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GEOMETRIC CONSTRUCTION OF THE CLASSICAL r-MATRIX FOR THE ELLIPTIC CALOGERO–MOSER SYSTEM

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Abstract. By applying the Hamiltonian reduction technique we derive a matrix first order differential equation that yields the classical r-matrices of the elliptic (Euler–) Calogero–Moser systems as well as their degenerations.

1. Introduction. The r-matrix of the elliptic Calogero–Moser (CM) model was first found in [1, 2] by direct computations. In [3] the same r-matrix was obtained from the r-matrix of the Euler–Calogero–Moser (ECM) model by the Hamiltonian reduction. In this lecture we derive, following [4], a new matrix first order differential equation on an unknown linear operator r acting on the space $\mathcal{F}$ of $sl(n,\mathbf{C})$-valued functions on a torus Σ_τ with a modular parameter τ:

$$X = [r(X), D] - k\bar{\partial}r(X) + Q(X), \tag{1}$$

where D and Q are constant diagonal matrices, $X = X(z,\bar{z}) \in \mathcal{F}$, and k is a number. We show that its solution r obeying a specific boundary condition is precisely the r-matrix of the elliptic CM model with a spectral parameter. The trigonometric and rational r-matrices correspond to degenerations of this equation when Σ_τ degenerates into a circle and into a point respectively. We observe that choosing another boundary condition one also gets the r-matrix of the elliptic ECM model [3] and its degenerations. In this sense eq.(1) can be treated as the generating equation for the family of dynamical r-matrices related to the CM systems.

The main tool we use to deduce (1) is the Hamiltonian reduction technique[1]. A hamiltonian action of a group G with a Lie algebra $\mathcal{G}$ on a symplectic manifold $\mathcal{P}$ gives rise to a moment map μ: $\mathcal{P} \to \mathcal{G}^*$, $\mathcal{G}^*$ is the dual to $\mathcal{G}$ with respect to a pairing $<,>$. Let

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[1]Our approach is inspired by the paper [7].

$\mathcal{J} \in \mathcal{G}^*$ and $G_{\mathcal{J}} \subset G$ be the isotropy subgroup of $\mathcal{J}$ under the coadjoint action. Then a quotient $\mathcal{P}_{red} = \mu^{-1}(\mathcal{J})/G_{\mathcal{J}}$ admits under some natural assumptions a symplectic structure [5]. The rational and trigonometric CM models are obtained according to this scheme by reducing the geodesic motions on the cotangent bundles of semi-simple Lie groups [6]. Their L-operators arise as the matrix functions of coordinates on the reduced phase space $\mathcal{P}_{red}$, while the Lax representation - as the equation of motion on $\mathcal{P}_{red}$. As usual r-matrices are defined by computing the Poisson bracket of two L-operators and arranging it in the form $\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]$, where $L_1 = L \otimes 1$, $L_2 = 1 \otimes L$. The computation of the Poisson bracket can be considerably simplified by using the Dirac bracket construction. According to [7], the Poisson bracket on the reduced phase space can be presented in the following convenient form

$$\{f, h\}_{red} = \{f, h\} - < \mathcal{J}, [V_f, V_h] >, \tag{2}$$

where f, h are functions on $\mathcal{P}$ whose restrictions on $\mu^{-1}(\mathcal{J})$ are $G_{\mathcal{J}}$-invariant and V_f denotes the solution of $< \mathcal{J}, [X, V_f] > = \xi_X f$, where ξ_X is the hamiltonian vector field generated by $X \in \mathcal{G}$.

The elliptic CM model with the spectral parameter can be obtained starting from an infinite-dimensional phase space. Namely, the Hamiltonian reduction procedure runs as follows [8, 9]. The infinite-dimensional phase space $\mathcal{P}$ is parametrized by the set $p = (\phi, c; A, k)$, where $\phi, A \in \mathcal{F}$, $c, k \in \mathbf{C}$ and is equipped with the standard Poisson (symplectic) structure:

$$\{f, h\} = \int_{\Sigma_\tau} d\bar{z}dz \mathrm{tr} \left(\frac{\delta f}{\delta \phi} \frac{\delta h}{\delta A} - \frac{\delta f}{\delta A} \frac{\delta h}{\delta \phi} \right) + \frac{\delta f}{\delta c} \frac{\delta h}{\delta k} - \frac{\delta f}{\delta k} \frac{\delta h}{\delta c}. \tag{3}$$

With a pairing

$$< (A, k), (\phi, c) > = \int_{\Sigma_\tau} d\bar{z}dz \mathrm{tr}(A\phi) + kc, \tag{4}$$

$\mathcal{P}$ can be identified with the cotangent bundle over the centrally extended sl current algebra (ϕ, c) on Σ_τ. The current group $\Sigma_\tau \times SL(n)$ acts on $\mathcal{P}$ as

$$(\phi(z, \bar{z}), c) \to (f(z, \bar{z})\phi(z, \bar{z})f^{-1}(z, \bar{z}), \quad c + \int_{\Sigma_\tau} d\bar{\eta}d\eta \ \mathrm{tr}\, \phi A\,), \tag{5}$$

$$(A(z, \bar{z}), k) \to (f(z, \bar{z})A(z, \bar{z})f^{-1}(z, \bar{z}) - k\bar{\partial} f(z, \bar{z})f^{-1}(z, \bar{z}), \quad k). \tag{6}$$

