-indexed system

$$o_{\alpha j} = \sum_\omega c_{j\omega} \prod_{\substack{k\in\alpha \\ i\in\omega}} \psi_{ki}$$

from the algebra $\mathcal{H}_{d,G}$. In formula (17) we have considered the appropriate iterations of the map ρ.

Motivated by the results of the above analysis, we shall define a crossproduct between $\mathcal{O}_d$ and $\mathcal{V}$ and reconstruct in such a way the bundle P. This construction is a generalization of the standard crossproduct construction considered in [2].

The algebra $\mathcal{V}$ is a bimodule over $\mathcal{O}_{d,G}$, in a natural manner. The bimodule structure is induced by the map γ. We shall denote by $\otimes_\gamma$ the corresponding bimodule tensor products between algebras $\mathcal{V}$ and $\mathcal{O}_d$.

The essential part of the construction is to prove the existence of the canonical flip-over map $\Psi\colon \mathcal{O}_d \otimes_\gamma \mathcal{V} \to \mathcal{V} \otimes_\gamma \mathcal{O}_d$. The consistency of the whole construction will be ensured by the compatibility relations between γ and ρ.

At first, let us forget for a moment about the relations in $\mathcal{O}_d$, and consider it as a free algebra. Let $\Psi\colon \mathcal{O}_d \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{O}_d$ be the linear map specified by

$$\Psi(\psi_{ki} \otimes f) = \sum_l \rho_{kl}(f) \otimes \psi_{li} \qquad \Psi(1 \otimes f) = f \otimes 1$$

$$\Psi(\psi_{ki}^* \otimes f) = \sum_{\alpha\beta} R_{\alpha k}^* \rho_{\alpha\beta}^m(f) \otimes \widehat{\psi}_{\beta i}$$

$$\Psi(\mu \otimes \mathrm{id}) = (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \Psi),$$

where $\mu\colon \mathcal{O}_d \otimes \mathcal{O}_d \to \mathcal{O}_d$ is the corresponding product map and we have put $R_{\alpha k} = \gamma(S_{\alpha k})$. Let us denote by the same symbol μ be the product in $\mathcal{V}$.

LEMMA 4. *We have*

$$\Psi(\mathrm{id} \otimes \mu) = (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \Psi)(\Psi \otimes \mathrm{id}). \tag{19}$$

P r o o f. It is sufficient to check the equality on elements of the form $\psi_{ki} \otimes f \otimes g$ and $\psi^*_{ki} \otimes f \otimes g$. Direct transformations give

$$\begin{aligned}\Psi(\psi_{ki} \otimes fg) &= \sum_l \rho(fg)_{kl} \otimes \psi_{li} = \sum_{lr} \rho(f)_{kr}\rho(g)_{rl} \otimes \psi_{li}\\ &= \sum_{lr} (\mu \otimes \mathrm{id})\Bigl\{\rho(f)_{kr} \otimes \rho(g)_{rl} \otimes \psi_{li}\Bigr\}\\ &= (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \Psi)(\Psi \otimes \mathrm{id})(\psi_{ki} \otimes f \otimes g)\end{aligned}$$

and similarly

$$\begin{aligned}\Psi(\psi^*_{ki} \otimes fg) &= \sum_{\alpha\beta} R^*_{\alpha k}\rho^m_{\alpha\beta}(fg) \otimes \widehat{\psi}_{\beta i} = \sum_{\alpha\beta\gamma} R^*_{\alpha k}\rho^m(f)_{\alpha\gamma}\rho^m(g)_{\gamma\beta} \otimes \widehat{\psi}_{\beta i}\\ &= \sum_{\alpha\beta\gamma} (\mu \otimes \mathrm{id})\Bigl\{R^*_{\alpha k}\rho^m_{\alpha\gamma}(f) \otimes \rho^m_{\gamma\beta}(g) \otimes \widehat{\psi}_{\beta i}\Bigr\}\\ &= (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \Psi)(\Psi \otimes \mathrm{id})(\psi^*_{ki} \otimes f \otimes g),\end{aligned}$$

which completes the proof. ∎

The map Ψ preserves the transformation properties of the elements from $\mathcal{O}_d$, in a natural manner. Let us now assume that the map Ψ is 'partially factorized' in the form $\Psi\colon \mathcal{O}_d \otimes \mathcal{V} \to \mathcal{V} \otimes_\gamma \mathcal{O}_d$. This allows us to consider the initial relations.

