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ON REPRESENTATION THEORY OF QUANTUM $SL_q(2)$ GROUPS AT ROOTS OF UNITY

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Abstract. Irreducible representations of quantum groups $SL_q(2)$ (in Woronowicz' approach) were classified in J.Wang, B.Parshall, Memoirs AMS 439 in the case of q being an odd root of unity. Here we find the irreducible representations for all roots of unity (also of an even degree), as well as describe "the diagonal part" of the tensor product of any two irreducible representations. An example of a not completely reducible representation is given. Non-existence of Haar functional is proved. The corresponding representations of universal enveloping algebras of Jimbo and Lusztig are provided. We also recall the case of general q. Our computations are done in explicit way.

0. Introduction. The quantum $SL(2)$ group is by definition a quantum group $(A, \triangle)$ that has the same representation theory as $SL(2)$, i.e. all nonequivalent irreducible representations are u^s, $s = 0, \frac{1}{2}, 1, \dots$ such that

$$\dim u^s = 2s + 1, \qquad u^t \mathbin{\bigcirc\!\!\!\!\top} u^s \approx \bigoplus_{\substack{r=|t-s|,\\ \text{step}=1}}^{t+s} u^r ,$$

$s, t = 0, \frac{1}{2}, 1, \dots$ and matrix elements of $u^{\frac{1}{2}}$ generate A as an algebra with unity I.

Putting $t, s = \frac{1}{2}$ in the above formula one can see that there must exist nonzero intertwiners $E \in \mathrm{Mor}(u^0, u^{\frac{1}{2}} \mathbin{\bigcirc\!\!\!\!\top} u^{\frac{1}{2}})$ and $E' \in \mathrm{Mor}(u^{\frac{1}{2}} \mathbin{\bigcirc\!\!\!\!\top} u^{\frac{1}{2}}, u^0)$. The operator E may be

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identified with a tensor $E \in K \otimes K$, whereas the operator E' with a tensor $E' \in K^* \otimes K^*$, where $K \approx \mathbf{C}^2$ is the carrier vector space of $u^{\frac{1}{2}}$.

The classification of quantum $SL(2)$ groups (described in the introduction of [16] and repeated here) is based on consideration of these tensors. There are three cases:

1. The rank of the symmetric part of E is 0. Then
$$E = e_1 \otimes e_2 - e_2 \otimes e_1 \,, \qquad E' = e^1 \otimes e^2 - e^2 \otimes e^1 \,,$$
where e_1, e_2 is a basis in K, while e^1, e^2 is the dual basis in K^*. This case corresponds to the undeformed (classical) $SL(2)$.
2. The rank of the symmetric part of E is 1. Then there exists a basis e_1, e_2 in K such that
$$E = e_1 \otimes e_2 - e_2 \otimes e_1 + e_1 \otimes e_1 \,, \qquad E' = -e^1 \otimes e^2 + e^2 \otimes e^1 + e^2 \otimes e^2 \,.$$
This case has been considered e.g. in [16].
3. The rank of the symmetric part E is 2. Then there exists $q \in \mathbf{C} \setminus \{0, 1, \text{roots of unity}\} \cup \{-1\}$ and a basis e_1, e_2 in K such that
$$E = e_1 \otimes e_2 - q\, e_2 \otimes e_1 \,, \qquad E' = e^1 \otimes e^2 - q\, e^2 \otimes e^1 \,.$$
This case has been considered in [15] and is recalled in section 1. For $q = 1$ one obtains the case 1.

In all these cases A is the algebra with unity I generated by matrix elements of $u = u^{\frac{1}{2}}$ satisfying the relations
$$(u \mathbin{\bigcirc\!\!\!\!\!\top} u)E = E \,, \quad E'(u \mathbin{\bigcirc\!\!\!\!\!\top} u) = E' \,.$$

When q is a root of unity and $q \neq \pm 1$, the representation theory of the case 3. is essentially different from that of $SL(2)$. In this case these objects are ambiguously called quantum $SL_q(2)$ groups at roots of unity. They are considered in the present paper in section 2.

The basic facts concerning quantum qroups and their representations are recalled in the appendix A. The quotient representations and the operation $\widetilde{\quad}$ are investigated in the appendix B.

0.1. *Basic notions.* The degree of a root of unity $q \in \mathbf{C}$ is the least natural number N such that $q^N = 1$. In the following we assume $N \geq 3$, i.e. $q = \pm 1$ are not roots of unity in our sense. We put
$$N_0 = \begin{cases} N \,, & \text{if } N \text{ is odd,} \\ \frac{N}{2} \,, & \text{if } N \text{ is even,} \\ +\infty \,, & \text{if } q \text{ is not a root of unity.} \end{cases}$$
We denote by $\mathbf{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$.