This action preserves the Poisson structure (3). The moment map is fixed to be

$$\mu(p) = k\bar{\partial}\phi + [A, \phi] = \mathcal{J}, \quad \mathcal{J} = \nu J \delta(z, \bar{z}), \tag{7}$$

where J denotes some element of the coadjoint sl orbit to be specified later (Sec.3) and ν is a coupling constant. Explicitly, $\mathcal{P}_{red}$ is described by a pair (L, D), where D is a constant diagonal matrix and L is a solution of (7) with $A = D$. L appears to be the L-operator of the elliptic CM system.

2. Generating equation. To start with we note that generically an element $A \in \mathcal{F}$ is gauge equivalent to a complex constant diagonal matrix $D(A)$ [9], i.e.

$$A = g(A)D(A)g(A)^{-1} - k\bar{\partial} g(A)g(A)^{-1}. \tag{8}$$

This matrix is defined up to the action of the elliptic affine Weyl group. Indeed, the gauge transformation with $f(z) = e^{2\pi i \frac{\bar{w}z - w\bar{z}}{\tau - \bar{\tau}}}$, where $w = diag(s_1, \ldots, s_n)$, $s_i = m_i + \tau n_i$, $m_i, n_i \in \mathbf{Z}$, $\operatorname{tr} w = 0$ leads to the substitution $D(A) \to D(A) + \frac{2\pi i k}{\tau - \bar{\tau}} w$. We fix $D(A)$ by choosing the fundamental Weyl chamber. The factor $g(A)$ is not uniquely defined. An element $\tilde{g}(A) = g(A)t$, where a diagonal matrix $t = t(z)$ is an entire function, also satisfies (8). Requiring $\tilde{g}(A)$ to be doubly periodic, we get that $t(z)$ is a constant matrix, i.e. an element of a maximal torus T of $SL(n)$. We also normalize g as $g(D) = 1$.

Let us assign to any $X \in \mathcal{F}$ a function F_X on $\mathcal{P}$:

$$F_X(p) = < \phi, g(A) X g(A)^{-1} >, \tag{9}$$

where $g(A)$ is some solution of (8). According to the choice (7), $G_{\mathcal{J}}$ acting on the surface $\mu^{-1}(\mathcal{J})$ coincides with a group of smooth mappings $\{g : \Sigma_\tau \to SL(n),\ g(0) \in H\}$, where H is the isotropy group of J. Now assume J to be such that $H \cap T = 0$. In this case there is no ambiguity in the choice of $g(A)$ for A restricted to $\mu^{-1}(\mathcal{J})$, i.e. on this surface F_X is well defined.

Functions F_X, $X \in \mathcal{F}$ are of interest due to their specific properties. The first one is that F_X is invariant with respect to (5),(6) with $g \in G_{\mathcal{J}}$, i.e. it can be viewed as a genuine function on the reduced phase space $\mathcal{P}_{red}$. The second property is that if we parametrize $\mathcal{P}_{red}$ by a pair (L, D), then owing to the normalization condition $g(D) = 1$ F_X restricted to $\mathcal{P}_{red}$ coincides with a matrix element $< L, X >$ of the L-operator. Thus, we can use (2) for F_X and F_Y to compute the Poisson bracket between the matrix elements of the L-operator. The calculations are straightforward. Let $f \circ A$ be a shorthand for the gauge transformation (6). Introducing a derivative $\nabla_X g(A) = \frac{d}{dt} g(e^{tX} \circ A)_{|t=0}$ of $g(A)$ at the point A along an orbit of gauge transformations, we find how the hamiltonian vector field ξ_X generated by X acts on F_Z:

$$\begin{aligned} \xi_X F_Z &= \frac{d}{dt} < e^{tX} \phi e^{-tX}, g(e^{tX} \circ A) Z g(e^{tX} \circ A)^{-1} >_{|t=0} \\ &= < \phi, g(A) \left[g(A)^{-1} \nabla_X g(A) - g(A)^{-1} X g(A), Z \right] g(A)^{-1} >, \end{aligned}$$

On $\mathcal{P}_{red}$ this formula takes the form $\xi_X F_Z = < L, [\nabla_X g(D) - X,\ Z] >$ and it provides the second term in (2):

$$< \mathcal{J}, [V_{F_X}, V_{F_Y}] > = < L, [\nabla_{V_{F_X}} g(D) - V_{F_X},\ Y] > . \tag{10}$$