LEMMA 5. *At the level of the partial factorization, it is possible to include consistently the defining relations for* $\mathcal{O}_d$ *in the game.*

P r o o f. We have to check that the defining relations 'pass' through the twist. Applying the definition of Ψ, commutation relations between ψ_{ki} and $\mathcal{V}$, and the fact that the resulting product is γ-relativized, we obtain

$$\begin{aligned}\Psi(\delta_{ij}1 \otimes f) = \sum_k \Psi(\psi^*_{ki}\psi_{kj} \otimes f) &= \sum_{kl} (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})(\psi^*_{ki} \otimes \rho(f)_{kl} \otimes \psi_{lj})\\ &= \sum_{kl\alpha\beta} R^*_{\alpha k}\rho^m_{\alpha\beta}\bigl[\rho(f)_{kl}\bigr] \otimes \widehat{\psi}_{\beta i}\psi_{lj}\\ &= \sum_{l\beta} fR^*_{\beta l} \otimes \widehat{\psi}_{\beta i}\psi_{lj} = \sum_{l\beta} f \otimes S^*_{\beta l}\widehat{\psi}_{\beta i}\psi_{lj}\\ &= f \otimes \delta_{ij}1,\end{aligned}$$

which completes the proof. ∎

Let us now assume that the following identity holds

$$\gamma(\psi)f = \sum_k f_k\gamma(\psi_k), \tag{20}$$

where $f \in \mathcal{V}$ and $\psi \in \mathcal{Q}_d$, while $\sum_k f_k \otimes \psi_k = \Psi(\psi \otimes f)$.

DEFINITION 2. Every pair (γ, ρ) satisfying all the above mentioned conditions is called a d-classifying map for M.

Let us consider the question of the projectability of twists to the tensor product over γ. Consistent factorization to the tensor products over γ will be possible only if the twist Ψ is $\mathcal{O}_{d,G}$-linear, on both sides.

LEMMA 6. *The map* $\Psi\colon \mathcal{O}_d \otimes \mathcal{V} \to \mathcal{V} \otimes_\gamma \mathcal{O}_d$ *is a homomorphism of* $\mathcal{O}_{d,G}$*-bimodules.*

Proof. Let us consider an arbitrary element $\psi \in \mathcal{O}_{d,G}$. We have then

$$\begin{aligned}\Psi(\psi x \otimes f) = \sum_k (\mathrm{id} \otimes \mu)\bigl[\Psi(\psi \otimes f_k) \otimes x_k\bigr] &= \sum_{kl} f_{kl} \otimes \psi_l x_k = \sum_{kl} f_{kl}\gamma(\psi_l) \otimes x_k\\ &= \sum_k \gamma(\psi) f_k \otimes x_k = \gamma(\psi)\Psi(x \otimes f),\end{aligned}$$

where $\Psi(x \otimes f) = \sum_k f_k \otimes x_k$ and $\Psi(\psi \otimes f_k) = \sum_l f_{kl} \otimes \psi_l$. Now, to prove the right $\mathcal{O}_{d,G}$-linearity, it is sufficient to consider homogeneous blocks x_{ki} from the algebra $\mathcal{H}_d$, where k, i are appropriate multiindexes. A direct computation gives

$$\begin{aligned}\Psi\bigl(x_{ki} \otimes f\gamma(\psi)\bigr) &= \sum_l (\mu \otimes \mathrm{id})\bigl[\rho^r_{kl}(f) \otimes \Psi\bigl(x_{li} \otimes \gamma(\psi)\bigr)\bigr]\\ &= \sum_{ln} \rho^r_{kl}(f)(\rho^r_{ln}\gamma)(\psi) \otimes x_{ni}\\ &= \sum_{ln} \rho^r_{kl}(f)\gamma[\tau^r_{ln}(\psi)] \otimes x_{ni}\\ &= \sum_{ln} \rho^r_{kl}(f) \otimes [\tau^r_{ln}(\psi)]x_{ni}\\ &= \sum_l \rho^r_{kl}(f) \otimes x_{li}\psi = \Psi(x_{ki} \otimes f)\psi,\end{aligned}$$

where r is the degree of elements x_{ki}. Hence, Ψ is $\mathcal{O}_{d,G}$-linear on both sides. ∎