0.2. *Results.* When q is a root of unity, then all nonequivalent irreducible representations of quantum $SL_q(2)$ group are $v^t \mathbin{\bigcirc\!\!\!\!\!\top} u^s$, $t = 0, \frac{1}{2}, 1, \ldots$, $s = 0, \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - \frac{1}{2}$. Here
$$u = u^{\frac{1}{2}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad v = v^{\frac{1}{2}} = \begin{pmatrix} \alpha^{N_0} & \beta^{N_0} \\ \gamma^{N_0} & \delta^{N_0} \end{pmatrix} .$$

Moreover, v is a fundamental representation of a quantum group isomorphic to $SL_{q'}(2)$, where $q' = q^{N_0^2} = \pm 1$. The following formulae hold

$$v^t \mathbin{\textcircled{\top}} v \approx v^{t-\frac{1}{2}} \oplus v^{t+\frac{1}{2}}\,, \quad \dim v^t = 2t+1\,,$$
$$u^s \mathbin{\textcircled{\top}} u \approx u^{s-\frac{1}{2}} \oplus u^{s+\frac{1}{2}}\,, \quad \dim u^s = 2s+1\,.$$

$t = 0, \frac{1}{2}, 1, \ldots,$ $s = \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - 1$. Let us summarize the main results of the paper. The basic decomposition of the representation $u^{\frac{N_0}{2}-\frac{1}{2}} \mathbin{\textcircled{\top}} u$ is described by

$$u^{\frac{N_0}{2}-\frac{1}{2}} \mathbin{\textcircled{\top}} u \approx \begin{pmatrix} u^{\frac{N_0}{2}-1} & * & * \\ 0 & v & * \\ 0 & 0 & u^{\frac{N_0}{2}-1} \end{pmatrix}.$$

Moreover, the elements denoted by three stars and the matrix elements of the representations $u^{\frac{N_0}{2}-1}$ and v are linearly independent. Thus the representation $u^{\frac{N_0}{2}-\frac{1}{2}} \mathbin{\textcircled{\top}} u$ is not completely reducible.

One has

$$v^t \mathbin{\textcircled{\top}} u^s \approx u^s \mathbin{\textcircled{\top}} v^t\,.$$

We also describe the "diagonal part" of tensor product of any two irreducible representations of $SL_q(2)$.

In sections 2.6—2.7 we describe representations of universal enveloping algebras of Jimbo and Lusztig corresponding to the irreducible representations of $SL_q(2)$.

Remark 0.1. The classification of irreducible representations of $SL_q(2)$ for N odd is given in [11]. In the present paper we consider also N even, prove our results in an explicit way and also show other results concerning representation theory of $SL_q(2)$ (see sections 2.1., 2.4.—2.7.). Description of v as a fundamental representation of $SL_{\pm 1}(2)$ is also contained in [11],[10].

1. The general case. In this section we recall the theory of quantum $SL_q(2)$ groups for general $q \in \mathbf{C} \setminus \{0\}$, see [15], [12] and [11].

In the present section $q \in \mathbf{C} \setminus \{0\}$ unless it is said otherwise.

We set $K = \mathbf{C}^2$ with canonical basis e_1, e_2. We fix linear mappings $E : \mathbf{C} \to K \otimes K$ and $E' : K \otimes K \to \mathbf{C}$ in the same way as in the case 3. of the classification of quantum $SL(2)$ groups in section 0.

$$E(1) = e_1 \otimes e_2 - q e_2 \otimes e_1\,, \tag{1.1}$$

$$\begin{aligned} E'(e_1 \otimes e_1) &= 0\,, & E'(e_2 \otimes e_2) &= 0\,, \\ E'(e_1 \otimes e_2) &= 1\,, & E'(e_2 \otimes e_1) &= -q\,. \end{aligned} \tag{1.2}$$

Definition 1.1. A is the universal algebra with unity generated by u_{ij}, $i, j = 1, 2$ satisfying

$$(u \mathbin{\textcircled{\top}} u)E = E \quad \text{and} \quad E'(u \mathbin{\textcircled{\top}} u) = E'\,, \tag{1.3}$$

where $u = (u_{ij})_{i,j=1}^2$.

Setting

$$u^{\frac{1}{2}} = u = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{2\times 2}(A)\,, \tag{1.4}$$

the relations (1.3) take the form

$$\begin{gathered} \alpha\beta = q\beta\alpha\,, \qquad \alpha\gamma = q\gamma\alpha\,, \\ \beta\delta = q\delta\beta\,, \qquad \gamma\delta = q\delta\gamma\,, \\ \beta\gamma = \gamma\beta\,, \\ \alpha\delta - q\beta\gamma = I\,, \qquad \delta\alpha - \frac{1}{q}\beta\gamma = I\,. \end{gathered} \tag{1.5}$$

Using (1.3) one can easily prove the following

Proposition 1.1. *There exists unique structure of Hopf algebra in A such that u is a representation* (*this representation is called fundamental*).