To obtain the bracket $\{F_X, F_Y\}_{red}$ we first calculate the quantities $\frac{\delta F_X}{\delta A_{ij}(\eta, \bar{\eta})}$ and $\frac{\delta F_X}{\delta \phi_{ij}(\eta, \bar{\eta})}$ and after substituting them into (3), we combine $\{F_X, F_Y\}$ with (10) according to (2). In this way we prove that there exists a linear operator r: $\mathcal{F} \to \mathcal{F}$ defined by

$$r(X)(z, \bar{z}) = \sum_{ij} \int_{\Sigma_\tau} d\bar{\eta} d\eta\ X_{ij}(\eta, \bar{\eta}) \frac{\delta g(A)}{\delta A_{ij}(\eta, \bar{\eta})}(z, \bar{z}) - \frac{1}{2}(\nabla_{V_{F_X}} g(D) - V_{F_X}) \tag{11}$$

and such that the Poisson bracket on $\mathcal{P}_{red}$ acquires the form

$$\{F_X, F_Y\}_{red} = < L, [r(X), Y] + [X, r(Y)] > . \tag{12}$$

Note that formula (11) reminds the one derived in [7] for the r-matrix of the trigonometric CM system without a spectral parameter.

In contrast to the Poisson bracket on $\mathcal{P}_{red}$, r-matrix (11) depends on the extension of F_X in the vicinity of $\mu^{-1}(\mathcal{J})$. We extend F_X in a way as to get the simplest form of r. To this end we assume that vector J is such that $sl(n)$ decomposes into the direct sum $\mathcal{H} \oplus \mathcal{T} \oplus \mathcal{C}$, where $\mathcal{H}$ and $\mathcal{T}$ are the Lie algebras of H and T respectively and $\mathcal{C}$ is an orthogonal to $\mathcal{H} \oplus \mathcal{T}$ with respect to the Killing metric. We also assume that $\mathcal{T}$ and $\mathcal{C}$ are two Lagrangian subspaces of the nondegenerate two-form $\omega(X,Y) = J([X,Y])$ defined on $\mathcal{T} \oplus \mathcal{C}$. These restrictions on J are similar to that in the finite-dimensional case [6, 7] and will be justified in the next section.

Recall that $g(A)(z,\bar{z}) = e^{X(z,\bar{z})}$ is defined up to the right multiplication by an element of T. We remove this ambiguity by choosing $X(z,\bar{z})$ to obey a boundary condition $X(0) \in \mathcal{H} \oplus \mathcal{C}$. In addition, if $A \in \mu^{-1}(\mathcal{J})$, then $X(0) \in \mathcal{H}$.

Having fixed $g(A)$, we find that $\nabla_X g(D)(z,\bar{z}) = (PX)(z,\bar{z})$, where P: $\mathcal{F} \to \mathcal{F}$ is a unit operator at all points of Σ_τ except zero and it projects $X(0)$ on $\mathcal{H} \oplus \mathcal{C}$ parallel to $\mathcal{T}$. For X such that $X(0) \in \mathcal{H} \oplus \mathcal{C}$ due to the singularity of $\mathcal{J}$ eq.(10) reduces to $0 =< L, [\nabla_X g(D) - X, Y] >= J([X(0), V_{F_Y}(0)])$. With our choice of J we conclude that $V_{F_Y}(0) \in \mathcal{H} \oplus \mathcal{C}$ for any Y. Hence, defining $g(A)$ as described above, we get from (11) the following r-matrix

$$r(X)(z,\bar{z}) = \sum_{ij} \int_{\Sigma_\tau} d\bar{\eta} d\eta \; X_{ij}(\eta,\bar{\eta}) \frac{\delta g(A)}{\delta A_{ij}(\eta,\bar{\eta})}(z,\bar{z}) \tag{13}$$

This formula has a transparent geometric meaning. Defining a time evolution of the field $A(t)$ as: $A(0) = D$ and $\frac{dA}{dt}_{|t=0} = X$, one has $r(X) = \frac{d}{dt} g(A(t))_{|t=0}$. Since eq.(8) is valid for any t, i.e.

$$A(t) = g(A)(t) D(A)(t) g(A)(t)^{-1} - k\bar{\partial} g(A)(t) g(A)(t)^{-1},$$

we differentiate it with respect to t and put $t = 0$. The result is equation (1), where $Q = \frac{d}{dt} D_{|t=0}$. For any smooth function $X(z,\bar{z}) \in \mathcal{F}$ it has a unique solution $r(X)$ obeying the boundary condition $r(X)(0) \in \mathcal{H} \oplus \mathcal{C}$. From (1) we also read off that the r-matrix is dynamical [10, 11] since it depends on D accumulating the coordinates on $\mathcal{P}_{red}$. We refer to (1) as to the factorization problem for sl connection. Hence, by construction the r-matrix of the elliptic CM model is defined as a unique solution of the factorization problem for sl connection obeying some specific boundary condition.