Let $\mathcal{Q} \subseteq \mathcal{O}_{d,G}$ be the space of elements ψ satisfying

$$\Psi\bigl(x\psi \otimes f\bigr) = \Psi\bigl(x \otimes \gamma(\psi)f\bigr) \tag{21}$$

for each $x \in \mathcal{O}_d$ and $f \in \mathcal{V}$. By definition, $\mathcal{Q}$ is a subalgebra of $\mathcal{O}_{d,G}$ and $1 \in \mathcal{Q}$.

LEMMA 7. (*i*) *The algebra* $\mathcal{Q}$ *contains the elements* $S^*_{\alpha k}$ *and we have* $\mathcal{H}_{d,G} \subseteq \mathcal{Q}$.

(*ii*) *The following equality holds*

$$\sum_r \Psi\bigl(p_{kr} \otimes \rho_{rl}(f)\bigr) = \rho_{kl}(f) \otimes 1, \tag{22}$$

for each $f \in \mathcal{V}$. *Here* $p_{kr} = \tau_{kr}(1)$ *are matrix elements of the associated projection in* $\mathrm{M}_d(\mathcal{O}_{d,G})$.

Proof. We shall prove here the second statement. A direct computation gives

$$\sum_r \Psi\bigl(p_{kr} \otimes \rho_{rl}(f)\bigr) = \sum_{r\alpha\beta i} (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})\Bigl(\psi_{ki} \otimes R^*_{\alpha r}\rho^m_{\alpha\beta}[\rho_{rl}(f)] \otimes \widehat{\psi}_{\beta i}\Bigr)$$

$$
\begin{aligned}
&= \sum_{rn\alpha\beta i} \rho_{kn}\Bigl\{R^*_{\alpha r}\rho^m_{\alpha\beta}\bigl[\rho_{rl}(f)\bigr]\Bigr\} \otimes \psi_{ni}\widehat{\psi}_{\beta i}\\
&= \sum_{n\beta i} \rho_{kn}[fR^*_{\beta l}] \otimes \psi_{ni}\widehat{\psi}_{\beta i}\\
&= \sum_{nr\beta i} \rho_{kr}(f) \otimes \tau_{rn}[S^*_{\beta l}]\psi_{ni}\widehat{\psi}_{\beta i} = \sum_{r\beta i} \rho_{kr}(f) \otimes \psi_{ri}S^*_{\beta l}\widehat{\psi}_{\beta i}\\
&= \sum_{ri} \rho_{kr}(f) \otimes \psi_{ri}\psi^*_{li} = \sum_{r} \rho_{kr}(f) \otimes p_{rl} = \rho_{kl}(f) \otimes 1.
\end{aligned}
$$

We have used the definition of the tensor product over γ, and the basic commutation relations in $\mathcal{V}$ and $\mathcal{O}_d$. ∎

LEMMA 8. *The algebra $\mathcal{Q}$ is closed under the action of operators τ_{kl}.*

Proof. Let us consider an arbitrary element $\varphi \in \mathcal{Q}$. We have then