Let

$$\alpha_k = \begin{cases} \alpha^k & \text{for } k \geq 0 \\ \delta^{-k} & \text{for } k < 0\,. \end{cases} \tag{1.6}$$

$$A_k = \operatorname{span}\{\alpha_s\beta^m\gamma^n \;:\; s\in\mathbf{Z},\ m,n\in\mathbf{N},\ |s|+m+n\leq k\}\,, \quad k\in\mathbf{N}\,. \tag{1.7}$$

Proposition 1.2. *Elements of the form*

$$\alpha_k\gamma^m\beta^n\,, \quad k\in\mathbf{Z}\,, \quad m,n\in\mathbf{N}\,, \tag{1.8}$$

form a basis of the algebra A. Moreover

$$\dim A_k = \sum_{l=0}^{k}(l+1)^2. \tag{1.9}$$

The proof is given in [17] (Proposition 4.2.).

Elements of the form

$$\alpha^k\delta^l\beta_m\,, \qquad k,l\in\mathbf{N}\,, \quad m\in\mathbf{Z} \tag{1.10}$$

are also a basis of the algebra A, where by definition

$$\beta_m = \begin{cases} \beta^m & \text{for } m \geq 0 \\ \gamma^{-m} & \text{for } m < 0\,. \end{cases} \tag{1.11}$$

This follows from the fact that each of the elements (1.10) of a given degree (a degree of an element (1.8) or (1.10) is the sum of absolute values of its indices) can be expressed as a finite linear combination of the elements (1.8) of the same or less degree and from the fact that the numbers of both kinds of elements of the same degree are equal.

The above consideration shows that

$$A_p = \operatorname{span}\left\{\alpha^k\delta^l\beta_m \;:\; k+l+|m|\leq p,\ \ k,l\in\mathbf{N},\ \ m\in\mathbf{Z}\right\} \quad \text{for} \ \ p\in\mathbf{N}\,.$$

In virtue of (1.3) $EE', \mathrm{id} \in Mor(u^{\textcircled{\scriptsize T}2}, u^{\textcircled{\scriptsize T}2})$. One has $(\mathrm{id}-EE')(e_i\otimes e_j) = e_j\otimes e_i$, $i,j=1,2$ for $q=1$. It means that $\mathrm{id}-EE'$ is equivalent to a transposition in this case. Using this intertwiner one can investigate symmetric and antisymmetric vectors (vectors such that $(\mathrm{id} - EE')v = \pm v$). For general q we are interested in intertwiners of the form $\sigma = \mathrm{id} + \lambda EE'$, satisfying the condition

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) \tag{1.12}$$

After some calculations one gets $\lambda = -q^{-2}, -1$. Taking $\lambda = -1$ one obtains

$$\sigma = \mathrm{id} - EE'$$

(the other value of λ corresponds to σ^{-1}). Using (1.1) and (1.2) one has

$$\begin{aligned} \sigma(e_1 \otimes e_1) &= e_1 \otimes e_1 \, , \\ \sigma(e_1 \otimes e_2) &= q\, e_2 \otimes e_1 \, , \\ \sigma(e_2 \otimes e_1) &= q\, e_1 \otimes e_2 + (1 - q^2)\, e_2 \otimes e_1 \, , \\ \sigma(e_2 \otimes e_2) &= e_2 \otimes e_2 \, . \end{aligned} \tag{1.13}$$

It can be easily found that

$$\sigma^2 = (1 - q^2)\sigma + q^2 \, , \tag{1.14}$$

i.e. $(\sigma - \mathrm{id})(\sigma + q^2) = 0$.

The eigenvalue 1 corresponds to symmetric vectors

$$e_1 \otimes e_1 \, , \quad q\, e_1 \otimes e_2 + e_2 \otimes e_1 \, , \quad e_2 \otimes e_2 \, ,$$

while the eigenvalue $-q^2$ corresponds to an antisymmetric vector

$$e_1 \otimes e_2 - q\, e_2 \otimes e_1 \, .$$

Let us define intertwiners

$$\sigma_k = \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-1 \text{ times}} \otimes\, \sigma \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{M-k-1 \text{ times}} \, , \quad k = 1, 2, \ldots, M - 1 \tag{1.15}$$

acting in $K^{\otimes M}$ $(M \in \mathbf{N})$.

