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SEMILINEAR RELATIONS AND $*$-REPRESENTATIONS OF DEFORMATIONS OF $so(3)$

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Abstract. We study a family of commuting selfadjoint operators $\mathbb{A} = (A_k)_{k=1}^n$, which satisfy, together with the operators of the family $\mathbb{B} = (B_j)_{j=1}^n$, semilinear relations $\sum_i f_{ij}(\mathbb{A})B_j g_{ij}(\mathbb{A}) = h(\mathbb{A})$, ($f_{ij}$, g_{ij}, $h_j : \mathbb{R}^n \to \mathbb{C}$ are fixed Borel functions). The developed technique is used to investigate representations of deformations of the universal enveloping algebra $U(so(3))$, in particular, of some real forms of the Fairlie algebra $U_q'(so(3))$.

Introduction. In last few years, due to applications to quantum groups, quantum homogeneous spaces and others, representations of $*$-algebras have attracted a considerable interest (see,for example, [3, 4, 8, 9, 10, 12, 15, 17, 21, 22, 28, 30, 34] and others).

We study representations of $*$-algebras generated by elements $b_1, \dots, b_n$ and relations:

$$P_i(b_1, \dots, b_n, b_1^*, \dots, b_n^*) = 0, \qquad i = 1, \dots, m.$$

Here P_i are polynomials over $\mathbb{C}$ in the non-commutative variables $b_1, \dots, b_n, b_1^*, \dots, b_n^*$. The problem of describing representations of the $*$-algebras into a $*$-algebra $L(H)$ of bounded operators on a Hilbert space H or into a $*$-algebra of unbounded operators can be reduced to that of operators $B_1, \dots, B_n$, which are connected by the relations

$$P_i(B_1, \dots, B_n, B_1^*, \dots, B_n^*) = 0, \qquad i = 1, \dots m.$$

In this article we will assume that the $*$-algebra contains a commutative subalgebra $\mathfrak{A}$ with selfadjoint generators a_k, $k = 1, \dots, m$, satisfying, together with the generators $b_1, \dots, b_n$, the semilinear relations:

$$\sum_i f_{ij}(a_1, \dots, a_m) b_j g_{ij}(a_1, \dots, a_m) = h_j(a_1, \dots, a_m), \qquad j = 1, \dots, n,$$

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where f_{ij}, g_{ij}, $h_j : \mathbb{R}^n \to \mathbb{C}$ are some polynomials. Then, to investigate their representations, we need to study the structure of commuting selfadjoint operators A_k, $k = 1, \dots, m$, and the operators $B_1 \dots, B_n$, which are connected by the following relations

$$\sum_i f_{ij}(A_1, \dots, A_m) B_j g_{ij}(A_1, \dots, A_m) = h_j(A_1, \dots, A_m), \qquad j = 1, \dots, n. \tag{1}$$

This article is devoted mainly to a study of such operators $\mathbb{A} = (A_k)_{k=1}^m$ and $\mathbb{B} = (B_k)_{k=1}^n$ (bounded and unbounded), the functions f_{ij}, g_{ij}, h_j are not assumed to be polynomials. The developed technique is used to give a description of all irreducible representations of deformations of $U(so(3))$, in particular, of some real forms of the Fairlie algebra.

In Section 1, following [25], we investigate spectral properties of bounded operators satisfying (1). In particular, criterions for the bounded operators $\mathbb{A}$, B to satisfy (1) are given (Theorems 1.2, 1.3). We note that the Kleinike-Shirokov theorem for normal operator A and the Fuglede-Putnam theorem follow from these results (see, for example [7, 23]). The problem of describing up to a unitary equivalence, all operators $\mathbb{A}$, $B \in L(H)$ satisfying (1) is "wild" in most cases (see Theorem 1.4), i.e., it contains as a subproblem the problem of describing pairs of selfadjoint operators without any relations ([14, 20]). Therefore, in Section 2, we study operators $\mathbb{A}$, B, which satisfy, in addition, some other relations, the form of which is suggested by many impotant examples.

The developed technique allows us to extend the list of $*$-algebras for which all representations may be described up to a unitary equivalence. In Section 3, as an example, we study representations of the $*$-algebras $A_{q,\mu}$ generated by elements $a_1 = a_1^*$, $a_2 = a_2^*$ and relations

$$\begin{aligned} &[a_1, [a_1, a_2]_q]_{q^{-1}} = \mu a_2, \\ &[a_2, [a_2, a_1]_q]_{q^{-1}} = \mu a_1. \end{aligned} \tag{2}$$

Here $[x, y]_q = xy - qyx$, $q \in \mathbb{R} \cup \mathbb{T}$, $q \neq 0$, $\mu \in \mathbb{R}$, $\mathbb{T}$ is the unit circle. In general, we can consider $\mu = 0, \pm 1$, for $A_{q,\mu} \simeq A_{q,1}$ if $\mu > 0$, $A_{q,\mu} \simeq A_{q,-1}$ if $\mu < 0$. Let us note that, for $\mu = q = 1$, the $*$-algebra coincides with a real form of the universal enveloping algebra $U(so(3))$, for $\mu = -q = 1$, it does so with a real form of the universal enveloping algebra of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra $so_{gr}(3)$. The latter is generated by selfadjoint elements $a_1 = a_1^*$, $a_2 = a_2^*$, $a_3 = a_3^*$ and quadratic relations $\{a_1, a_2\} = a_3$, $\{a_2, a_3\} = a_1$, $\{a_3, a_1\} = a_2$, here $\{x, y\} = xy + yx$ (see [24]).

Non-involutive algebra with generators a_1, a_2 satisfying (2) for $\mu = -1$ was determined by D.B. Fairlie ([4]) as q-analogue of the Lie algebra $so(3)$. Following [13], we denote it by $U_q'(so(3))$. It is easily seen that the considered $*$-algebras $A_{q,1}$, $A_{q,-1}$ are its real forms, the first algebra is compact, the second one is non-compact.

Some reprsentations of the Fairlie algebra have been investigated by different authors. All irreducible representations of $U(so_{gr}(3)) = A_{-1,1}$ was described by M.F. Gorodniy and G.B. Podkolzin in 1984 ([6]). Even in this case there arise irreducible representations, which do not have analogs in the classical case $U(so(3))$. Some series of finite-dimensional representations were studied by D.B. Fairlie ([4]) and M. Havlicek, A.U. Klymyk, E.Pelantova ([13]). The aim of this paper is to describe all irreducible representations of the above $*$-algebras $A_{q,\mu}$ up to unitary equivalence. We study both bounded and unbounded representations, give definition of the "integrable" representations. We also consider behavior of the irreducible representations as $q \to \pm 1$.

We note that irreducible representations of another real form of the Fairlie algebra $U_q(so(2,1))$ defined by the involution $a_1^* = -a_1$, $a_2^* = a_2$ was described by O.M. Gavrilik, A.U.Klimyk in [5].

Using the technique of semilinear relations one can describe representations of real forms of the quantum algebra $U_q(sl(3))$ and other objects (see [25],[31]).

1. Representations of semilinear relations

1.1. Class of relations. Let H be a complex separable Hilbert space, f_{ij}, g_{ij}, h_i, $i = 1, \dots, m$, $j = 1, \dots, l$, be bounded Borel functions defined on $D \subset \mathbb{R}^n$. A family of bounded commuting selfadjoint operators $\mathbb{A} = (A_k)_{k=1}^n$ and an operator $B \in L(H)$ which satisfy relations

$$\sum_{i=1}^{m} f_{ij}(\mathbb{A}) B g_{ij}(\mathbb{A}) = h_j(\mathbb{A}), \qquad j = 1, \dots, l, \tag{3}$$

is called a representation of system of semilinear relations (3). Clearly, if f_{ij}, g_{ij}, h_i are polynomials, then to equations (3) there corresponds a quotion algebra $\mathfrak{A}$ of the free $*$-algebra with generators $a_k = a_k^*$, $k = 1, \dots, n$, b, b^* with respect to the two-sided ideal generated by the elements $\sum_{i=1}^{m} f_{ij}(a_1, \dots, a_n) b g_{ij}(a_1, \dots, a_n) - h_j(a_1, \dots, a_n)$, $j = 1, \dots, l$. In this case the study of the family $\mathbb{A} = (A_k)_{k=1}^n$, ($A_k = A_k^*$, $[A_i, A_j] = 0$) and the operator $B \in L(H)$ satisfying relation (3) is equivalent to that of $*$-representations $\pi : \mathfrak{A} \to L(H)$.

Note that such notions as a unitarily equivalent representation of a $*$-algebra, an irreducible (indecomposable) representation, a factor-representation (a $*$-algebras of type I, not of type I), and others have the sense accepted in the theory of representations (see, for examples, [11, 16, 28]), and are naturally carried over to $*$-representations of relations (3).

The aim of this section is to study the structure of representations of system (3). Namely, we investigate their spectral properties (subsection 1.2), and the possibility of describing all irreducible representations up to a unitary equivalence (subsection 1.3).

R e m a r k 1.1. To study unbounded representations of relations (3) it is necessary to define the meaning of operator equalities (3). The question on "correct" definition of relations (3) it is investigated for some special semilinear relations (Section 2).

The study of bounded representations of system (3) can be reduced to a study of representations $\mathbb{A} = (A_k)_{k=1}^n$, $B \in L(H)$ of system of homogeneous relations

$$\sum_{i=1}^{m} f_{ij}(\mathbb{A}) B g_{ij}(\mathbb{A}) = 0, \qquad j = 1, \dots, l. \tag{4}$$

Namely, any representation $\mathbb{A}$, B of (3) is of the form $B = B' + \varphi(\mathbb{A})$, where $\mathbb{A}$, B' is a representation of (4) and φ is some Borel function (Proposition 1.1,[25]). Therefore, from now on we restrict ourselves to considering only system of homogeneous relations (4).