$$
\begin{aligned}
\Psi\bigl(x\tau_{kl}(\varphi) \otimes f\bigr) &= \sum_{\alpha\beta i} (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})\Bigl(x\psi_{ki}\varphi \otimes R^*_{\alpha l}\rho^m_{\alpha\beta}(f) \otimes \widehat{\psi}_{\beta i}\Bigr)\\
&= \sum_{\alpha\beta i} (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})\Bigl(x\psi_{ki} \otimes \gamma(\varphi)R^*_{\alpha l}\rho^m_{\alpha\beta}(f) \otimes \widehat{\psi}_{\beta i}\Bigr)\\
&= \sum_{\alpha\beta ni} (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})\Bigl(x\psi_{ki} \otimes R^*_{\alpha n}\rho^m_{\alpha\beta}\bigl[(\rho_{nl}\gamma)(\varphi)f\bigr] \otimes \widehat{\psi}_{\beta i}\Bigr)\\
&= \sum_{ni} (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \Psi)\bigl(x\psi_{ki} \otimes \psi^*_{ni} \otimes (\rho_{nl}\gamma)(\varphi)f\bigr)\\
&= \sum_{r} \Psi\bigl(xp_{kr} \otimes \rho_{rl}\gamma(\varphi)f\bigr) = \Psi\bigl(x \otimes \rho_{kl}\gamma(\varphi)f\bigr).
\end{aligned}
$$

The last equality in the above sequence of transformations is justified by the previous lemma. ∎

From the sequence of the previous lemmas, it follows that

$$
\mathcal{Q} = \mathcal{O}_{d,G}. \tag{23}
$$

In other words, the domain of the map Ψ can be factorized through γ. In such a way we obtain the fully factorized $\mathcal{O}_{d,G}$-bimodule map

$$
\Psi\colon \mathcal{O}_d \otimes_\gamma \mathcal{V} \rightarrow \mathcal{V} \otimes_\gamma \mathcal{O}_d. \tag{24}
$$

The properties of this map are summarized in the following

PROPOSITION 9. (*i*) *The following identities hold*

$$
\Psi(\mathrm{id} \otimes \mu) = (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \Psi)(\Psi \otimes \mathrm{id}) \tag{25}
$$

$$
\Psi(\mu \otimes \mathrm{id}) = (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \Psi) \tag{26}
$$

$$
\Psi(\psi_{ki} \otimes f) = \sum_{l} \rho_{kl}(f) \otimes \psi_{li} \tag{27}
$$

$$
\Psi(\psi^*_{ki} \otimes f) = \sum_{\alpha\beta} R^*_{\alpha k}\rho^m_{\alpha\beta}(f) \otimes \widehat{\psi}_{\beta i}. \tag{28}
$$

(*ii*) *The map* Ψ *is bijective, and*

$$\Psi^{-1} = *\Psi* \tag{29}$$

where $*\colon \mathcal{V} \otimes_\gamma \mathcal{O}_d \leftrightarrow \mathcal{O}_d \otimes_\gamma \mathcal{V}$ *is the canonical conjugation map.*

Proof. We have to prove property (*ii*). Let us assume that the element $\psi \in \mathcal{O}_d$ possesses the property that equality $\Psi{*}\Psi = *$ holds on all elements of the form $\psi \otimes f$. Evidently, the space $\mathcal{L}$ of elements ψ contains the unit $1 \in \mathcal{O}_d$ and it is a left $\mathcal{O}_{d,G}$-submodule of $\mathcal{O}_d$. It turns out that $\mathcal{L}$ is also a right $\mathcal{H}_d$-submodule of $\mathcal{O}_d$. Indeed,

$$\begin{aligned}
(\Psi{*}\Psi)(\psi\psi_{ki} \otimes f) &= \sum_l (\Psi*)(\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})\bigl(\psi \otimes \rho_{kl}(f) \otimes \psi_{li}\bigr)\\
&= \sum_l \Psi(\mu \otimes \mathrm{id})\bigl[\psi_{li}^* \otimes (*\Psi)\bigl(\psi \otimes \rho_{kl}(f)\bigr)\bigr]\\
&= \sum_l (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})\bigl[\psi_{li}^* \otimes (\Psi{*}\Psi)\bigl(\psi \otimes \rho_{kl}(f)\bigr)\bigr]\\
&= \sum_l (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})(\psi_{li}^* \otimes \rho_{lk}(f^*) \otimes \psi^*)\\
&= \sum_{l\alpha\beta} R_{\alpha l}^* \rho_{\alpha\beta}^m \rho_{lk}(f^*) \otimes \widehat{\psi}_{\beta i}\psi^* = \sum_\beta f^* R_{\beta k}^* \otimes \widehat{\psi}_{\beta i}\psi^*\\
&= \sum_\beta f^* \otimes S_{\beta k}^* \widehat{\psi}_{\beta i}\psi^* = f^* \otimes (\psi\psi_{ki})^*.
\end{aligned}$$