This definition and the properties (1.14) and (1.12) of σ imply the relations of Hecke algebra:

$$\left.\begin{aligned} &\sigma_k \sigma_l = \sigma_l \sigma_k \, , \\ &\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \, , \\ &{\sigma_k}^2 = (1 - q^2)\sigma_k + q^2 \end{aligned}\right\} \quad \begin{aligned} &|k - l| \geq 2 \, , \\ &k, l = 1, \ldots, M - 1 \, . \end{aligned} \tag{1.16}$$

Let

$$E_k = \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-1 \text{ times}} \otimes E \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{M-k-1 \text{ times}} \, , \quad k = 1, 2, \ldots, M - 1 \, . \tag{1.17}$$

$$E'_k = \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-1 \text{ times}} \otimes E' \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{M-k-1 \text{ times}} \, , \quad k = 1, 2, \ldots, M - 1 \, . \tag{1.18}$$

Let us define a subspace of symmetric vectors

$$K^{\frac{M}{2}} = K^{\otimes_{\mathrm{sym}} M} = \{x \in K^{\otimes M} \, : \, \sigma_k x = x \, , \quad k = 1, \ldots, M - 1\} \, . \tag{1.19}$$

The intertwiner E_k is an injection and therefore

$$K^{\frac{M}{2}} = \{x \in K^{\otimes M} : E'_k x = 0 \, , \quad k = 1, \ldots, M - 1\} \, . \tag{1.20}$$

$K^{\frac{M}{2}}$ is equal to the intersection of kernels of intertwiners E'_k, $k = 1, \ldots, M-1$, hence it is an invariant subspace of $K^{\otimes M}$. Thus one can define the subrepresentation

$$u^{\frac{M}{2}} = u^{\textcircled{T} M}\Big|_{K^{\frac{M}{2}}} . \tag{1.21}$$

In particular, $u^0 = I$, $u^{\frac{1}{2}} = u$.

Each permutation $\pi \in \Pi(M)$ can be written as follows

$$\pi = t_{i_1} \ldots t_{i_m} , \quad \begin{array}{c} 1 \le i_1, \ldots, i_m \le M-1 , \\ i_1, \ldots, i_m \in \mathbf{N} , \end{array} \tag{1.22}$$

where $t_j = (j, j+1)$ is a transposition and $m = m(\pi)$ is minimal. It is known that $m(\pi)$ is equal to the number of π-inversions. The intertwiner

$$\sigma_\pi = \sigma_{i_1} \ldots \sigma_{i_m} \tag{1.23}$$

does not depend on a choice of a minimal decomposition (1.22), which can be obtained from (1.16) (cf [14], page 154).

Let us define an operator of q-symmetrization $S_M : K^{\otimes M} \to K^{\otimes M}$ given by (cf. [5])

$$S_M = \sum_{\pi \in \Pi(M)} q^{-2\ m(\pi)} \sigma_\pi . \tag{1.24}$$

Proposition 1.3. $(\sigma_k - \mathrm{id}) S_M = 0$ *for* $k = 1, \ldots, M-1$.

Proof. Let us call a permutation $\pi \in \Pi(M)$ "good", if $\pi^{-1}(k) < \pi^{-1}(k+1)$ for fixed k. A "bad" permutation is meant to be not a "good" one. If $\pi = t_{i_1} \ldots t_{i_l}$ is a minimal decomposition of a "good" permutation into transpositions, then a minimal decomposition of the "bad" permutation $t_k \pi$ is $t_k t_{i_1} \ldots t_{i_l}$ and $m(t_k\pi) = m(\pi) + 1$, $\sigma_{t_k \pi} = \sigma_k \sigma_\pi$. In such a way "good" and "bad" permutations correspond bijectively. According to (1.24), one has

$$\begin{aligned} S_M = & \underbrace{\sum_{\substack{\pi \in \Pi(M) \\ \pi \text{ is } ''\text{good}''}} q^{-2\ m(\pi)} \sigma_\pi}_{\substack{\| \\ S_M^{(0)}}} + \sum_{\substack{\pi' \in \Pi(M) \\ \pi' \text{ is } ''\text{bad}''}} q^{-2\ m(\pi')} \sigma_{\pi'} \\ = & (\mathrm{id} + q^{-2} \sigma_k)\, S_M^{(0)} , \end{aligned}$$

where $S_M^{(0)}$ is defined in the first line of the formula. Using (1.16) one can check the equation $\sigma_k S_M = S_M$. **Q.E.D.**

Corollary 1.4. $\operatorname{Im} S_M \subset K^{\frac{M}{2}}$.

Let us define Fact_x for $x \in \mathbf{C}$ as follows

$$\mathrm{Fact}_x(M) = \sum_{\pi \in \Pi(M)} x^{2\ m(\pi)} . \tag{1.25}$$

Using the mathematical induction one can prove

Proposition 1.5. $\mathrm{Fact}_x(M) = \begin{cases} \prod_{k=1}^{M} \frac{1-x^{2k}}{1-x^2} , & x \neq \pm 1 \\ M! , & x = \pm 1 . \end{cases}$

Corollary 1.6. $\mathrm{Fact}_{\frac{1}{q}}(M) = 0$ *if and only if* $M \geq N_0$.

Using (1.24), Proposition 1.3. and (1.25), one can obtain

PROPOSITION 1.7. $S_M^2 = \mathrm{Fact}_{\frac{1}{q}}(M) S_M$.