To system of semilinear relations (4) there correspond

a) the characteristic binary relation:

$$\Gamma = \{(t,s) \in \mathbb{R}^n \times \mathbb{R}^n \mid \Phi_j(t,s) = \sum_{i=1}^{n} f_{ij}(t) g_{ij}(s) = 0, j = 1, \dots, l\},$$

$\Phi_i(t,s)$ is called the characteristic function of the i-th relation;

b) an oriented graph (D, Γ) where an edge $\underset{t}{\bullet} \longrightarrow \underset{s}{\bullet}$ belongs to the graph if and only if $(t,s) \in \Gamma$.

Sometimes, for brevity, we denote the graph by Γ and call it the graph of system of relations (4).

Remark 1.2. If a family $\mathbb{A} = (A_k)_{k=1}^n$ and a selfadjoint operator B is a solution of system of equations (4), then $\mathbb{A}$, B satisfy also the following relations

$$\sum_{i=1}^{m} \overline{g_{ij}(\mathbb{A})} B_j \overline{f_{ij}(\mathbb{A})} = 0, \qquad j = 1, \dots, l. \tag{5}$$

Therefore, with a representation $\mathbb{A}$, $B = B^*$ of (4), it is naturally to connect the binary relation

$$\Gamma_s = \{(t,s) \in \mathbb{R}^n \times \mathbb{R}^n \mid (t,s) \in \Gamma, (s,t) \in \Gamma\},$$

where Γ is the binary relation of (4).

Throughout the paper, $\Gamma|_M$ denotes subgraph of the graph Γ with the property that vertices of $\Gamma|_M$ are points of M and its edges are that of Γ connecting the points of M. In the case when $\Phi(t,s) = 0$ is equivalent to $\Phi(s,t) = 0$, the graph Γ together with the edge $\bullet \longrightarrow \bullet$ also contains the edge $\bullet \longleftarrow \bullet$, hence the graph can be considered as non-oriented.

1.2. Support of representation. Let $\mathbb{A}$ be a family of commuting selfadjoint operator, $E_{\mathbb{A}}(\cdot)$ be the joint resolution of the identity for $\mathbb{A}$.

Definition 1.1. We say that a subset $\mathcal{F} \subset \mathbb{R}^n \times \mathbb{R}^n$ $\mathbb{A}$-*supports* an operator $B \in L(H)$ if

$$E_{\mathbb{A}}(\alpha) B E_{\mathbb{A}}(\beta) = 0$$

for any pair (α, β) of Borel sets such that $(\alpha \times \beta) \cap \mathcal{F} = \emptyset$.

It is not difficult to prove that there exists the smalest closed set $\mathcal{F}$ supporting B (take the complement to the union of all open $\alpha \times \beta$). We will denote this set by $\mathrm{supp}_{\mathbb{A}}(\mathcal{F})$. If the joint spectrum of the family $\mathbb{A}$ is finite, i.e., $\sigma(\mathbb{A}) = \{\lambda_1, \dots, \lambda_s\} \subset \mathbb{R}^n$, then that $\mathcal{F}$ $\mathbb{A}$-supports an operator B means $P_i B P_j = 0$ for any $(\lambda_i, \lambda_j) \notin \mathcal{F}$, where P_i is the projection on the eigenspace corresponding to λ_i. Under the assumption $\sigma(\mathbb{A}) = \{\lambda_1, \dots, \lambda_s\}$ we have that $\mathbb{A}$, $B \in L(H)$ determine a representation of relation (4) if and only if $P_i B P_j = 0$ as soon as $(\lambda_i, \lambda_j) \notin \Gamma$, or, which is the same, $\mathrm{supp}_{\mathbb{A}} B \subset \Gamma$, where Γ is the characteristic function corresponding to (4). This immediately follows from the equalities

$$P_r (\sum_{i=1}^{m} f_{ij}(\mathbb{A}) B g_{ij}(\mathbb{A})) P_k = \Phi_j(\lambda_r, \lambda_k) P_r B P_k, \quad r, k = 1, \dots, s, i = 1, \dots, l.$$

For general bounded representations we have the following necessary condition.

THEOREM 1.1. *If* $\mathbb{A}$, B *is a representation of relation* (4)*, then* $\operatorname{supp}_{\mathbb{A}}(B) \subset \Gamma$*, where* Γ *is the binary relation corresponding to* (4)*.*

The proof of the theorem is more complicated, we refer the reader to [25]. Let us note that from the above theorem it follows the Kleinike-Shirokov theorem under the assumption $A = A^*$, i.e., if $A = A^*$, $B \in L(H)$ satisfy relation $[A, [A, B]] = 0$, then A, B commute.

In the general case the inverse statement to that of Theorem 1.1 is not true (see example 1.1). However, it turns out that for a broad class of equations (4), its characteristic binary relation completely determines the solution space, i.e., the set of all operators $\mathbb{A}$-supported by Γ coinsides with the set

$$\{B \in L(H) \mid \sum_{i=1}^{n} f_{ij}(\mathbb{A}) B g_{ij}(\mathbb{A}) = 0, j = 1, \dots, l\}.$$

We first consider the case when the family $\mathbb{A}$ consists of at most two operators. The following theorem was proved in [25].

THEOREM 1.2. *Let* f_{ij}, g_{ij} *be polynomials. For a pair* $\mathbb{A}$, B *to define a representation of relation* (4) *it is necessary and sufficient that*

$$supp_{\mathbb{A}}(B) \subset \Gamma.$$

In particular, from this theorem it easily follows that, if two polynomial relations have the same graph and $\mathbb{A}$ consists of at most two operators, then their bounded representations coinside.

R e m a r k 1.3. The theorem remains valid if we require that $g_{ij} \in \mathrm{Lip}_1 \sigma(\mathbb{A})$ in the case $\mathbb{A} = (A_1, A_2)$ and $g_{ij} \in \mathrm{Lip}_{1/2} \sigma(\mathbb{A})$ when $\mathbb{A}$ consisits of only one operator A. If the functions f_{ij}, g_{ij} are continuous, then the condition $\operatorname{supp}_A B \subset \Gamma$ is not sufficient for (A, B) to be a representation of relation (3). The corresponding example was constructed in [29].

The condition of smoothness of the functions g_{ij} can be replaced by the condition on the graph of the relations: if $\operatorname{supp}_A(B) \subset F \equiv \{(t, s) \mid s = \varphi(t)\} \subset \mathbb{R}^2$, where φ is a bounded Borel function, then the pair A, B gives a representation of any relation whose graph contains F.

R e m a r k 1.4. In [25] the statement analogous to that of Theorem 1.2 was proved also for the following operator relations:

$$\sum_{k=1}^{n} f_k(M) B g_k(N) = 0, \tag{6}$$

where f_k, g_k are polynomials and M, N are arbitrary bounded normal operators. Namely, for M, N, B to satisfy (4) it is necessary and sufficient that

$$E_M(\alpha) B E_N(\beta) = 0$$

for any α, $\beta \in \mathfrak{B}(\mathbb{C})$, $\alpha \times \beta \cap \{(t, s) \in \mathbb{C} \times \mathbb{C} \mid \sum_{k=1}^{n} f_k(t) g_k(s) = 0\} = \emptyset$.

In particular, it is easy to see that this implies the Fuglede-Putnam theorem (see [23]).

If the family $\mathbb{A}$ consists of more then two operators, then the statement of Theorem 1.2, generally speaking, is not true. The following example is constructed by using the arguments analogous to that given in [29].

EXAMPLE 1.1. Let Δ be operator acting on $L(H)$ in the following way

$$\Delta(B) = \sum_{k=1}^{3} [A_k, [A_k, B]] - B.$$

It is easy to show that the characteristic function of the relation $\Delta^2(B) = 0$ equals to $\Phi^2(t,s)$, where $\Phi(t,s)$ is that of relation $\Delta(B) = 0$. Hence, characteristic binary relations of both relations coinside. At the same time, the equations

$$\Delta(B) = 0 \quad \text{and} \quad \Delta^2(B) = 0$$

are not equivalent, i.e., there exists a solution of the second equation which does not satisfy the first one.

The following result is true without any restriction on the dimension of the family $\mathbb{A}$.

THEOREM 1.3. *Let $\mathbb{A} = (A_k)_{k=1}^n$, and Γ be such that there exists a decomposition of $\sigma(\mathbb{A})$ into Borel sets P_i, $\sigma(\mathbb{A}) = \cup_{i=1}^n P_i$, such that each $P_i \times P_j \cap \Gamma$, $i, j = 1, 2, \ldots$, is a graph of a mapping, i.e., $P_i \times P_j \cap \Gamma = \{(t, \varphi_{ij}(t)\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Then $\mathbb{A}$, B satisfy (4) if and only if*

$$supp_{\mathbb{A}}(B) \subset \Gamma.$$

Let us note that the condition of the theorem holds, for example, if the set $\{\mu \in D \mid (\lambda, \mu) \in \Gamma\}$, $\sigma(\mathbb{A}) \in D$ is finite for any $\lambda \in \sigma(\mathbb{A})$.

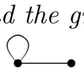
All results remain true if $B = B^*$, but with Γ_s instead of Γ.

1.3. Classification of representations. When studying representations of an algebraic structure, an important problem is that of describing, up to a unitary equivalence, all irreducible representations. The possibility of unitary classification of all indecomposable representations of system of semilinear relations (4) depends on the structure of the corresponding graph. As a rule, this problem is "wild", i.e., it contains as a subproblem the problem of describing, up to a unitary equivalence, pairs of selfadjoint operators without any relations([20]). In this case, for brevity, we say that system of semilinear relations (4) is "wild". We have the following criterion.