3. Elliptic r-matrix. In this section we show how to solve (1). To this end we first analyze an equation on Σ_τ:

$$\bar{\partial}\mathcal{E}(z,\bar{z}) = \delta(z,\bar{z}) \tag{14}$$

In the vicinity of the origin eq.(14) defines a meromorphic function with a first order pole with the residue $1/2\pi i$. We define a solution of (14) as a meromorphic function having simple poles at the points of the period lattice $\mathbf{Z}\tau_1 + \mathbf{Z}\tau_2$ ($\tau_1 = 1, \tau_2 = \tau$) with residues $1/2\pi i$ and satisfying the quasiperiodicity condition:

$$\mathcal{E}(z + \tau_k) = \mathcal{E}(z) + C_{\tau_k}, \tag{15}$$

where C_{τ_k} are complex numbers. Note that $\mathcal{E}(z)$ can not be doubly periodic since there is no elliptic functions with first order poles. The difference of two solutions $\mathcal{E}$ and $\tilde{\mathcal{E}}$ of

(14) is a holomorphic but non-periodic function ψ (poles and residues of $\mathcal{E}$'s coincide) with

$$\psi(z+\tau_k) = \psi(z) + \delta_k, \quad \delta_k = C_{\tau_k} - \tilde{C}_{\tau_k}. \tag{16}$$

Recall that the numbers C_k are not arbitrary. They obey Legendre's identity $C_1\tau_2 - C_2\tau_1 = 1$, which originates from integrating $\mathcal{E}(z)$ around the pole at the origin [13]. Therefore, we get $\delta_1\tau_2 - \delta_2\tau_1 = 0$. The only holomorphic function obeying (16) with δ_k constrained as above is $\psi(z) = \alpha z + \beta$, $\alpha = \delta_1$, $\beta \in \mathbf{C}$. Hence, any two solutions of (14) are related as $\tilde{\mathcal{E}}(z) = \mathcal{E}(z) + \alpha z + \beta$. The Weierstrass zeta-function $\zeta(z) = \frac{1}{z} + \sum_{n,m\in\mathbf{Z}}\left(\frac{1}{z-\omega_{nm}} + \frac{1}{\omega_{nm}} + \frac{z}{\omega_{nm}^2}\right)$, $\omega_{nm} = n\tau_1 + m\tau_2$, satisfies the properties listed above and therefore represents a peculiar solution of (14). Thus, we conclude that any meromorphic function $\mathcal{E}(z)$ with only simple poles at the points of the period lattice $\mathbf{Z}\tau_1 + \mathbf{Z}\tau_2$ with the residues $1/2\pi i$ and obeying (15) is of the form

$$\mathcal{E}(z) = \frac{1}{2\pi i}\zeta(z) + \alpha z + \beta. \tag{17}$$

When $\beta = 0$ these functions are odd $\mathcal{E}(-z) = -\mathcal{E}(z)$.

Introduce the notation:

$$X = \sum_{ij} x_{ij}(z,\bar{z})e_{ij}, \quad Y = r(X) = \sum_{ij} y_{ij}(z,\bar{z})e_{ij}, \quad D = \sum_i d_i e_{ii}, \quad Q = \sum_i q_i e_{ii}, \tag{18}$$

where e_{ij} are the matrix unities, then eq.(1) is equivalent to the system

$$\bar{\partial} y_{ii} = \frac{1}{k} t_i \tag{19}$$

$$\bar{\partial} y_{ij} + \frac{d_{ij}}{k} y_{ij} = -\frac{1}{k} x_{ij}, \quad i \neq j, \tag{20}$$

where $t_i(z,\bar{z}) = q_i - x_{ii}(z,\bar{z})$ and $d_{ij} = d_i - d_j$.

By forming the convolution of the fundamental solution $\mathcal{E}(z)$ with the r.h.s. of (19) we restore the diagonal part Y_{diag} of Y up to a constant matrix $h \in \mathcal{T}$: $Y_{diag}(z,\bar{z}) = \sum_i \int_{\Sigma_\tau} \frac{d\bar{\eta}d\eta}{k}\ \mathcal{E}(z-\eta)t_i(\eta,\bar{\eta}) + h$. Requiring Y to be doubly periodic, we determine the unknown matrix Q:

$$Q = \frac{1}{2i\Sigma_\tau}\int_{\Sigma_\tau} d\bar{\eta}d\eta \quad x_{ii}(\eta,\bar{\eta})e_{ii}. \tag{21}$$

To solve eq.(20) by a similar device we need a doubly periodic solutions $\mathcal{E}_{ij}$ of the equation

$$\bar{\partial}\mathcal{E}_{ij}(z,\bar{z}) + \frac{d_{ij}}{k}\mathcal{E}_{ij}(z,\bar{z}) = \delta(z,\bar{z}), \quad i \neq j. \tag{22}$$

Eq.(22) coincides in essence with the one defining the L-operator of the elliptic CM model and it has a unique doubly periodic solution given by [9]

$$\mathcal{E}_{ij}(z,\bar{z}) = \frac{1}{2\pi i} e^{\frac{d_{ij}}{k}(z-\bar{z})} \frac{\sigma(z + \frac{d_{ij}}{\pi k}\mathrm{Im}\tau)}{\sigma(z)\sigma(\frac{d_{ij}}{\pi k}\mathrm{Im}\tau)} \equiv w_{ij}(-z,-\bar{z}), \tag{23}$$

where $\sigma(z)$ is the Weierstrass sigma-function.