PROPOSITION 1.8. $\dim K^{\frac{M}{2}} = M + 1$.

Proof. For a given element $x \in K^{\otimes M}$ one can write a decomposition

$$x = \sum_{i_1,\ldots,i_M=1,2} x_{i_1,\ldots,i_M} e_{i_1} \otimes \ldots \otimes e_{i_M} \,, \tag{1.26}$$

where $x_{i_1,\ldots,i_M} \in \mathbf{C}$. According to (1.20), the statement $x \in K^{\frac{M}{2}}$ is equivalent to: $\forall k = 1, 2, \ldots, M-1 \;\; E'_k x = 0$, which can be replaced by: for all $k = 1, 2, \ldots, M-1$ and for all $x_{i_1,\ldots,i_M}$ such that $i_k = 1$ and $i_{k+1} = 2$ the following holds

$$x_{\ldots \overset{k}{\vee}{1}2\ldots} = q\, x_{\ldots \overset{k}{\vee}{2}1\ldots} \,, \tag{1.27}$$

where $\overset{k}{\vee}$ denotes the k–th position of an index. It means that all the coefficients $x_{i_1,\ldots,i_M}$ can be uniquely computed from the coefficients $x_{1\ldots1}$, $x_{1\ldots12}$, …, $x_{2\ldots2}$. One can conclude now that the thesis holds. **Q.E.D.**

PROPOSITION 1.9. *The linear span W_l of $e_{j_1} \otimes \ldots \otimes e_{j_M}$, $j_1, \ldots, j_M = 1, 2$ with a given number l of m such that $j_m = 1$, is invariant w.r.t. the intertwining operators σ_k, $k = 1, 2, \ldots, M-1$ as well as w.r.t. the operator S_M.*

The above proposition can be directly obtained from (1.15), (1.13) and the definition (1.24) of S_M.

PROPOSITION 1.10. *The following inequalities hold*

1. $\dim \mathrm{Im}\, S_M \geq M + 1$ *for $M \in \mathbf{N}$ such that $M < N_0$,*

2. $\dim \mathrm{Im}\, S_{N_0} \geq N_0 - 1$ *(for $N_0 < \infty$).*

Proof. Let us fix M. Using Proposition 1.9. one can see that in a decomposition of

$$S_M(\underbrace{e_1 \otimes \ldots \otimes e_1}_{k \text{ times}} \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{M-k \text{ times}}) \,, \quad k = 0, \ldots, M \tag{1.28}$$

there are only the elements of the basis $e_{i_1} \otimes \ldots \otimes e_{i_M}$ that have the number of e_1 equal to k.

It can be easily computed that an element $\underbrace{e_1 \otimes \ldots \otimes e_1}_{k \text{ times}} \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{M-k \text{ times}}$ has the coefficient equal to

$$\mathrm{Fact}_{\frac{1}{q}}(k)\, \mathrm{Fact}_{\frac{1}{q}}(M-k) \,.$$

For $M < N_0$ all these coefficients are nonzero and the elements (1.28) are linearly independent. In the case $M = N_0$ one gets coefficients equal 0 only for $k = 0$ and $k = N_0$. **Q.E.D.**

Using Corollary 1.4., Proposition 1.10. and Proposition 1.8., we get

COROLLARY 1.11. *If M is a natural number such that $M < N_0$ then*

$$\dim \mathrm{Im}\, S_M = M + 1 \,,$$

$$\mathrm{Im}\, S_M = K^{\frac{M}{2}} \,.$$

LEMMA 1.12. *Let* φ *be the intertwiner defined by*

$$\varphi : K^{s-\frac{1}{2}} \longrightarrow K^s \otimes K\,,$$
$$\varphi(x) = (S_{2s} \otimes \mathrm{id})(x \otimes E(1))\,, \quad x \in K^{s-\frac{1}{2}}\,,$$

for given $s = 0, \frac{1}{2}, 1, \ldots$ *such that* $2s < N_0 - 1$. *Moreover, let the representation* $u^{s-\frac{1}{2}}$ *be irreducible. Then*

1. $\ker\varphi = \{0\}$,

2. $\mathrm{Im}\,\varphi$ *corresponds to the representation* $u^{s-\frac{1}{2}}$,

3. $\mathrm{Im}\,\varphi \cap K^{s+\frac{1}{2}} = \{0\}$.