THEOREM 1.4. *System of semilinear relations (4) is "wild" if and only if the corresponding graph Γ contains the subgraphs:*

a)  *or b)* *(and with any other orientation) for the representations with arbitrary B;*

and the graph Γ_s contains the subgraphs:

c)  *, or d)* *for the representations with $B = B^*$.*

Otherwise, any indecomposable representation is one- or two-dimensional.

EXAMPLE 1.2. The relation

$$[A, [A, B]_q]_{q^{-1}} = \mu B, \qquad B = B^*,$$

is "wild" for any parameters $q \in \mathbb{R} \cup \mathbb{T}$, $q \neq 0$, $\mu \in \mathbb{R}$, except for $q \in \mathbb{T}$, $\mu \leq 0$. This follows from the fact that, only under this conditions, the equation

$$\Phi(t,s) \equiv t^2 + (q+q^{-1})ts + s^2 - \mu = 0$$

does not have two different solutions (t,s_1), (t,s_2) for any $t \in \mathbb{R}$.

Remark 1.5. It is easy to prove that under the assumption $B = B^*$, system (3) is not "wild" if and only if any its representation is that of the dynamical relation $AB = BF(A)$, where F is some Borel real function.

2. Polynomial from the left relations and dynamical systems

2.1. Symmetric representations of polynomial from the left semilinear relations. In this section we study representations $A = A^*$, $B = B^*$ (symmetric representations) of the following polynomial from the left relation

$$P(A,B) \equiv \sum_{k=1}^{n} A^k B \alpha_k(A) = 0, \tag{7}$$

where α_k are real polynomials, $\alpha_n(A) \equiv 1$.

The characteristic function of the above relation is of the form $\Phi(t,s) = \sum_{k=1}^{n} t^k \alpha_k(s)$. One can prove that there exist Borel functions F_k, $k = 1,\dots,n$, such that

$$\Phi(t,s) = \prod_{k=1}^{n} (t - F_k(s)).$$

By Theorem 1.3, we can restrict ourselves to considering relation (7) such that F_k, $k = 1,\dots,n$, are real and may be defined on some subset $D \subset \mathbb{R}$, and F_k, F_j are not identically equal for $k \neq j$.

Remark 2.1. From Theorem 1.1 it easily follows that if $\Delta \subset \mathbb{R}$ is invariant with respect to F_i, $i = 1,\dots,n$, then $E_A(\Delta)$ is such with respect to the operators of representations A, B.

First we show that the study of (7) can be reduced to the study of several dynamical relations:

$$AB_k = B_k F_k(A), \quad k = 1,\dots,n. \tag{8}$$

Before formulating the corresponding theorem we note that, without loss of generality, we can assume that $(P(A,B))^* = P(A,B)$, because $A = A^*$, $B = B^*$ determines a representation of (7) if and only if $\operatorname{supp}_A B \in \Gamma_s$, where $\Gamma_s = \{(t,s) \mid \Phi(t,s) = \Phi(s,t) = 0\}$. Hence $\Phi(t,s) = \overline{\Phi(s,t)}$. From the symmetricity of the characteristic function it follows easily that for any $\lambda \in D$, $k \leq n$, there exists $l \leq n$ such that $F_l(F_k(\lambda)) = \lambda$.

THEOREM 2.1. *To every representation $A = A^*$, $B = B^*$ of semilinear relation (7) satisfying the above conditions there corresponds a unique representation A, $B_k^{(1)}$, $B_k^{(2)}$, $k = 1,\dots,n$, of the relations*

$$AB_k^{(i)} = B_k^{(i)} F_k(A), \qquad k = 1,\dots,n, \ i = 1,2, \tag{9}$$

such that the operators $B_k^{(1)}$, $k = 1, \dots, n$, are selfadjoint,

$$B_k^{(2)} E_A(\{\lambda \mid \exists s < k : F_s(\lambda) = F_k(\lambda)\} \cup \{\lambda \mid \exists s \le k : F_s(F_k(\lambda)) = \lambda\}) = 0,$$
$$B_k^{(1)} E_A(\{\lambda \mid \exists s < k : F_s(\lambda) = F_k(\lambda)\}) = 0,$$

and $B = \sum_{k=1}^n B_k^{(1)} + \sum_{k=1}^n (B_k^{(2)} + (B_k^{(2)})^*)$. Moreover the representation A, B, B^* is irreducible if and only if so is the representation A, $B_k^{(1)}$,$B_k^{(2)}$, $k = 1, \dots, n$.

A detailed proof of the theorem is given in [25].

2.2. Additional relations. If no additional conditions are assumed on the operators $A = A^*$, $B = B^*$, then, by Theorem 1.4, we have that relation (7) is "wild" as soon as it differs from the dynamical relation

$$AB = BF(A),$$

where F is a fixed Borel function.

Here, following [25], we will study symmetric representations of (7) under the condition that the operators of the representation $A = A^*$, $B = B^*$ satisfy some other additional relations. For more general case when B need not be selfadjoint, we refer the reader to [25].

Additional relations between operators A, B lead to that between operators $B_k^{(i)}$ in the decomposition. For simplicity, from now on we will assume that the characteristic function of (7) is of the form $\Phi(t, s) = (t - F_1(s))(t - F_2(s))$, $F_i : D \to D$, D is a subset of $\mathbb{R}$. Just this case will be needed to study of representations of deformations of $so(3)$ in Section 3.

Note that from Theorem 2.1 it follows that $B_k^{(2)} = 0$. Put $X = B_1^{(2)}$, $Y_i = B_i^{(1)}$, $i = 1, 2$. It is not difficult to show that we can restrict ourselves to considering the case $\{\lambda \in D \mid F_i(F_i(\lambda)) = \lambda, F_i(\lambda) \ne \lambda\} = \emptyset$, $i = 1, 2$, hence $Y_i = Y_i E_A(\{\lambda \mid F_i(\lambda) = \lambda\})$, $i = 1, 2$.

Consider the relation of the form

$$\sum_k \psi_k(A) B g_k(A) B \varphi_k(A) = h(A), \tag{10}$$

where ψ_k, g_k, φ_k, h are bounded Borel function defined on the spectrum of A.

Proposition 2.1. *Let $A = A^*$, $B = B^*$ be a representation of (7). If the operators A, B satisfy additionally relation (10), then the corresponding operators A, Y_1, Y_2, X are connected by the relations:*

$$XY_1 = X^*Y_2 = 0, \quad Y_1 X s_{11}(A) = Y_2 X^* s_{22}(A) = 0, \quad Y_1Y_2 = Y_2Y_1 = 0, \tag{11}$$
$$X^*X s_{12}(A) + XX^* s_{21}(A) + Y_1^2 s_{11}(A) + Y_2^2 s_{22}(A) = h(A), \tag{12}$$

where $s_{ij}(A) = \sum_k \psi_k(F_j(F_i(A))) g_k(F_j(A)) \varphi_k(A)$.

Conversely, if A, Y_1, Y_2, X, in addition, satisfy (11), (12), then A, $B = Y_1 + Y_2 + X + X^*$ is a solution of (10).

We will study the following two cases:

1) $\{\lambda \in \sigma(A) \mid F_i(\lambda) = \lambda\} = \emptyset$, $i = 1, 2$, and either $\operatorname{Ker} s_{12}(A) = \{0\}$ or $\operatorname{Ker} s_{21}(A) = \{0\}$;

2) either $\{\lambda \in \sigma(A) \mid F_1(\lambda) = \lambda\} = \emptyset$ and $\operatorname{Ker} s_{12}(A) = \{0\}$ or $\{\lambda \in \sigma(A) \mid F_2(\lambda) = \lambda\} = \emptyset$ and $\operatorname{Ker} s_{21}(A) = \{0\}$.

Consider the first case. For convenience, assume that $\operatorname{Ker} s_{12}(A) = \{0\}$. Since $\{\lambda \in \sigma(A) \mid F_i(\lambda) = \lambda\} = \emptyset$, $i = 1, 2$, we have $Y_1 = Y_2 = 0$. From (8), (12) it follows that

$$AX = XF_1(A), \qquad X^*X = -XX^*s_{21}(A)s_{12}^{-1}(A) + h(A)s_{12}^{-1}(A). \tag{13}$$

Let $X = U|X|$ be the polar decomposition of the operator X ($\operatorname{Ker} X = \operatorname{Ker} U$). By [19, 32], the problem of unitary classification of the triple $(A,\, X,\, X^*)$ satisfying (13) reduces to that of the triple $(A,\, |X|,\, U)$ such that $A = A^*$, $|X| \geq 0$, U is a partial isometry, and

$$\begin{aligned} |X|U &= UG(|X|, A), \\ AU &= UF_1(A), \end{aligned} \tag{14}$$

where $G(x, y) = -xs_{21}(F_1(y))s_{12}^{-1}(F_1(y)) + h(F_1(y))s_{12}^{-1}(F_1(y))$, moreover U is a centred operator, i.e., $[U^k(U^*)^k, U^m(U^*)^m] = 0$, $[(U^*)^kU^k, (U^*)^mU^m] = 0$ and $[(U^*)^kU^k, U^m(U^*)^m] = 0$ for any m, $k \in \mathbb{N}$.

Let now $\{\lambda \in \sigma(A) \mid F_1(\lambda) = \lambda\} = \emptyset$, $\{\lambda \in \sigma(A) \mid F_2(\lambda) = \lambda\} \neq \emptyset$, and $\operatorname{Ker} s_{12}(A) = \{0\}$. Hence $Y_1 = 0$, $Y_2 \neq 0$. An easy computation shows that A, Y_2, X satisfy the relations:

$$\begin{gathered} [A, Y_2] = 0, \quad [|X|, Y_2] = 0, \quad [A, |X|] = 0, \\ AU = UF_1(A), \quad |X|U = UG(|X|, A), \quad Y_2U = 0, \end{gathered} \tag{15}$$

where $G(x, y) = -xs_{21}(F_1(y))s_{12}^{-1}(F_1(y)) + h(F_1(y))s_{12}^{-1}(F_1(y))$, moreover the operator U is centred. Note that in this case $\operatorname{Ker} U^* \neq \{0\}$.