Hence, $\mathcal{L} = \mathcal{O}_d$, which completes the proof. ∎

As a direct consequence of the above proposition, we see that formulas

$$(g \otimes \psi)(f \otimes \varphi) = g\Psi(\psi \otimes f)\varphi \tag{30}$$

$$(f \otimes \psi)^* = \Psi(\psi^* \otimes f^*) \tag{31}$$

consistently define a *-algebra structure on the space

$$\mathcal{B} = \mathcal{V} \otimes_\gamma \mathcal{O}_d.$$

Furthermore, a natural action δ of G on $\mathcal{O}_d$ is naturally projectable to a comodule *-algebra map $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$. Let $i\colon \mathcal{V} \to \mathcal{B}$ be the canonical inclusion map, given by $i(f) = f \otimes 1$. By construction, i is a unital *-homomorphism.

Proposition 10. *The triplet* $P = (\mathcal{B}, i, F)$ *is a quantum principal* G*-bundle over the quantum space* M*.*

Proof. Let us consider the vertical integration map $h_M\colon \mathcal{B} \to \mathcal{B}$ given by $h_M = (\mathrm{id} \otimes h)F$, where $h\colon \mathcal{A} \to \mathbb{C}$ is the Haar measure [10] of G. We have

$$h_M i(f) = f \qquad h_M(\mathcal{B}) = \mathcal{V},$$

as follows from the definitions of $\mathcal{B}$ and F. Hence i is injective and its image coincides with the F-fixed point subalgebra of $\mathcal{B}$. The freeness of F is a direct consequence of the freeness of δ. ∎

Therefore, we have the following natural correspondence

$$\left\{\begin{matrix}\text{Quantum principal } G\text{-bundles } P\\ \text{having complexity level } d\\ \text{equipped with a } d\times d \text{ matrix } B\end{matrix}\right\} \leftrightarrow \Bigl\{d\text{-classifying maps } (\rho,\gamma)\Bigr\}.$$

The above correspondence simplifies if we factorize down to the appropriate homotopy classes. Namely, it turns out [7] that the homotopy class of B, as well as the homotopy class of the classifying map (ρ,γ) are stable, if we stay in the framework of the same homotopy class of P and allow d to take arbitrarily high values (if necessary). Factorizing through the homotopy equivalence we obtain the correspondence

$$\left\{\begin{matrix}\text{Homotopy classes of quantum principal}\\ G\text{-bundles } P \text{ over } M\\ \text{having finite complexity levels}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Homotopy classes}\\ \text{of classifying maps } (\rho,\gamma)\end{matrix}\right\}.$$

4. Quantum characteristic classes. In this section we shall consider a quantum generalization [5,6] of the classical Weil homomorphism, and discuss its relations with the quantum classifying spaces.

Let Γ be a bicovariant [11] first-order *-calculus over G. Let us consider the universal graded-differential algebra Ω built over the vector space Γ_{inv} of left-invariant elements of Γ. We assume that $d(1)=0$, and that the elements of Γ_{inv} have degree one in Ω. The natural *-involution on Γ_{inv} naturally extends to Ω, so that the differential $d\colon\Omega\to\Omega$ is hermitian. By construction, we have $H(\Omega)=\mathbb{C}$.

To fix ideas, we shall assume that the higher-order calculus on G is given by the universal [3] differential envelope $\Gamma^\wedge$ of Γ. Another natural choice is to assume that the higher-order calculus is based on the braided exterior [11] algebra $\Gamma^\vee$, associated to Γ.