Proof. We compute

$$\begin{aligned}
&\varphi(e_1 \otimes \ldots \otimes e_1)\\
&\quad = S_{2s}(e_1 \otimes \ldots \otimes e_1) \otimes e_2 - q\, S_{2s}(e_1 \otimes \ldots \otimes e_1 \otimes e_2) \otimes e_1\\
&\quad = \mathrm{Fact}_{\frac{1}{q}}(2s)\, e_1 \otimes \ldots \otimes \underline{e_1 \otimes e_2} -\\
&\qquad\qquad q\left[\mathrm{Fact}_{\frac{1}{q}}(2s-1)\, e_1 \otimes \ldots \otimes e_1 \otimes \underline{e_2 \otimes e_1} + \ldots\right] \neq 0\,.
\end{aligned}$$

Thus irreducibility of $u^{s-\frac{1}{2}}$ implies 1. The result of 2. is now obvious.

$\mathrm{Im}\,\varphi$ corresponds to the irreducible representation $u^{s-\frac{1}{2}}$. Thus if 3. would not hold then $\mathrm{Im}\,\varphi \subset K^{s+\frac{1}{2}}$, because $K^{s+\frac{1}{2}}$ is $u^s \bigcirc\!\!\!\!\top\, u$ invariant (see (1.20)). This implies $\varphi(e_1 \otimes \ldots \otimes e_1) \in K^{s+\frac{1}{2}}$. Applying (1.26) and (1.27) to the underlined elements of $x = \varphi(e_1 \otimes \ldots \otimes e_1)$ one gets

$$\mathrm{Fact}_{\frac{1}{q}}(2s) = q\,(-q)\mathrm{Fact}_{\frac{1}{q}}(2s-1)\,,$$

which is impossible (see Proposition 1.5). This contradiction shows 3. **Q.E.D.**

PROPOSITION 1.13. *Let the representations* u^0, $u^{\frac{1}{2}}, \ldots, u^s$ *be irreducible for fixed* $s \in \frac{\mathbf{N}}{2}$ *such that* $\frac{1}{2} \le s < \frac{N_0}{2} - \frac{1}{2}$. *Then*

1. $u^s \bigcirc\!\!\!\!\top\, u \approx u^{s-\frac{1}{2}} \oplus u^{s+\frac{1}{2}}$,

2. the representation $u^{s+\frac{1}{2}}$ *is irreducible.*

Proof. 1. Using (1.20) one has

$$K^{s+\frac{1}{2}} \subset K^s \otimes K\,.$$

Applying the intertwiner φ of Lemma 1.12. one gets

$$K^{s+\frac{1}{2}} \oplus \mathrm{Im}\,\varphi \subset K^s \otimes K\,. \tag{1.29}$$

The dimensions of both sides of the inclusion are the same. This proves 1.

2. Analogously as in 1. one has $u^i \bigcirc\!\!\!\!\top\, u \approx u^{i-\frac{1}{2}} \oplus u^{i+\frac{1}{2}}$, $i = \frac{1}{2}, 1, \frac{1}{2}, \ldots, s$. Using the mathematical induction, $u^{\bigcirc\!\!\!\!\top\, l}$ can be decomposed into a direct sum of some copies of $u^{\frac{l}{2}}$, $u^{\frac{l}{2}-1}$, …, $u^{\frac{1}{2}}$ or u^0, $l = 0, 1, \ldots, 2s+1$.

Therefore

$$\begin{aligned}
\dim A_{2s+1} &= \dim \mathrm{span}\left\{(u^{\bigcirc\!\!\!\!\top\, l})_{km} \,:\, l = 0, 1, \ldots, 2s+1\,,\ \ k, m = 1, \ldots, 2^l\right\}\\
&\le \dim \mathrm{span}\left\{u^{\frac{l}{2}}_{km} \,:\, l = 0, 1, \ldots, 2s+1\,,\ \ k, m = 1, \ldots, (l+1)\right\}
\end{aligned} \tag{1.30}$$

Comparing this with (1.9) one obtains that the matrix elements of the representation $u^{s+\frac{1}{2}}$ must be linearly independent and the representation $u^{s+\frac{1}{2}}$ must be irreducible. **Q.E.D.**

Using the last proposition and the mathematical induction one can prove

THEOREM 1.14. *The representations u^s, $s \in \frac{\mathbf{N}}{2}$, $s < \frac{N_0}{2}$ are irreducible and the following decomposition holds*

$$u^s \bigcirc\!\!\!\!\top\, u \approx u^{s-\frac{1}{2}} \oplus u^{s+\frac{1}{2}}, \quad s < \frac{N_0}{2} - \frac{1}{2}.$$

COROLLARY 1.15. *In particular the representation $u^{\bigcirc\!\!\!\!\top\, M}$ is a direct sum of some copies of representations $u^{\frac{M}{2}}$, $u^{\frac{M}{2}-1}$, …, $u^{\frac{1}{2}}$ or u^0 for $M \in \mathbf{N}$ such that $M < N_0$.*

Using the above theorem and Proposition A.2. in [7] one gets

COROLLARY 1.16. *Each representation of $SL_q(2)$ is a direct sum of some copies of u^s, $s \in \mathbf{N}/2$, for $q \in \mathbf{C} \setminus \{0, \text{roots of unity}\}$.*

2. The case of roots of unity. The complex number q is assumed to be a root of unity all over the section. In this case N_0 is finite (see section 0) and $N_0 \geq 2$.