Due to different applications representations of relations (14), (15), they have been investigated in [19, 32]. Put $\mathbb{F}(x_1, x_2) = (G(x_1, x_2), F_1(x_2))$. Denote by $\mathbb{F}^{(k)}$ the k-th iteration of $\mathbb{F}$. By [19, 32], if (A, X, X^*, Y_2) is an irreducible representation of (14), then the spectral measure of the family of commuting selfadjoint operators $(|X|^2,\, A)$ is concentrated on $\Omega_{\mathbf{x}}$, where $\mathbf{x} = (x_1, x_2)$, $\Omega_{\mathbf{x}} = \{\mathbb{F}^{(k)}(x_1, x_2) \mid k \geq 0,\ (\mathbb{F}^{(k)})_1(x_1, x_2) > 0, F_2(x_2) = x_2\}$ or $\Omega_{\mathbf{x}} = \{\mathbb{F}^{(k)}(x_1, x_2) \mid 0 \leq k \leq m-1, (\mathbb{F}^{(k)})_1(x_1, x_2) > 0, k \leq m, (\mathbb{F}^{(m)})_1(x_1, x_2) = 0, F_2(x_2) = x_2\}$. Let $\mu^2(\mathbf{x})s_{11}(x_2) + x_1s_{12}(x_2) = h(x_2)$. Representations act on $l_2(\Omega_{\mathbf{x}})$ by the formulas:

$$\begin{gathered} Ae_{\mathbf{y}} = y_2e_{\mathbf{y}}, \quad |X|e_{\mathbf{y}} = \sqrt{y_1}e_{\mathbf{y}}, \quad Y_2e_{\mathbf{x}} = \mu(\mathbf{x})e_{\mathbf{x}}, \\ Ue_{\mathbf{y}} = \begin{cases} e_{\mathbb{F}(\mathbf{y})} & \text{if } \mathbb{F}(\mathbf{y}) \in \Omega_{\mathbf{x}}, \\ 0 & \text{if } \mathbb{F}(\mathbf{y}) \notin \Omega_{\mathbf{x}}. \end{cases} \end{gathered}$$

The possibility of describing all irreducible representations of relation (15) depends on the structure of the dynamical system $(\sigma(|X|^2, A), \mathbb{F}(x, y))$. Let us note that in this case $\mathbb{F}$ is one-to-one. If the dynamical system has a measurable section (a set intersecting each orbit Ω in a single point) or for any orbit Ω, there exists $\mathbf{x} = (x_1, x_2) \in \Omega$ such that $x_1 \leq 0$, then the spectral measure of the family $(|X|^2, A)$ is concentrated on a subset Ω^0 of an orbit $\Omega_{\mathbf{x}} = \{\mathbb{F}^{(k)}(x_1, x_2) \mid k \in \mathbb{Z}\}$. The orbit $\Omega_{\mathbf{x}}$ satisfies the conditions:

either $(\mathbb{F}^{(k)})_1(x_1,x_2) > 0$, $k \in \mathbb{Z}$, then $\Omega^0 = \Omega_{\mathbf{x}}$;
or $(\mathbb{F}^{(k)})_1(x_1,x_2) > 0$, $k \geq 0$, and $(\mathbb{F}^{(-1)})_1(x_1,x_2) = 0$, then $\Omega^0 = \{\mathbb{F}^{(k)}(x_1,x_2), k \geq 0\}$;
or $(\mathbb{F}^{(-k)})_1(0,x_2) > 0$, $k \in \mathbb{N}$, then $\Omega^0 = \{\mathbb{F}^{(-k)}(0,x_2), k \in \mathbb{N} \cup \{0\}\}$;
or $(\mathbb{F}^{(k)})_1(x_1,x_2) > 0$, $0 \leq k \leq m-1$, and $(\mathbb{F}^{(-1)})_1(x_1,x_2) = 0$, $(\mathbb{F}^{(m)})_1(x_1,x_2) = 0$, then $\Omega^0 = \{\mathbb{F}^{(k)}(x_1,x_2), 0 \leq k \leq m\}$. Representations act in $l_2(\Omega^0)$ by the formulas:

$$Ae_{\mathbf{y}} = y_2 e_{\mathbf{y}}, \quad |X|e_{\mathbf{y}} = \sqrt{y_1}\, e_{\mathbf{y}},$$
$$Ue_{\mathbf{y}} = \begin{cases} e_{\mathbb{F}(\mathbf{y})} & \text{if } \mathbb{F}(\mathbf{y}) \in \Omega^0, \\ 0 & \text{if } \mathbb{F}(\mathbf{y}) \notin \Omega^0. \end{cases}$$

Other cases that arise in study of representations of deformations of $so(3)$ are considered in Section 3.

2.3. Unbounded representations. Studying representations of different $*$-algebras there arises a necessity to study unbounded operators satisfying (7). For this purpose we need to give sense to the operator equality

$$\sum_{k=1}^{n} A^k B \alpha_k(A) = 0.$$

Here we assume that α_k, $k = 1, \dots, n$ are polynomials,

$$\sum_{k=1}^{n} t^k \alpha_k(s) = (t - F_1(s))^{m_1} (t - F_2(s))^{m_2}, \quad m_1, m_2 \in \mathbb{N} \cup \{0\}.$$

Following [25], we make the following definition

Definition 2.1. We say that symmetric operators A, B satisfy relation (7) if there exists a dense set Φ such that

1) Φ is invariant with respect to A, B, B^*, $E_A(\Delta)$, $\Delta \in \mathfrak{B}(\mathbb{R})$;
2) $\Phi \subset H_b(A)$, $D(B) \supset H_b(A)$, where $H_b(A)$ is the set of all vectors bounded for the operator A;
3) (7) holds on Φ.

Applying the arguments analogous to that given in Theorem 6.3,[25], one can prove the following theorem.

Theorem 2.2. *Let conditions (1)–(2) hold for $\Phi \subset H$. Then the following statements are equivalent:*

1) $\sum_{k=1}^{n} A^k B \alpha_k(A)\varphi = 0$ *for all* $\varphi \in \Phi$,
2) $E_A(\alpha) B E_A(\beta)\varphi = 0$ *for all* $\varphi \in \Phi$, *for all* α, $\beta \in \mathfrak{B}(\mathbb{R}), \alpha \times \beta \cap \Gamma = \emptyset$.

If A, B satisfy, in addition, relation (10), where φ_k, ψ_k, g_k, h are polynomials, then, in the first considered case, we have $B\varphi = X\varphi + X^*\varphi$ and relations (14) hold on Φ, $D(X), D(X^*) \supset H_b(A)$, irreducible representations being described by formulas (16); in the second case, $B\varphi = X\varphi + X^*\varphi + Y\varphi$, $\varphi \in \Phi$, and A, X, X^*, Y satisfy (15), $D(X)$, $D(X^*) \supset H_b(A)$, irreducible representations are of the form (16).

3. Representations of deformations of $SO(3)$

3.1. $*$-Algebras $A_{q,\mu}$. Here we consider $*$-algebras $A_{q,\mu}$ which are generated by selfadjoint elements $a_1 = a_1^*$, $a_2 = a_2^*$ and relations

$$[a_1, [a_1, a_2]_q]_{q^{-1}} = \mu a_2, \qquad [a_2, [a_2, a_1]_q]_{q^{-1}} = \mu a_1, \tag{16}$$

where $[x, y]_q = xy - qyx$, $q \in \mathbb{R} \cup \mathbb{T}$, $q \neq 0$, $\mu \in \mathbb{R}$. If $q = \mu = 1$, then relations (16) determine the universal enveloping of the Lie algebra $so(3)$ or that of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ graded $so_{gr}(3)$ if $q = -\mu = -1$.

First note that by using the transformation $a_1 \leftrightarrow \frac{1}{\sqrt{\mu}} a_1$, if $\mu > 0$, and $a_1 \leftrightarrow \frac{1}{\sqrt{-\mu}} a_1$, if $\mu < 0$, relation (16) can be reduced to the following

$$[a_1, [a_1, a_2]_q]_{q^{-1}} = \operatorname{sign}\mu a_2, \tag{17}$$

$$[a_2, [a_2, a_1]_q]_{q^{-1}} = \operatorname{sign}\mu a_1. \tag{18}$$

The non-involutive algebra generated by the elements a_1, a_2 and relations (16) for $\mu = -1$ is the q-analogue of the universal enveloping algebra of the Lie algebra $U(so(3))$ defined by Fairlie ([4]). The $*$-algebras $A_{q,1}$, $A_{q,-1}$ are its real forms, the first is compact, the second is non-compact. Some represesentations of the Fairlie algebra have been studied in [4, 5, 13]. Our aim is to study all irreducible representations of $A_{q,\mu}$ up to a unitary equivalence by using the technique developed in the previous sections.

Let us note that relations (17), (18) are symmetric. Moreover the first relation (17) is semilinear with respect to a_2, the second is quadratic with respect to a_2 as considered in Section 2. Let π be a representation of $A_{q,\mu}$ in a Hilbert space H. Put $A_i = \pi(a_i)$, $i = 1, 2$. To study unbounded representations of relations (17), (18), we must define their meaning. Following Section 2, we give

Definition 3.1. Symmetric operators A_1, A_2 form a representation of (17), (18) if there exists a dense set $\Phi \subset H$ such that:

1) Φ is invariant with respect to A_1, A_2, $E_{A_1}(\Delta)$, $\Delta \in \mathfrak{B}(\mathbb{R})$;
2) Φ consists of bounded vectors for A_1, $(\Phi \subset H_b(A_1))$, $D(A_2) \supset H_b(A_1)$;
3) relations (17), (18) hold on Φ.