Combining all the pieces together we can write a general solution $Y(z)$ of eq.(1):

$$Y = h + \int_{\Sigma_\tau} \frac{d\bar{\eta}d\eta}{k} \left(\mathcal{E}(z-\eta) \sum_i t_i(\eta,\bar{\eta})e_{ii} - \sum_{i\neq j} w_{ij}(\eta - z, \bar{\eta}-\bar{z})x_{ij}(\eta,\bar{\eta})e_{ij} \right). \quad (24)$$

At this point we specify J. In the elliptic case [9] one should choose the following representative J on the coadjoint $sl(n)$ orbit

$$J = 1 - u \otimes s^\dagger, \quad (25)$$

where u is a vector in $\mathbf{C}^n$ and $\bar{s}_i = 1/u_i$. Then eq.(7) defines the following L-operator:

$$L = \sum_i p_i e_{ii} - \nu \sum_{i\neq j} \frac{u_i}{u_j} \mathcal{E}(z,\bar{z})_{ij} e_{ij}. \quad (26)$$

The momentum part in L follows from the diagonal part of (7) with $J_{ii} = 0$. Later on we point out the connection of (26) with Krichever's L-operator [12].

The Lie algebra $\mathcal{H}$ of the isotropy group H of J is determined by ($X \in \mathcal{H}$):

$$u_i(s^\dagger X)_j - (Xu)_i \bar{s}_j = 0. \quad (27)$$

Choosing in (27) $i = j$, one gets $(s^\dagger X)_i = \frac{\bar{s}_i}{u_i}(Xu)_i$ and thereby (27) reduces to $\frac{(Xu)_i}{u_i} = \frac{(Xu)_j}{u_j} = \lambda$, where $\lambda \in \mathbf{C}$. One also has $(s^\dagger X)_i = \frac{\bar{s}_i}{u_i}\lambda u_i = \lambda \bar{s}_i$. Thus, we find

$$\mathcal{H} = \{X \in sl(n,\mathbf{C}) : \quad Xu = \lambda u, \quad s^\dagger X = \lambda s^\dagger, \quad \lambda \in \mathbf{C}\}. \quad (28)$$

From (28) we can read off that $\mathcal{H} \cap \mathcal{T} = 0$. Since the real dimension of $sl(n,\mathbf{C})$ is $2(n^2-1)$ and $\mathcal{H}$ is defined by $4n-4$ equations, we get $\dim \mathcal{H} = 2(n^2-1) - (4n-4) = 2(n-1)^2$.

Decompose $sl(n)$ into the direct sum $\mathcal{H} \oplus \mathcal{T} \oplus \mathcal{C}$, as above. To describe $\mathcal{C}$ explicitly we introduce a matrix C

$$C = z \otimes s^\dagger - u \otimes y^\dagger$$

depending on two vectors $z, y \in \mathbf{C}^n$. Let $X \in \mathcal{H}$ and $Xu = s^\dagger X = 0$, then $\mathrm{Tr}(XC) = 0$ by the cyclic property of the trace. On the other hand, if $X \in sl(n)$ we have $\mathrm{Tr}(XC) = \sum_{ik}(z_i \bar{s}_k - u_i \bar{y}_k)x_{ki}$ and consequently for C to be orthogonal to any $X = (x_i\delta_{ij}) \in \mathcal{T}$, we get $z_i\bar{s}_i - u_i\bar{y}_i = \beta$ for any i, where β is arbitrary. Orthogonality of $\mathcal{H}$ and C also implies: $0 = \mathrm{Tr}(JC) = \beta n(1 - < s^\dagger, u >) = \beta n(1-n)$ that gives $\beta = 0$. Thus, $C \in \mathcal{C}$ if $\bar{y}_i = \frac{\bar{s}_i}{u_i} z_i = \frac{1}{u_i^2} z_i$. We put $\sum_i \frac{z_i}{u_i} = 0$ to have the correct dimension of $\mathcal{C}$: $\dim \mathcal{C} = 2(n-1)$. Note also that $\mathcal{T}$ and $\mathcal{C}$ form a pair of complementary Lagrangian subspaces with respect to $\omega(X,Y) =< J, [X,Y] >$ defined on $\mathcal{T} \oplus \mathcal{C}$.

Now we find that $(Cu)_i = nz_i$ and $(s^\dagger C)_i = -\frac{n}{u_i^2} z_i$. This allows us to describe the action of a generic element $X \in \mathcal{H} \oplus \mathcal{C}$ on u and s:

$$\begin{aligned} (Xu)_i &= \lambda u_i + n z_i, \\ (s^\dagger X)_i &= \tfrac{\lambda}{u_i} - \tfrac{n z_i}{u_i^2}. \end{aligned} \quad (29)$$

Summing up the second lines in (29) and taking into account $\sum \frac{z_i}{u_i} = 0$, we find λ: $\lambda = \frac{1}{n}\sum_i \frac{(Xu)_i}{u_i}$. Solving (29) for z_i, we arrive at

PROPOSITION 1. *Let X be an arbitrary element of $\mathcal{H}\oplus\mathcal{C}$. Then the following relation*

$$u_i(s^\dagger X)_i+(Xu)_i\frac{1}{u_i}=\frac{2}{n}\sum_j\frac{(Xu)_j}{u_j} \tag{30}$$

is valid for any i.