2.1. *The basic decomposition.* From the proof of Lemma 1.12 one can see that a decomposition of $u^{\frac{N_0}{2}-\frac{1}{2}} \bigcirc\!\!\!\!\top\, u$ may be completely different from the one in Theorem 1.14. The aim of the present subsection is to find it.

Let L be the subspace of $K^{\otimes N_0}$ given by the formula

$$L = \operatorname{span}\left\{e_k \otimes \ldots \otimes e_k \in K^{\otimes N_0} : k = 1, 2\right\}. \tag{2.1}$$

One can see that $\dim L = 2$ and $L \subset K^{\frac{N_0}{2}}$.

LEMMA 2.1. *Let φ be the following intertwiner*

$$\varphi : K^{\frac{N_0}{2}-1} \longrightarrow K^{\frac{N_0}{2}-\frac{1}{2}} \otimes K,$$
$$\varphi(x) = (S_{N_0-1} \otimes \mathrm{id})(x \otimes E(1))$$

for $x \in K^{\frac{N_0}{2}-1}$. Then
1. $\ker \varphi = \{0\}$,
2. $\operatorname{Im} \varphi$ *corresponds to the irreducible representation* $u^{\frac{N_0}{2}-1}$,
3. $K^{\frac{N_0}{2}} = \operatorname{Im} \varphi \oplus L$.

P r o o f. 1. and 2. See Lemma 1.12.

3. One can easily check (in different manners for $k = 1, 2, \ldots, N_0 - 2$ and for $k = N_0 - 1$) that

$$E'_k \varphi(e_1 \otimes \ldots \otimes e_1) = 0 \quad \text{for} \quad k = 1, 2, \ldots, N_0 - 1.$$

It means that

$$\varphi(e_1 \otimes \ldots \otimes e_1) \in K^{\frac{N_0}{2}}. \tag{2.2}$$

Moreover, the above element is different from zero.

According to 2. $\mathrm{Im}\,\varphi$ corresponds to an irreducible representation and therefore

$$\mathrm{Im}\,\varphi \subset K^{\frac{N_0}{2}}\,. \tag{2.3}$$

Decomposing an element of $\mathrm{Im}\,\varphi$ into elements of the basis $e_{i_1}\otimes\ldots\otimes e_{i_{N_0}}$, $i_1,\ldots,i_{N_0}=1,2$, one can see that the elements $e_k\otimes\ldots\otimes e_k$, $k=1,2$ have the coefficients equal 0 (we use Proposition 1.9). Thus

$$\mathrm{Im}\,\varphi\cap L=\{0\}\,.$$

Calculating the dimensions one can prove

$$K^{\frac{N_0}{2}}=\mathrm{Im}\,\varphi\oplus L\,.$$

Q.E.D.

Remark 2.1. Let $\hat S_M$, $M\in\mathbf{N}$ be an intertwiner defined as follows

$$\hat S_M = S_M|_{K^{\frac{M}{2}-\frac12}\otimes K}$$
$$\hat S_M : K^{\frac{M}{2}-\frac12}\otimes K\longrightarrow K^{\frac{M}{2}}$$

One can prove

$$\mathrm{Im}\,\varphi=\mathrm{Im}\,\hat S_{N_0}=\mathrm{Im}\,S_{N_0}\subset K^{\frac{N_0}{2}}=\ker\hat S_{N_0}\subset\ker S_{N_0}\,,$$

which gives (cf Proposition 1.7) the equalities $\hat S^2_{N_0}=0$, $S^2_{N_0}=0$. The situation for $M<N_0$ was completely different: S_M was proportional to a projection and hence

$$\mathrm{Im}\,S_M\oplus\ker S_M=K^{\otimes M}\,,$$
$$\mathrm{Im}\,\hat S_M\oplus\ker\hat S_M=K^{\frac{M}{2}-\frac12}\otimes K\,.$$

Proposition 2.2. *Let*

$$v=\begin{pmatrix}\alpha^{N_0} & \beta^{N_0}\\ \gamma^{N_0} & \delta^{N_0}\end{pmatrix}. \tag{2.4}$$

Then v is a quotient irreducible representation of a subrepresentation of $u^{\frac{N_0}{2}-\frac12}\bigcirc\!\!\!\!\top\, u$, corresponding to the quotient space $K^{\frac{N_0}{2}}/\mathrm{Im}\,\varphi\approx L$.

Proof. In virtue of (1.21) $K^{\frac{N_0}{2}}$ is $u^{\frac{N_0}{2}-\frac12}\bigcirc\!\!\!\!\top\, u$-invariant subspace.