Let $\widetilde{\varpi}\colon\Omega\to\Omega\mathbin{\widehat{\otimes}}\Gamma^\wedge$ be the unital graded-differential homomorphism uniquely characterized by the property

$$\widetilde{\varpi}(\vartheta)=\varpi(\vartheta)+1\otimes\vartheta$$

for each $\vartheta\in\Gamma_{inv}$. Here $\varpi\colon\Gamma_{inv}\to\Gamma_{inv}\otimes\mathcal{A}$ is the corresponding adjoint action (the restriction to Γ_{inv} of the right action $\varrho_\Gamma\colon\Gamma\to\Gamma\otimes\mathcal{A}$). The map $\widetilde{\varpi}$ is hermitian, and we have

$$(\widetilde{\varpi}\otimes\mathrm{id})\widetilde{\varpi}=(\mathrm{id}\otimes\widehat{\phi})\widetilde{\varpi},$$

where $\widehat{\phi}\colon\Gamma^\wedge\to\Gamma^\wedge\mathbin{\widehat{\otimes}}\Gamma^\wedge$ is the graded-differential extension of the coproduct map (acting on Γ as a direct sum of left/right action maps $\ell_\Gamma\colon\Gamma\to\mathcal{A}\otimes\Gamma$ and ϱ_Γ).

Let us now consider the subalgebra $\daleth\subseteq\Omega$ consisting of all the elements invariant under the action $\widetilde{\varpi}$. This algebra generally possesses non-trivial cohomology classes.

Definition 3. The elements of the graded *-algebra $H(\daleth)$ are called *universal characteristic classes* associated to G and Γ.

Let us consider an arbitrary $P = (\mathcal{B}, i, F)$ quantum principal G-bundle over M, and let $\Omega(P)$ be an arbitrary differential calculus over P, in the sense of [4]. This means that $\Omega(P)$ is a graded-differential *-algebra extending $\mathcal{B}$, generated by $\mathcal{B}$, and such that there exists a graded-differential extension of $\widehat{F}\colon \Omega(P) \to \Omega(P) \mathbin{\widehat{\otimes}} \Gamma^\wedge$ of the right action F. Let $\Omega(M) \subseteq \Omega(P)$ be a graded-differential *-subalgebra consisting of $\widehat{F}$-invariant elements. The elements of this subalgebra play the role of differential forms on the base space M.

Finally, let us consider an arbitrary connection ω on P. By definition [4], this means that $\omega\colon \Gamma_{inv} \to \Omega(P)$ is a first-order hermitian map satisfying

$$\widehat{F}\omega(\vartheta) = \varpi(\vartheta) + 1 \otimes \vartheta$$

for each $\vartheta \in \Gamma_{inv}$. By universality of the algebra Ω, this map admits the unique extension $\widehat{\omega}\colon \Omega \to \Omega(P)$, which is a graded-differential algebra homomorphism. Furthermore, $\widehat{\omega}$ is hermitian, and it intertwines maps $\widetilde{\varpi}$ and $\widehat{F}$. This intertwining property implies that $\widehat{\omega}(\urcorner) \subseteq \Omega(M)$.

LEMMA 11. *The induced cohomology map $W\colon H(\urcorner) \to H(M)$ is independent of the choice of ω.* ∎

The above lemma justifies Definition 3. The main idea is to define characteristic classes as generic cohomology classes expressed algebraically via the connection ω and its differential $d\omega$. The map W is a general quantum counterpart of the Weil homomorphism.

Let us now apply the construction to the universal bundle, QE^d_G over the classifying space QB^d_G, assuming that the calculus on QE^d_G is based on the universal envelope. Let $W_G\colon H(\urcorner) \to H(\mathrm{QB}^d_G)$ be the corresponding Weil homomorphism.

PROPOSITION 12. *The map W_G is bijective. Therefore, we can naturally identify cohomology algebras $H(\urcorner)$ and $H(\mathrm{QB}^d_G)$.* ∎

On the other hand, for an arbitrary complexity-d bundle P with a differential structure $\Omega(P)$, the natural map $\gamma\colon \mathcal{O}_d \to \mathcal{B}$ uniquely extends to a differential algebra homomorphism $\gamma\colon \Omega(\mathrm{QE}^d_G) \to \Omega(P)$, which satisfies $\gamma\bigl[\Omega(\mathrm{QB}^d_G)\bigr] \subseteq \Omega(M)$. This is a consequence of the fact that γ intertwines $\widehat{\delta}$ and $\widehat{F}$.

Let $\gamma_*\colon H(\mathrm{QB}^d_G) \to H(M)$ be the corresponding cohomology map.