We use Proposition 1 to fix an element h. To this end we require $Y(0)$ to be an element of $\mathcal{H}\oplus\mathcal{C}$. Then by substituting $Y(0)$ in (30) we completely determine h:

$$h=-\int_{\Sigma_\tau}\frac{d\bar\eta d\eta}{k}\mathcal{E}(-\eta)\sum_i t_ie_{ii}+\sum_{i\neq j}\int_{\Sigma_\tau}\frac{d\bar\eta d\eta}{2k}\left(\frac{u_j}{u_i}w_{ij}x_{ij}+\frac{u_i}{u_j}w_{ji}x_{ji}\right)\left(e_{ii}-\frac{1}{n}I\right).$$

Thus, we arrive at

PROPOSITION 2. *There is a unique solution $Y(z,\bar z)$ of eq.(1) obeying the boundary condition $Y(0)\in\mathcal{H}\oplus\mathcal{C}$:*

$$\begin{aligned}Y(z,\bar z)=&\sum_i\int_{\Sigma_\tau}\frac{d\bar\eta d\eta}{k}(\mathcal{E}(z-\eta)-\mathcal{E}(-\eta))(q_i-x_{ii}(\eta,\bar\eta))e_{ii}\\&-\sum_{i\neq j}\int_{\Sigma_\tau}\frac{d\bar\eta d\eta}{k}w_{ij}(\eta-z,\bar\eta-\bar z)x_{ij}(\eta,\bar\eta)e_{ij}\\&+\sum_{i\neq j}\int_{\Sigma_\tau}\frac{d\bar\eta d\eta}{2k}\left(\frac{u_j}{u_i}w_{ij}(\eta,\bar\eta)x_{ij}(\eta,\bar\eta)+\frac{u_i}{u_j}w_{ji}(\eta,\bar\eta)x_{ji}(\eta,\bar\eta)\right)\left(e_{ii}-\frac{1}{n}I\right),\end{aligned} \tag{31}$$

where q_i and w_{ij} are given by (21) and (23) respectively.

Using the explicit form (17) of $\mathcal{E}(z)$ and taking into account (21) it is easy to reduce the first line in (31) to

$$\int_{\Sigma_\tau}\frac{d\bar\eta d\eta}{2ik}\left(\frac{\Phi(z,\bar z)}{\Sigma_\tau}-\frac{(\zeta(z-\eta)+\zeta(\eta))}{\pi}\right)\sum_i x_{ii}(\eta,\bar\eta)e_{ii},$$

where we have introduced a function $\Phi(z,\bar z)=\int_{\Sigma_\tau}\frac{d\bar\eta d\eta}{2\pi i}\,(\zeta(z-\eta)+\zeta(\eta))$. Hence, despite $\alpha z+\beta$ enters (17) the solution of (1) does not depend on it. We get from (31) the following

THEOREM 1. *The r-matrix corresponding to L-operator (26) is the following $sl(n)\otimes sl(n)$-valued function on $\Sigma_\tau\times\Sigma_\tau$*

$$\begin{aligned}r(z,\eta)=&\left(\frac{\Phi(z,\bar z)}{2ik\Sigma_\tau}-\frac{\zeta(z-\eta)+\zeta(\eta)}{2\pi ik}\right)\sum_i e_{ii}\otimes e_{ii}-\frac{1}{k}\sum_{i\neq j}w_{ij}(\eta-z,\bar\eta-\bar z)e_{ij}\otimes e_{ji}\\&+\frac{1}{2k}\sum_{i\neq j}\left(e_{ii}-\frac{1}{n}I\right)\otimes\left(\frac{u_j}{u_i}w_{ij}(\eta,\bar\eta)\;e_{ji}+\frac{u_i}{u_j}w_{ji}(\eta,\bar\eta)\;e_{ij}\right).\end{aligned} \tag{32}$$

L-operator as well as r-matrix (32) depend on a vector $u\in\mathbf{C}^n$. However, by conjugating L with the matrix e^U, $U_{ij}=u_i\delta_{ij}$ this dependence may be removed. The corresponding r-matrix is given by (32) with all $u_i=1$.

Now it is the time to state a connection of (32) with the r-matrix of the elliptic CM model found in [1, 2]. Without loss of generality we can assume that the integration domain Σ_τ has the vertices at the points $\pm\frac{1}{2}\pm\frac{\tau}{2}$. Then by the oddness of ζ-function one has $\int_{\Sigma_\tau} d\eta d\bar\eta\ \zeta(\eta) = 0$ and therefore $\Phi(z,\bar z)$ reduces to

$$\Phi(z,\bar z) = \int_{\Sigma_\tau} \frac{d\bar\eta d\eta}{2\pi i}\ \zeta(z-\eta). \tag{33}$$