3.2. Representations of $A_{q,0}$. 1) Let $q \in \mathbb{R} \setminus \{0, \pm 1\}$. Taking into account that the characteristic function of semilinear relation (17) is of the form $\Phi(t, s) = (t - qs)(t - q^{-1}s)$, we have, by Remark 2.1, that the representation space H is decomposed into a direct sum of the subspaces $H_0 = \operatorname{Ker} A_1$, $H_1 = (\operatorname{Ker} A_1)^{\perp}$, which are invariant with respect to A_1, A_2. In H_0 the operator $A_1 = 0$, hence any irreducible representation in the subspace is one dimensional. In H_1, by Theorem 2.1, the operator A_2 can be represented in the form $A_2 = X + X^*$, where $A_1 X = q X A_1$. From Proposition 2.1 it follows that X, X^* satisfy, in addition, the relation $X^* X = q^{-2} X X^*$. Since the dynamical system $(\lambda, \mu) \to (q\lambda, q^{-2}\mu)$ has a measurable section $\tau \times \mathbb{R}^+$, where

$$\tau = \begin{cases} (-|q|, -1] \cup \{0\} \cup [1, |q|) & \text{if } |q| > 1, \\ [-1, -|q|) \cup \{0\} \cup (|q|, 1] & \text{if } |q| < 1, \end{cases}$$

we have, by [32], that any irreducible reprentation is connected with an orbit of the dynamical system. Namely, the following proposition holds:

Proposition 3.1. *Any irreducible representation of relation (17), (18) is one of the following:*

1) one-dimensional: $A_1 = (\lambda)$, $A_2 = (\mu)$, $\lambda,\ \mu \in \mathbb{R}$, $\lambda\mu = 0$;

2) infinite-dimensional in $l_2(\mathbb{Z})$:

$$A_1 e_k = \lambda q^k e_k, \quad A_2 e_k = q^{-k}\mu e_{k+1} + q^{-(k-1)}\mu e_{k-1},$$

where $(\lambda, \mu) \in \tau \times \mathbb{R}^+$, $\lambda \neq 0$.

2) Let $q \in \mathbb{T} \setminus \{\pm 1\}$. Since any connected components of the graph corresponding to semilinear relation (17) are $\bullet$, $\overset{\circlearrowleft}{\bullet}$, we have that any irreducible representation is one-dimensional:

$$A_1 = \lambda, \quad A_2 = \mu, \quad \lambda, \mu \in \mathbb{R}, \quad \lambda\mu = 0.$$

3) Let $q = \pm 1$. If $q = 1$, then relations (17), (18) are of the form

$$[a_1, [a_1, a_2]] = 0, \qquad [a_2, [a_2, a_1]] = 0.$$

By Definition 2, we have $[A_1, A_2] = 0$ on Φ, hence they commute in the sense of resolutions of the identity. Thus, any irreducible representation is one-dimensional:

$$A_1 = \lambda, \quad A_2 = \mu, \quad \lambda, \mu \in \mathbb{R}.$$

It should be noted that under another definition of unbounded representations of $A_{q,\mu}$ there exist representations different from the described ones.

If $q = -1$, then as above, we have $\{A_1, A_2\} = 0$ on Φ. By [15], any irreducible representation is either 1) one-dimensional: $A_1 = \lambda$, $A_2 = \mu$, $\lambda\mu = 0$, $\lambda, \mu \in \mathbb{R}$; or 2) two-dimensional:

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad \lambda > 0,\ \mu > 0. \tag{19}$$

Note that for $q = 1$ ($q = -1$), the infinite-dimensional representation from Propsition 3.1 is decomposed into the direct sum of one-dimensional irreducible reprsentation (respectively, two-dimensional irreducible representations of the form (19)).

3.3. Representations of $A_{q,\mu}$, $\mu > 0$. 1) $q \in \mathbb{R} \setminus \{0, \pm 1\}$. First consider the case $q > 0$. Characteristic function corresponding to (17) is of the form

$$\Phi(t, s) = t^2 - (q + q^{-1})ts + s^2 - 1 = (t - F_1(s))(t - F_2(s)),$$

where $F_{1(2)}(s) = sch\sigma \pm \sqrt{s^2 sh^2\sigma + 1}$, here $sh\sigma = \frac{q-q^{-1}}{2}$. Define $\dfrac{sh\sigma A_0}{sh\sigma} = A_1$. It is easy to check that $supp_{A_0} A_2 \subset \{(t, s) \mid (t - s - 1)(t - s + 1) = 0\}$. By Theorem 2.1, the operator A_2 can be represented in the form $A_2 = X + X^*$ such that

$$A_0 X = X(A_0 + 1).$$

Besides, X, X^* are connected by the relation

$$X^* X = F(XX^*, A_0),$$

where $F(x, y) = x\frac{ch((y-1)\sigma)}{ch((y+1)\sigma)} - \frac{sh(y\sigma)}{2sh\sigma ch((y+1)\sigma)}$. The dynamical system $(x, y) \to (F(x, y+1), y+1)$ has the measurable section $\mathbb{R}^+ \times [0, 1)$. Thus, by [32], we have the following proposition.

Proposition 3.2. *Let $q = e^{\sigma}$, $\sigma \in \mathbb{R} \setminus \{0\}$. Any irreducible representation is finite-dimensional and unitarily equivalent to one of the following: $H = \mathbb{C}^n$*

$$A_1 e_k = \frac{sh((2k-n-1)\sigma/2)}{sh\sigma} e_k, \qquad A_2 e_k = \begin{cases} \alpha_1 e_2, & k = 1, \\ \alpha_k e_{k+1} + \alpha_{k-1} e_{k-1}, & k \neq 1, n, \\ \alpha_{n-1} e_{n-1}, & k = n, \end{cases}$$

where $\alpha_k = \sqrt{\frac{sh(k\sigma)sh((n-k)\sigma)}{4sh^2\sigma ch((2k-n-1)\sigma/2)ch((2k-n+1)\sigma/2)}}$, $k = 1, \ldots, n-1$.

Let now $q < 0$. Characteristic function corresponding to (17) is the same but with $F_{1(2)}(s) = -sch\sigma \pm \sqrt{s^2 sh^2\sigma + 1}$. As above, put $\frac{sh\sigma A_0}{sh\sigma} = A_1$. Then $supp_{A_0} A_2 \subset \tilde{\Gamma} \equiv \{(t,s) \mid (t+s-1)(t+s+1) = 0\}$. Let

$$\tilde{F}_i(x) = \begin{cases} -(x+(-1)^i), & \text{if } x \in (-\frac{1}{2}+2k, \frac{1}{2}+2k),\ k \in \mathbb{Z}, \text{ or} \\ & x = \frac{1}{2} - 2p, -\frac{1}{2} - 2(p+1), p \in \mathbb{N}, \\ -(x-(-1)^i), & \text{otherwise,} \end{cases} \qquad i = 1, 2.$$

If $x \neq \pm\frac{1}{2}$, then $\tilde{F}_1(\tilde{F}_2(x)) = \tilde{F}_2(\tilde{F}_1(x)) = x$, $\tilde{F}_2(\pm\frac{1}{2})) = \pm\frac{1}{2}$. It is easily seen that $\tilde{\Gamma} = \{(t,s) \mid (t - \tilde{F}_1(s))(t - \tilde{F}_2(s)) = 0\}$ and the sets $M_1 = \{(-1)^k(\frac{1}{2}+k), k \in \mathbb{N} \cup \{0\}\}$, $M_2 = \{(-1)^k(-\frac{1}{2}-k), k \in \mathbb{N} \cup \{0\}\}$ are invariant with respect to F_i, $i = 1, 2$. Thus, the representation space H is decomposed into a direct sum of subspaces H_i, $i = 1, 2, 3$, which are invariant with respect to A_0, A_2 or, what is the same, with respect to A_1, A_2. Here $H_1 = E_{A_0}(M_1)H$, $H_2 = E_{A_0}(M_2)H$, $H_3 = (H_1 \oplus H_2)^{\perp}$. In H_i, $i = 1, 2$, the operator A_2 can be represented in the form $A_2 = X + X^* + Y$, $Y = Y^* : H(\pm\frac{1}{2}) \to H(\pm\frac{1}{2})$, $X^*|_{H(\pm\frac{1}{2})} = 0$ such that

$$A_0 X = X\tilde{F}_1(A_0).$$

Moreover, if (A_1, A_2) is irreducible, then, by results of subsection 2.2, $Y|_{H(\pm\frac{1}{2})} = \mu I$, $\mu \in \mathbb{R}$ and

$$X^*X = F_{\mu}(XX^*, A_0),$$

where

$$F_{\mu}(x,y) = \begin{cases} x\frac{ch((y\mp1)\sigma)}{ch((y\pm1)\sigma)} \mp \frac{sh(y\sigma)}{2sh\sigma ch((y\pm1)\sigma)} & \text{if } y = \pm(-1)^k(\frac{1}{2}+k), k \in \mathbb{N}, \\ \mu^2\frac{ch(\sigma/2)}{ch(3\sigma/2)} - \frac{sh(\sigma/2)}{2sh\sigma ch(3\sigma/2)} & \text{if } y = \pm\frac{1}{2}. \end{cases}$$

In H_3, we have $A_2 = X + X^*$ and

$$A_0 X = X\tilde{F}_1(A_0),$$
$$X^*X = F(XX^*, A_0),$$

where $F(x,y) = x\frac{ch((y-1)\sigma)}{ch((y+1)\sigma)} - \frac{sh(y\sigma)}{2sh\sigma ch((y+1)\sigma)}$, if $y \in (-\frac{1}{2}+2k+1, \frac{1}{2}+2k+1)$, otherwise $F(x,y) = x\frac{ch((y+1)\sigma)}{ch((y-1)\sigma)} + \frac{sh(y\sigma)}{2sh\sigma ch((y-1)\sigma)}$. The set $\mathbb{R}^+ \times (-\frac{1}{2}, \frac{1}{2})$ is a measurable section of the dynamical system $\mathbb{R}^+ \times \mathbb{R} \setminus (M_1 \cup M_2) \ni (x,y) \to (F(x, \tilde{F}_1(y)), \tilde{F}_1(y))$. Thus, by arguments given in Section 2, we have the following list of irreducible representations.