LEMMA 13. *Under the above assumptions we have*

$$W = \gamma_* W_G. \tag{32}$$

In other words, the quantum Weil homomorphism is induced by an arbitrary classifying map. ∎

For the end of this section, let us mention that there exists another [4–6], nonequivalent, approach to defining universal characteristic classes, which is from the conceptual point of view closer to the original Weil construction. This approach is applicable only to bundles equipped with differential structures admitting very special connections (called regular and multiplicative [4]). In this case it is possible to construct, with the help of the curvature of an arbitrary regular and multiplicative connection, a natural map $w\colon \mathrm{I}(\Sigma) \to HZ(M)$, where $HZ(M)$ is the cohomology of the graded centre of $\Omega(M)$ and $\mathrm{I}(\Sigma)$ is the G-invariant part of the braided-symmetric algebra Σ built over Γ_{inv}, relative to the natural [11] braid operator $\sigma\colon \Gamma_{inv}^{\otimes 2} \to \Gamma_{inv}^{\otimes 2}$ associated to Γ. The algebra $\mathrm{I}(\Sigma)$ is always *commutative*, and it plays the role of the algebra of invariant polynomials over the Lie algebra of the classical structure group. The algebras $\mathrm{I}(\Sigma)$ and $H(\urcorner)$ are generally non-isomorphic, however they can be related [5] through a special long exact sequence.

5. Concluding remarks. The spaces corresponding to algebras $\mathcal{O}_{d,G}$ are classifying only in the relative sense, they work for bundles having a fixed degree of complexity. The 'true' quantum classifying space QB_G, together with its universal bundle QE_G, can be constructed [7] by taking the appropriate topological inverse limit of algebras $\mathcal{O}_{d,G}$. However, effectively we always work with some $\mathcal{O}_{d,G}$, for sufficiently large d.

Let us assume that G is a classical compact Lie group. Interestingly, the corresponding classifying space QB_G is still a quantum object. However, the classical classifying space B_G can be obtained by forcing the commutativity of the algebras $\mathcal{O}_{d,G}$. Equivalently, the space B_G is identified with the *classical part* of QB_G. The difference in the natures of two classifying spaces is responsible for the existence of purely quantum bundles with the classical structure group G. Such bundles can be even constructed over classical smooth manifolds (an example of a quantum line bundle of this kind is considered in [5]).

The presented classification theory radically differs from its classical counterpart in the following point. In classical theory homotopic bundles over the same manifold are isomorphic. In quantum theory this is no longer the case, and it may happen for example that a classical bundle is homotopic to a non-classical quantum bundle. However, it turns out [7] that homotopic bundles give the same characteristic classes (under the appropriate technical assumptions concerning the differential calculus).

It is important to mention that quantum characteristic classes depend, besides on the bundle P, also on specifications of corresponding differential structures (in particular, the calculus $\Omega(M)$ on the base explicitly depends on the calculi on P and G). As far as classifying spaces are concerned, the dependence is reduced to the choice of the calculus over G, since it is natural to assume that the calculus on QE_G is based on the appropriate universal envelope. The choice of the higher-order calculus on G may have an essential influence to the algebra of universal characteristic classes. For example, if we assume that the calculus on G is given by the universal envelope of $\mathcal{A}$, the algebra $\Omega(\mathrm{QB}_G)$ will be acyclic. In contrast to that, if we assume that the higher-order calculus is given by the braided [11] exterior algebra (of the universal first-order calculus) then, generally, the cohomology of the corresponding $\Omega(\mathrm{QB}_G)$ will be highly non-trivial. An interesting manifestation of this phenomena is the existence [5] of characteristic classes having a 'purely quantum' origin, associated to quantum bundles with classical M and G.

The above described ways of defining quantum characteristic classes are not unique. Another natural possibility is to first factorize all differential algebras through the graded commutators subcomplex, and then to pass to the cohomology classes. This thinking is along the lines of cyclic homology theory [1,9]. Cyclic homology classes of the classifying space are naturally mapped [8], via classification maps, into cyclic homology classes of the corresponding quantum spaces. Moreover, such a definition does not depend of differential structures.

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