Eq.(33) means that $\Phi(z,\bar z)$ is a solution of the equation $\bar\partial\Phi(z,\bar z) = 1$, i.e. $\Phi(z,\bar z) = \bar z + f(z)$, where $f(z)$ is an entire function. The monodromy properties of ζ define the ones for $\Phi(z,\bar z)$: $\Phi(z+\tau_k,, \bar z+\bar\tau_k) = \Phi(z,\bar z) + \frac{\Sigma_\tau}{\pi}C_{\tau_k}$. For $f(z)$ this gives

$$f(z+\tau_k) - f(z) = \frac{\Sigma_\tau}{\pi}C_{\tau_k} - \bar\tau_k. \tag{34}$$

Following the same lines as above we conclude that the only entire function obeying (34) is $f(z) = \alpha z + \beta$ with $\alpha = \frac{C_\tau/\pi - \bar\tau}{\tau}$. The constant β is equal to zero by the oddness of $\Phi(z)$. Thus, we get for $\Phi(z,\bar z)$ the following explicit expression

$$\Phi(z,\bar z) = \bar z - z + \frac{C_1}{\pi}\mathrm{Im}\tau\ z. \tag{35}$$

In [1] Krichever's L-operator [12]: $L^{Kr} = \sum_i p_i e_{ii} - \frac{\nu}{2\pi i}\sum_{i\neq j} G_{ij}(z)e_{ij}$, where $G_{ij}(z) = 2\pi i e^{\frac{d_{ij}}{k}\Phi(z,\bar z)}\mathcal{E}_{ij}(z,\bar z)$, was used to find the corresponding r-matrix. We see that L^{Kr} is related to (26) by the similarity transformation $L^{Kr}(z) = W(z,\bar z)L(z,\bar z)W(z,\bar z)^{-1}$, with $W(z,\bar z) = e^{\frac{D}{k}\Phi(z,\bar z)}$. Calculating $\{W_2, L_1\}$ with the help of the canonically conjugated variables $\{P,D\} = \frac{1}{2i\Sigma_\tau}\sum_i e_{ii}\otimes e_{ii}$, $P = \sum_i p_i e_{ii}$ on $\mathcal{P}_{red}$, we can show [4] that the r-matrix for L^{Kr} is just the one found in [1, 2].

4. Trigonometric r-matrix. Consider the cotangent bundle to the centrally extended current algebra $S^1 \to su(n)$. It is known [14] that reducing it by the action of the current group $S^1 \to SU(n)$ and imposing the moment $\mathcal{J} = J\delta(\varphi)$ with $J = i\nu\sum_{i\neq j}(e_{ij}+e_{ji})$ one left with the phase space (L,D) of the trigonometric CM model, where L is the corresponding L-operator:

$$L(\varphi) = i\sum_i p_i e_{ii} + \frac{\nu}{2k}\sum_{i\neq j}\frac{e^{\frac{i}{k}d_{ij}(\pi-\varphi)}}{\sin\frac{\pi}{k}d_{ij}}e_{ij}, \tag{36}$$

Retracing the same steps as in the elliptic case one gets the following equation on the r-matrix (see [4] for details)

$$X = [Y,D] - kY' + Q, \quad X,Y\in S^1\times su(n), \quad D,Q\in\mathcal{T}, \tag{37}$$

where this time $\mathcal{T}$ is a maximal torus of $su(n)$. Clearly, eq.(37) may be viewed as a degeneration of (1) that corresponds to a degeneration of Σ_τ into a circle. In this section we solve (37) explicitly and thereby recover the r-matrix of the trigonometric CM model [11].

The root decomposition of $su(n)$ elements X, Y, D and Q is given by (18) with coefficients obeying the unitary condition $x_{ji} = -x^*_{ij}$, etc. From (37) one finds two equations on diagonal and nondiagonal parts of Y respectively. Imposing the periodicity condition: $Y(0) = Y(2\pi)$, we reconstruct Y up to an element $h\in\mathcal{T}$:

$$Y(\varphi) = h + \frac{1}{k}\sum_{ij}\int_0^{2\pi} d\varphi' w_{ij}(\varphi,\varphi')x_{ij}(\varphi')e_{ij}, \tag{38}$$

where

$$w_{ij}(\varphi,\varphi') = \left(\frac{ie^{-\frac{i\pi d_{ij}}{k}}}{2\sin\frac{\pi}{k}d_{ij}} - \theta(\varphi,\varphi')\right) e^{-\frac{id_{ij}}{k}(\varphi-\varphi')}, \quad w_{ii}(\varphi,\varphi') = \frac{\varphi}{2\pi} - \theta(\varphi,\varphi') \tag{39}$$

and $\theta(\varphi,\varphi')$ is the Heviside function.