Proposition 3.3. *Let $q = -e^{\sigma}$, $\sigma \in \mathbb{R} \setminus \{0\}$. Any irreducible representation is unitarily equivalent to one of the following:*

1) $H = \mathbb{C}^n$,

$$A_1 e_k = \frac{(-1)^k sh((2k+(-1)^j)\sigma/2)}{sh\sigma} e_k, \quad A_2 e_k = \begin{cases} \alpha_0 e_1 + \frac{(-1)^i sh(n\sigma)}{2sh\sigma ch(\sigma/2)} e_0, & k = 0, \\ \alpha_k e_{k+1} + \alpha_{k-1} e_{k-1}, & k \neq 0, n-1, \\ \alpha_{n-2} e_{n-2}, & k = n-1, \end{cases}$$

where $\alpha_k = \sqrt{\frac{sh((n-k-1)\sigma)sh((n+k+1)\sigma)}{4sh^2\sigma ch((2k+1)\sigma/2)ch((2k+3)\sigma/2)}}$, $k = 0, \dots, n-2$, $i, j = 0, 1$;

2) $H = \mathbb{C}^n$, $n = 2p+1$,

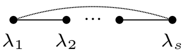
$$A_1 e_k = \frac{(-1)^k sh((2k-n+1)\sigma/2)}{sh\sigma} e_k, \quad A_2 e_k = \begin{cases} \alpha_0 e_1, & k = 0, \\ \alpha_k e_{k+1} + \alpha_{k-1} e_{k-1}, & k \neq 0, n-1, \\ \alpha_{n-2} e_{n-2}, & k = n-1, \end{cases}$$

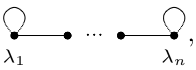
where $\alpha_k = \sqrt{\frac{sh((k+1)\sigma)sh((n-k-1)\sigma)}{4sh^2\sigma ch((2k-n+1)\sigma/2)ch((2k-n+3)\sigma/2)}}$, $k = 0, \dots, n-2$.

2) Let $q \in \mathbb{T} \setminus \{\pm 1\}$. Put $\sigma = arccos\frac{q+q^{-1}}{2}$. Here one should distinguish two cases: a) $\sigma \in \pi\mathbb{Q}$, b) $\sigma \notin \pi\mathbb{Q}$.

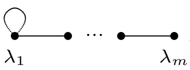
Let $\sigma = \pi\frac{k}{n}$ ($\frac{k}{n}$ is an irreducible fraction), $s = \begin{cases} n, & k \text{ is even}, \\ 2n, & k \text{ is odd}. \end{cases}$

Then, in contrast to the above cases, all connected components of the graph corresponding to semilinear relation (17) are of the form:

I. , where $\lambda_m = \frac{\sin((x+m)\sigma)}{\sin\sigma}$, $x\sigma \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{\frac{\pi(2l+1)+m\sigma}{2} \mid m, l \in \mathbb{Z}\}$,

II. , where $\lambda_m = -\frac{\cos(\sigma/2+(m-1)\sigma)}{\sin\sigma}$, k is odd;

III. , where $m \leq n$, $\lambda_i = \frac{\sin((x+i)\sigma)}{\sin\sigma}$, $i = 1, \dots, m$, $\lambda_i \neq \lambda_j$, $1 \leq i < j \leq m$;

IV. , where $m \leq n$ and either $\lambda_i = \frac{\cos(\sigma/2+(i-1)\sigma)}{\sin\sigma}$, $i = 1, \dots, m$, or $\lambda_i = -\frac{\cos(\sigma/2+(i-1)\sigma)}{\sin\sigma}$, $i = 1, \dots, m$, $\lambda_i \neq \lambda_j$,$1 \leq i \neq j \leq m$.

Since all the connected components are finite we have that $\sigma(A_1)$ is finite as soon as (A_1, A_2) is irreducible, the spectral measure concentrating on the set of points of a component (see Proposition 3.6,[25]). If the support of an irreducible representation is the graph II, then, by Theorem 2.1, we have $A_2 = X + X^* + Y_1 + Y_2$, where $Y_1 = Y_1^* : H_{\lambda_1} \to H_{\lambda_1}$, $Y_2 = Y_2^* : H_{\lambda_n} \to H_{\lambda_n}$, H_{λ_i} is the eigenspace of A_1 corresponding to λ_i (here $F_1(\lambda_i) = \lambda_{i+1}$, $i \neq n$, $F_1(\lambda_n) = \lambda_n$, $F_2(\lambda_i) = \lambda_{i-1}$, $i \neq 1$, $F_2(\lambda_1) = \lambda_1$). If $\sigma(A_1) = \{\dfrac{cos(\frac{\pi}{2}+s\sigma)}{\sin\sigma} \mid s = 0, \dots, n\}$, then the support of the representation is graph III. By Theorem 2.1, $A_2 = X + X^*$, (here $F_1(\lambda_i) = \lambda_{i+1}$, $i \neq n$, $F_2(\lambda_i) = \lambda_{i-1}$, $i \neq 1$), but neither $\operatorname{Ker} s_{12}(A_1) = \{0\}$ nor $\operatorname{Ker} s_{21}(A_1) = \{0\}$ (see Section 2). If support of the irreducible representation is graph IV with $m = n$, $\lambda_n = \frac{1}{\sin\sigma}$, we have $A_2 = Y_1 + X + X^*$, $Y_1 = Y_1^* : H_{\lambda_1} \to H_{\lambda_1}$, ($F_1(\lambda_i) = \lambda_{i+1}$, $i \neq n$, $F_2(\lambda_i) = \lambda_{i-1}$, $i \neq 1$, $F_2(\lambda_1) = \lambda_1$), but $\operatorname{Ker} s_{12}(A_1) \neq \{0\}$. In all these cases, in contrast to the above ones, there arises irreducible representations such that $\sigma(A_1)$ is not simple. For a detailed investigation we refer the reader to [27]. The following proposition gives the full description of irreducible representations (A_1, A_2) for $\sigma \in \pi\mathbb{Q}$. We will denote by I_n the identity in n-dimensional space.

Proposition 3.4. *Let $\sigma = \pi\frac{k}{n}$, $\sigma \neq \pi l$. Any irreducible representation of (17), (18) is unitarily equivalent to one of the following:* 1) $H = \mathbb{C}^s$,

$$A_1 e_m = \frac{\sin((x+m)\sigma)}{\sin\sigma} e_m, \qquad A_2 e_m = \begin{cases} \alpha_0 e_1 + e^{i\phi}\alpha_{s-1}e_{s-1}, & m = 0, \\ \alpha_m e_{m+1} + \alpha_{m-1}e_{m-1}, & m \neq 0, s-1, \\ \alpha_{s-2}e_{s-2} + e^{-i\phi}\alpha_{s-1}e_0, & m = s-1, \end{cases}$$

where $\alpha_m = \sqrt{\frac{4y\sin^2\sigma\cos(x\sigma)\cos((x+1)\sigma) - \sin m\sigma\sin((2x+(m+1))\sigma)}{4\sin^2\sigma\cos((x+(m+1))\sigma)\cos((x+m)\sigma)}}$, $(x,y) \in \{(x,y) \in M_1 \times \mathbb{R}^+ \mid \alpha_m > 0, m = 0,\dots,s-1\}$, $\phi \in [0,2\pi)$, $\sigma M_1 = [-\pi/2,\pi/2]\setminus\{\frac{\pi(2l+1)+m\sigma}{2} \mid l,m \in \mathbb{Z}\}$;

2) *$H = \mathbb{C}^n$, k is odd,*

$$A_1 e_m = -\frac{\cos(\frac{\sigma}{2} + (m-1)\sigma)}{\sin\sigma} e_m, \qquad A_2 e_m = \begin{cases} (-1)^i\lambda e_1 + \mu_1 e_2, & m = 1, \\ \mu_m e_{m+1} + \mu_{m-1}e_{m-1}, & m \neq 1, n-1, \\ \mu_{n-1}e_{n-1} + (-1)^j\lambda e_n, & m = n, \end{cases}$$

where $\mu_m = \sqrt{\frac{\sin^2(m\sigma) - 4\lambda^2\sin^2(\sigma/2)\sin^2\sigma}{4\sin^2\sigma\sin((2m-1)\sigma/2)\sin((2m+1)\sigma/2)}}$, $i,j = 0,1$, $\lambda \in \{\lambda \in \mathbb{R} \mid \mu_m > 0, m = 1,\dots,n-1\}$;

3) *$H = \mathbb{C}^{2n}$, k is odd,*

$$A_1 = \begin{pmatrix} \lambda_1 I_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n I_2 \end{pmatrix}, A_2 = \begin{pmatrix} Y_1 & \mu_1 I_2 & & \\ \mu_1 I_2 & 0 & \ddots & \\ & \ddots & \ddots & \mu_{n-1}I_2 \\ & & \mu_{n-1}I_2 & Y_2 \end{pmatrix},$$

where $\lambda_m = -\frac{\cos(\frac{\sigma}{2} + (m-1)\sigma)}{\sin\sigma}$, $Y_1 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, $Y_2 = \lambda\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$, μ_m *is the coefficient defined above,* $\lambda \in \{\lambda \in \mathbb{R}^+ \mid \mu_m > 0, m = 1,\dots,n-1\}$, $\varphi \in [0,2\pi)$;