Since $su(n) = \mathcal{H}\oplus\mathcal{T}\oplus\mathcal{C}$, where $\mathcal{H}$ is a maximal proper Lie subgroup of $su(n)$ and $\mathcal{C}$ is an orthogonal to $\mathcal{H}\oplus\mathcal{T}$, just as in the elliptic case we can fix h by requiring $Y(0)$ to be an element of $\mathcal{H}\oplus\mathcal{C}$. From the results of [7] it follows that this requirement is equivalent to the set of relations $(1\le i\le n)$:

$$\sum_j \mathrm{Im}Y_{ij}(0) = \frac{1}{n}\sum_{i\neq j}\mathrm{Im}Y_{ij}(0). \tag{40}$$

Substituting $Y(0)$ in (40), one finds h that makes (40) true:

$$h = \frac{1}{2k}\sum_{i\neq j}\int_0^{2\pi} d\varphi' w_{ij}(0,\varphi')x_{ij}(\varphi')\left((\frac{1}{n}-e_{ii}) + (\frac{1}{n}-e_{jj})\right). \tag{41}$$

Combining (38) and (41), we finally get

Proposition 3. *Eq.(37) has a unique solution $Y(\varphi)$ obeying the constraint $Y(0)\in\mathcal{H}\oplus\mathcal{C}$ and it is given by (38) and (41).*

As a direct consequence of this lemma we get that r-matrix of the trigonometric CM system is the following function on $S^1\times S^1$

$$r(\varphi,\varphi') = \frac{\varphi-\pi}{2\pi k}\sum_i e_{ii}\otimes e_{ii} + \frac{i}{2k}\sum_{i\neq j}\frac{\cos\frac{\pi d_{ij}}{k}}{\sin\frac{\pi d_{ij}}{k}}e^{-i\frac{d_{ij}}{k}(\varphi-\varphi')}\; e_{ij}\otimes e_{ij} \tag{42}$$

$$-\frac{1}{2k}\sum_{i\neq j}\left(e_{ii}-\frac{1}{n}\right)\otimes\left(\frac{e^{-i\frac{d_{ij}}{k}}}{1-e^{-\frac{2\pi i d_{ij}}{k}}}e_{ij} - \frac{e^{i\frac{d_{ij}}{k}}}{e^{\frac{2\pi i d_{ij}}{k}}}e_{ji}\right) + \frac{1}{2k}s(\varphi,\varphi').$$

In (42) we have introduced a matrix s:

$$s(\varphi,\varphi') = \left(\sum_{i\neq j}e^{-\frac{id_{ij}}{k}(\varphi-\varphi')}e_{ij}\otimes e_{ji} + \sum_i e_{ii}\otimes e_{ii}\right)\epsilon(\varphi-\varphi'), \tag{43}$$

where $\epsilon(\varphi-\varphi') = [1-2\theta(\varphi-\varphi')]$ is the sign function.

By direct calculations [4] one can prove the following

Proposition 4. *Matrix s leads to the trivial Poisson bracket on the reduced phase space, i.e. the following relation is satisfied*

$$[s_{12}(\varphi,\varphi'), L(\varphi)\otimes I] - [s_{21}(\varphi',\varphi), I\otimes L(\varphi')] = 0. \tag{44}$$

Corollary. *r-matrix of the trigonometric CM model is given by* (42) *with* $s(\varphi,\varphi') = 0$.

On $\mathcal{P}_{red}$ the variables (P,X) are canonically conjugated: $\{P_1,X_2\} = -\frac{1}{2\pi}\sum_i e_{ii}\otimes e_{ii}$. L-operator (36) as well as r-matrix (42) depend on the parameter φ. However, this parameter may be removed by the similarity transformation $L\to\tilde{L} = Q(\varphi)L(\varphi)Q(\varphi)^{-1}$, where $Q(\varphi) = e^{-\frac{i}{k}X(\pi-\varphi)}$. One can easily show that the r-matrix corresponding to the

$\tilde{L}$ coincides with the one first found in [11] and then derived in [7] by the Hamiltonian reduction applied to the cotangent bundle $T^*SL(n)$.

5. Concluding remarks. A few remarks are in order. Assuming in (24) $h = 0$ and choosing $\mathcal{E}(z) = \frac{1}{2\pi i}\zeta(z)$, we find a matrix

$$r(z,\eta) = \left(\frac{\Phi(z,\bar{z})}{2ik\Sigma_\tau} - \frac{\zeta(z-\eta)}{2\pi ik}\right)\sum_i e_{ii}\otimes e_{ii} - \frac{1}{k}\sum_{i\neq j} w_{ij}(\eta - z, \bar{\eta} - \bar{z})e_{ij}\otimes e_{ji}$$

that turns out to be an r-matrix for the L-operator

$$L = \sum_i (p_i - f_{ii}\zeta(z))e_{ii} - \nu\sum_{i\neq j}\mathcal{E}_{ij}(z)f_{ij}e_{ij}$$

of the elliptic ECM model containing the additional dynamical variables f_{ij} [3]. Thus, eq.(1) also covers the ECM system being a spin extension of the CM model and gives a suggestion that the ECM r-matrix can be obtained by the Hamiltonian reduction.

If Σ_τ degenerates into a point, eq.(1) takes the form $X = [r(X), D] + Q$. One can easily show that it defines the r-matrices of the rational CM and ECM systems without spectral parameter.

Since eq.(1) is Lie-algebraic it hopefully may be used to find spectral-dependent r-matrices for CM models related to the other root systems. We also suppose that the study of possible deformations of eq.(1) is a good starting point to develop the r-matrix approach [15] to the Ruijsenaars systems [16] being relativistic extensions of the CM models.

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