4) *$H = \mathbb{C}^n$, k is odd,*

$$A_1 e_m = -\frac{\cos((m-1)\sigma)}{\sin\sigma} e_m, \quad A_2 e_m = \begin{cases} \alpha_1 e_2, & m = 1, \\ \alpha_m e_{m+1} + \alpha_{m-1}e_{m-1}, & m \neq 1, n, \\ \alpha_{n-1}e_{n-1}, & m = n, \end{cases}$$

where $\alpha_1 = \alpha_{n-1} = \frac{1}{\sqrt{2}\sin\sigma}$, $\alpha_m = \frac{1}{2\sin\sigma}$, $m = 2,\dots,n-1$;

5) *$H = \mathbb{C}^{2(n-1)}$, k is odd,*

$$A_1 = \begin{pmatrix} \lambda_1 I_1 & & & 0 \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n I_1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & X_1^* & & & \\ X_1 & 0 & X_2^* & & \\ & X_2 & \ddots & & \\ & & & \ddots & X_{n-1}^* \\ & & & X_{n-1} & 0 \end{pmatrix},$$

where $\lambda_m = -\frac{\cos((m-1)\sigma)}{\sin\sigma}$, $X_1 = \begin{pmatrix} \frac{1}{\sqrt{2}\sin\sigma}\cos\varphi \\ \frac{1}{\sqrt{2}\sin\sigma}\sin\varphi \end{pmatrix}$, $X_{n-1} = (\frac{1}{\sqrt{2}\sin\sigma}, 0)$, $X_i = \frac{1}{2\sin\sigma}I_2$, $i = 2, \dots, n-2$, $\varphi \in [0, \pi)$*;*

6*)* $H = \mathbb{C}^n$*,* k *is even,*

$$A_1 e_m = (-1)^{k/2}\frac{\cos((2m-1)\sigma/2)}{\sin\sigma}e_m, \qquad A_2 e_m = \begin{cases} \alpha_1 e_2 + \frac{(-1)^i}{2\sin\sigma}e_1, & m = 1, \\ \alpha_m e_{m+1} + \alpha_{m-1}e_{m-1}, & m \neq 1, n, \\ \alpha_{n-1}e_{n-1}, & m = n, \end{cases}$$

where $\alpha_m = \frac{1}{2\sin\sigma}$, $i = 0, 1$*;*

7*)* $H = \mathbb{C}^{2n-1}$*,* k *is even,*

$$A_1 = \begin{pmatrix} \lambda_1 I_2 & & & 0 \\ & \ddots & & \\ & & \lambda_{n-1}I_2 & \\ 0 & & & \lambda_n I_1 \end{pmatrix}, \; A_2 = \begin{pmatrix} Y & X_1^* & & \\ X_1 & 0 & & \\ & & \ddots & X_{n-1}^* \\ & & X_{n-1} & 0 \end{pmatrix},$$

where $\lambda_m = \dfrac{\cos((2m-1)\sigma/2)}{\sin\sigma}$, $Y = \frac{1}{2\sin\sigma}\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$, $X_i = \frac{1}{2\sin\sigma}I_2$, $i = 1, \dots, n-2$, $X_{n-1} = (\frac{1}{\sqrt{2}\sin\sigma}, 0)$, $\varphi \in [0, 2\pi)$*;*

8*)* $H = \mathbb{C}^l$*,*

$$A_1 e_m = (-1)^i\frac{(\cos(2m-1)\sigma/2)}{\sin\sigma}e_m, \qquad A_2 e_m = \begin{cases} \alpha_1 e_2 + \frac{(-1)^j\sin l\sigma}{2\sin(\sigma/2)\sin\sigma}e_1, & m = 1, \\ \alpha_m e_{m+1} + \alpha_{m-1}e_{m-1}, & m \neq 1, l, \\ \alpha_{l-1}e_{l-1}, & m = l, \end{cases}$$

where $\alpha_m = \sqrt{-\frac{\sin((l-m)\sigma)\sin((l+m)\sigma)}{4\sin^2\sigma\sin((2m-1)\sigma/2)\sin((2m+1)\sigma/2)}}$*),* $l \in \{l \in \mathbb{N} \mid \alpha_m > 0, 1 \leq m < l\}$, $i, j = 0, 1$*;*

9*)* $H = \mathbb{C}^l$*,*

$$A_1 e_m = \frac{\sin((x+m)\sigma)}{\sin\sigma}e_m, \qquad A_2 e_m = \begin{cases} \alpha_1 e_2, & m = 1, \\ \alpha_m e_{m+1} + \alpha_{m-1}e_{m-1}, & m \neq 1, l, \\ \alpha_{l-1}e_{l-1}, & m = l, \end{cases}$$

where $\alpha_1 = \sqrt{-\frac{\sin((x+1)\sigma)}{2\sin\sigma\cos((x+2))\sigma)}}$, $\alpha_m = \sqrt{-\frac{\sin(m\sigma)\sin((2x+(m+1))\sigma)}{4\sin^2\sigma\cos((x+m)\sigma)\cos((x+(m+1))\sigma)}}$, $m \neq 1$, $(x, l) \in \{(x, l) \in \mathbb{R}\times\mathbb{N} \mid \sigma x \neq \frac{\pi}{2} - l\sigma + \pi s, \sin l\sigma\sin((2x+(l+1))\sigma) = 0, \sin(\frac{m-l}{2})\sigma\cos((x+\frac{m+lp}{2})\sigma) \neq 0, \alpha_m > 0, m = 1, \dots, l-1, s \in \mathbb{Z}\}$.

If $\sigma \notin \pi\mathbb{Q}$, then for any set M which is invariant with respect to F_1, F_2, we have $\overline{M} = [-\frac{1}{\sin\sigma}, \frac{1}{\sin\sigma}]$. Therefore, irreducible representations of relation (17) need not to be concentrated on the trajectory $\Omega = \{F_1^{(k)}(F_2^{(m)}(\lambda)) \mid k, m \in \mathbb{Z}\}$. It is clear that the points of the trajectory can be parametrized in the following way: $\Omega = \{\frac{\sin((x+k)\sigma)}{\sin\sigma} \mid k \in \mathbb{Z}\}$, $x \in [-\frac{\pi}{2\sigma}, \frac{\pi}{2\sigma}]$. Here we describe irreducible representations connected with these trajectories.

Proposition 3.5. *Let* $\sigma \notin \pi\mathbb{Q}$*. Then, any irreducible representation* A_1*,* A_2 *with* $\sigma(A_1) \subset \{\frac{\sin((x+k)\sigma)}{\sin\sigma} \mid k \in \mathbb{Z}\}$ *is unitarily equivalent to one of the following:*

1*)*

$$Ae_k = \frac{\sin((x+k)\sigma)}{\sin\sigma}e_k, \qquad Be_k = \alpha_k e_{k+1} + \alpha_{k-1}e_{k-1},$$

where the collection $\{e_k\}$ *forms either*

- *a basis in* $l_2(\mathbb{N})$ *and either* $(x+k)\sigma \neq \pm\frac{\pi}{2} + 2\pi l$, $\frac{\pi\pm\sigma}{2} + \pi l$,

$$\alpha_k = \sqrt{-\frac{\sin(k\sigma)\sin((2x+(k+1))\sigma)}{4\sin^2\sigma\cos((x+k)\sigma)\cos((x+(k+1))\sigma)}},$$

$k \in \mathbb{N}$, $l \in \mathbb{Z}$, $x \in \{\mathbb{R} \mid \alpha_k > 0, k > 0\}$, *or* $\sigma(x+1) = \pm\frac{\pi}{2} + 2\pi l$, $\alpha_1 = \frac{1}{\sqrt{2}\sin\sigma}$, $\alpha_k = \frac{1}{2\sin\sigma}$, $k > 1$*; or*

- *a basis in* $\mathbb{C}^n$ *and*

$$\alpha_k = \sqrt{\frac{\sin(k\sigma)\sin((n-k))\sigma)}{4\sin^2\sigma\cos((2k-n-1)\sigma/2)\cos((2k-n+1))\sigma)}},$$

$k = 1,\ldots,n-1$, $n \in \{n \in \mathbb{N} \mid \alpha_k > 0, k = 1,\ldots,n-1\}$, *or*

- *a basis in* $l_2(\mathbb{Z})$ *and*

$$\alpha_k = \sqrt{\frac{4y\sin^2\sigma\cos(x\sigma)\cos((x+1)\sigma) - \sin k\sigma\sin((2x+(k+1))\sigma)}{4\sin^2\sigma\cos((x+(k+1))\sigma)\cos((x+k)\sigma)}},$$

$x\sigma \neq \frac{\pi(2l+1)+m\sigma}{2}$, $l, m \in \mathbb{Z}$, $(x,y) \in \{(x,y) \in \mathbb{R}\times\mathbb{R}^+ \mid \alpha_k > 0, k \in \mathbb{Z}\}$*;*

2*)*

$$A_1 e_m = (-1)^i \frac{cos(2m-1)\sigma/2}{\sin\sigma} e_m, \qquad A_2 e_m = \begin{cases} \alpha_1 e_2 + \mu e_1, & m = 1, \\ \alpha_m e_{m+1} + \alpha_{m-1} e_{m-1}, & m \neq 1, l, \end{cases}$$

where the collection $\{e_m\}$ *forms either*

- *a basis in* $\mathbb{C}^l$ *and* $\mu = (-1)^j \frac{\sin(l\sigma)}{2\sin(\sigma/2)\sin\sigma}$, α_m *coinside with* α_m *from Proposition 3.4, 8), $l \in \{l \in \mathbb{N} \mid \alpha_m > 0, 1 \leq m < l\}$, $i, j = 0, 1$, or*
- *a basis in* $l_2(\mathbb{N})$ *and*

$$\alpha_m = \sqrt{\frac{-4\mu^2\sin^2\sigma\sin^2(\sigma/2) + \sin^2(m\sigma)}{4\sin^2\sigma\sin((2m-1)\sigma/2)\sin((2m+1)\sigma/2)}},$$

$(l,\mu) \in \{(l,\mu) \in \mathbb{N}\times\mathbb{R} \mid \alpha_m > 0, m \in \mathbb{N}\}$, $i, j = 0, 1$.

3) Let $q = \pm 1$. If $q = 1$, then $A_{q,\mu}$ coinsides with a real form of the universal enveloping algebra $U(so(3))$, the full description of its irreducible representation can be found , for example, in [35]. If $q = -1$, then $A_{q,\mu}$ is a real form of the universal enveloping algebra of the graduated algebra $U(so(3))_{gr}$ studied in [6]. Representations of such algebras can be described by using the technique of semilinear relations.

We note that 1) any irreducible representation described in Proposition 3.2 (Proposition 3.3) converges to the corresponding representation of the algebra $U(so(3))$ as $q \to 1$ (to $U(so_g(3))$ as $q \to -1$); 2) for q enough close to 1 or -1 or, what is the same, for σ enough close to 0 or π, there are no representations of the forms 1) $-$ 3). Besides, there are no infinite-dimensional representations given in Proposition 3.5. Nevetherless, for any $\sigma = \pi\frac{k}{n}$, k is odd, $\sigma \neq 0, \neq \pi$, there exist representations of the form 4) $-$ 7), moreover

$$||A_1||, ||A_2|| \to \infty \quad \text{as } \sigma \to 0, \pi.$$

3.4. Representations of $A_{q,\mu}$, $\mu < 0$. 1) Let $q \in \mathbb{T}$. Taking into account that $\Phi(t,s) = t^2-(q+q^{-1})ts+s^2+1$, we have that all connected components of the graph corresponding to relation (17) are $\underset{\lambda}{\bullet}$. Hence, by Theorem 1.4, any irreducible representation of (17) is one-dimensional. From (18) it follows that any irreducible representation of $A_{q,\mu}$ is trivial $A_1 = 0$, $A_2 = 0$.

2) Let $q \in \mathbb{R} \setminus \{\pm 1, 0\}$, $q > 0$. The characteristic function of semilinear relation (17) is of the form $\Phi(t,s) = (t - F_1(s))(t - F_2(s))$, where $F_{1(2)}(s) = sch\sigma \pm \sqrt{s^2 sh^2\sigma - 1}$, $ch\sigma = \frac{q+q^{-1}}{2}$, $\sigma > 0$. It is easy to check that the sets $M_1 = \{\lambda \in \mathbb{R} \mid \lambda \le -\frac{1}{sh\sigma}\}$, $M_2 = \{\lambda \in \mathbb{R} \mid \lambda \ge \frac{1}{sh\sigma}\}$ are invariant with respect to the functions F_i, $i = 1,2$, hence $H_i = E_A(M_i)H$, $i = 1,2$, being invariant with respect to the operators of representations acting in H. Besides, all conected components of $\Gamma|_{\mathbb{R}\setminus(M_1\cup M_2)}$ are of the form $\underset{\lambda}{\bullet}$, all irreducible representations in $H_3 = (H_1 \oplus H_2)^{\perp}$ being trivial. As in the case $q < 0$, $\mu > 0$ the problem of describing all irreducible representations, up to a unitary equivalence, in H_i, $i = 1,2$, is reduced to that of the triple (A_0, X, X^*) satisfying the relations:

$$A_0 X = X(A_0 + I), \qquad X^*X = F(XX^*, A_0),$$

where

$$F(x,y) = \begin{cases} \frac{xsh\sigma(y-1)}{sh\sigma(y+1)} + \frac{ch\sigma y}{2sh\sigma sh\sigma(y+1)} & \text{if } y \ne 1/2, \\ -\frac{sh(\sigma/2)\mu^2}{sh(3\sigma/2)} + \frac{ch(\sigma/2)}{2sh\sigma sh(3\sigma/2)} & \text{if } y = 1/2, \end{cases}$$

and

$$A_1 = (-1)^i \frac{ch\sigma A_0}{sh\sigma}, \quad A_2 = X + X^* + \mu P_0, \quad X^*(P_0 + P_1) = 0$$

(here P_0, P_1 are the projections on the eigenspaces of the operator A with the eigenvalue $\{(-1)^i \frac{ch(\sigma/2)}{sh\sigma}\}$, $\{0\}$, respectively, $i = 0,1$, $\mu \in \mathbb{R}$). For a details we refer the reader to [27]. Since the dynamical system $(x,y) \to (F(x,y+1), y+1)$ has the measurable section $\mathbb{R}^+ \times [0,1)$, one can describe all irreducible representations up to a unitary equivalence. Analogously we can describe irreducible representations for $q < 0$.

Proposition 3.6. *All irreducible representations of $A_{q,\mu}$ are unitarily equivalent to one of the following:*

I. $q = e^{\sigma}$, $\sigma > 0$

1) $H = l_2(\mathbb{Z}^+)$

$$A_1 e_k = (-1)^i \frac{ch\sigma(k+1/2)}{sh\sigma} e_k, \qquad A_2 e_k = \begin{cases} \alpha_0 e_1 + \mu e_0, & k = 0, \\ \alpha_k e_{k+1} + \alpha_{k-1} e_{k-1}, & k > 0, \end{cases}$$

where $\alpha_k = \sqrt{\frac{sh^2(k+1)\sigma - 4sh^2\sigma sh^2(\sigma/2)\mu^2}{4sh^2\sigma sh((2k+1)\sigma/2)sh((2k+3)\sigma/2)}}$, $\mu^2 < \frac{1}{4sh^2(\sigma/2)}$, $i = 0,1$.

2) $H = l_2(\mathbb{Z}^+)$

$$A_1 e_k = (-1)^i \frac{ch(\sigma(x+k))}{sh\sigma} e_k, \qquad A_2 e_k = \begin{cases} \alpha_0(x) e_1, & k = 0 \\ \alpha_k(x) e_{k+1} + \alpha_{k-1}(x) e_{k-1}, & k > 0, \end{cases}$$

where $\alpha_k(x) = \sqrt{\frac{sh((k+1)\sigma)sh((2x+k)\sigma)}{4sh^2\sigma sh((x+k+1)\sigma)sh((x+k)\sigma)}}$, $x > 0$, $x \ne \frac{1}{2}$, and $\alpha_0(0) = \frac{1}{\sqrt{2}sh\sigma}$, $\alpha_k(0) = \frac{1}{2sh\sigma}$, $k > 0$, $i = 0,1$.

3) $H = l_2(\mathbb{Z})$

$$A_1 e_k = (-1)^i \frac{ch(\sigma(x+k))}{sh\sigma} e_k, \qquad A_2 e_k = \alpha_k(x,y) e_{k+1} + \alpha_{k-1}(x,y) e_{k-1},$$

where $\alpha_k(x,y) = \sqrt{\frac{4ysh^2\sigma sh(x\sigma)sh((x+1)\sigma)+sh(k\sigma)sh((2x+k+1)\sigma)}{4sh^2\sigma sh((x+k)\sigma)sh((x+k+1)\sigma)}}$ *and* $(x,y) \in \{(x,y) \in (0,1) \times \mathbb{R}^+ \mid \alpha_k(x,y) > 0, k \in \mathbb{Z}\}$, $i = 0, 1$.

4) zero one-dimensional representation.

II. $q = -e^\sigma$, $\sigma > 0$,

1) $H = l_2(\mathbb{Z}^+)$

$$A_1 e_k = (-1)^{k+i} \frac{ch(\sigma(x+k))}{sh\sigma} e_k, \quad A_2 e_k = \begin{cases} \alpha_0(x) e_1, & k = 0 \\ \alpha_k(x) e_{k+1} + \alpha_{k-1}(x) e_{k-1}, & k > 0, \end{cases}$$

where $\alpha_k(x) = \sqrt{\frac{sh((k+1)\sigma)sh((2x+k)\sigma)}{4sh^2\sigma sh((x+k+1)\sigma)sh((x+k)\sigma)}}$, $x > 0$, $\alpha_0(0) = \frac{1}{\sqrt{2}sh\sigma}$, $\alpha_k(0) = \frac{1}{2sh\sigma}$, $k > 0$, $i = 0, 1$.

2) $H = l_2(\mathbb{Z})$

$$A_1 e_k = (-1)^{k+i} \frac{ch(\sigma(x+k))}{sh\sigma} e_k, \qquad A_2 e_k = \alpha_k(x,y) e_{k+1} + \alpha_{k-1}(x,y) e_{k-1},$$

$\alpha_k(x,y) = \sqrt{\frac{ysh(x\sigma)sh((x+1)\sigma)}{sh((x+k)\sigma)sh((x+k+1)\sigma))} + \frac{sh(k\sigma)sh((2x+k+1)\sigma)}{4sh^2\sigma sh((x+k)\sigma)sh((x+k+1)\sigma)}}$, $(x,y) \in \{(x,y) \in (0,1) \times \mathbb{R}^+ \mid \alpha_k(x,y) > 0, k \in \mathbb{Z}\}$, $i = 0, 1$.

3) zero one-dimensional representation.

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