Using Lemma 2.1. one can see that $K^{\frac{N_0}{2}}/\mathrm{Im}\,\varphi\approx L$ corresponds to a representation. Its matrix elements are the matrix elements of the representation $u^{\bigcirc\!\!\!\!\top\, N_0}$ that appear at the intersections of columns and rows corresponding to $e_1\otimes\ldots\otimes e_1$ and $e_2\otimes\ldots\otimes e_2$ and are given by (2.4). The elements of v are elements of the basis (1.8) of the algebra A and therefore v is irreducible. **Q.E.D.**

Lemma 2.3. *Let us define an intertwiner*

$$\hat E'_{m-1}=E'_{m-1}|_{K^{\frac{m}{2}-\frac12}\otimes K}\,,$$
$$\hat E'_{m-1}:K^{\frac{m}{2}-\frac12}\otimes K\longrightarrow K^{\otimes(m-2)}\,,$$

$m=2,3,\ldots$. *Then*

1. $\ker\hat E'_{m-1}=K^{\frac{m}{2}}$, *2.* $\operatorname{Im}\hat{E}'_{m-1} = K^{\frac{m}{2}-1}$,

3. $\operatorname{Im}\hat{E}'_{m-1}$ *corresponds to the representation* $u^{\frac{m}{2}-1}$.

Proof. 1. follows from (1.20) and the definition of $\hat{E}'_{m-1}$.

2. One can see (cf 1.27) that

$$\operatorname{Im}\hat{E}'_{m-1} \subset K^{\frac{m}{2}-1}. \tag{2.5}$$

One has

$$\dim(K^{\frac{m}{2}-\frac{1}{2}} \otimes K) = \dim\ker\hat{E}'_{m-1} + \dim\operatorname{Im}\hat{E}'_{m-1}.$$

Thus $\dim\operatorname{Im}\hat{E}'_{m-1} = m-1$ (see Proposition 1.8.). That and (2.5) prove 2.

3. follows from (1.21). **Q.E.D.**

COROLLARY 2.4. *The following decomposition holds* (*cf Appendix B.*)

$$u^s \widetilde{\bigoplus} u \approx \widetilde{u^{s-\frac{1}{2}}} \oplus \widetilde{u^{s+\frac{1}{2}}} \quad \textit{for } s = \tfrac{1}{2}, 1, \ldots.$$

Proof. We notice $K^{\frac{m}{2}-\frac{1}{2}} \otimes K /_{\ker\hat{E}'_{m-1}} \approx \operatorname{Im}\hat{E}'_{m-1}$ and set $s = \frac{m}{2} - \frac{1}{2}$. **Q.E.D.**

COROLLARY 2.5. *The quotient space* $(K^{\frac{N_0}{2}-\frac{1}{2}} \otimes K)/_{K^{\frac{N_0}{2}}}$ *corresponds to the representation* $u^{\frac{N_0}{2}-1}$, *which is irreducible.*

THEOREM 2.6.

1. $u^{\frac{N_0}{2}-\frac{1}{2}} \bigoplus u \approx \begin{pmatrix} u^{\frac{N_0}{2}-1} & * & * \\ 0 & v & * \\ 0 & 0 & u^{\frac{N_0}{2}-1} \end{pmatrix}$.

2. $\widetilde{u^{\frac{N_0}{2}}} \approx u^{\frac{N_0}{2}-1} \oplus v$.

3. All elements denoted by three stars are linearly independent from each other as well as from the matrix elements of the representations $u^{\frac{N_0}{2}-1}$ *and* v.

Proof. 1. and 2. We use Lemma 2.1., Proposition 2.2. and Corollary 2.5.

3. Using Theorem 1.14. one can see that the representation $u^{\bigoplus N_0}$ decomposes into some copies of $u^{\frac{N_0}{2}-\frac{1}{2}} \bigoplus u$, $u^{\frac{N_0}{2}-1}$, $u^{\frac{N_0}{2}-2}$, …, $u^{\frac{1}{2}}$ or u^0.

The following is obvious

$$A_{N_0} = \operatorname{span}\left\{(u^{\bigoplus k})_{ij} \ : \ k = 0, 1, \ldots, N_0, \quad i, j = 1, 2, \ldots, 2^k\right\},$$

where A_k is defined by (1.7).

According to Corollary 1.15. and 1. of Theorem 2.6. one has

$$A_{N_0} \ = \ \operatorname{span}\left\{ \begin{pmatrix}\text{elements denoted}\\ \text{by three stars}\end{pmatrix}, v_{i'j'},\ u^{\frac{k}{2}}_{ij} \ : \ \begin{array}{c} i', j' = 1, 2, \\ k = 0, 1, 2, \ldots, N_0 - 1, \\ i, j = 1, 2, \ldots, k+1 \end{array} \right\}.$$

Comparing the dimensions of both sides one gets 3. (see (1.9)). **Q.E.D.**

2.2. *Irreducible representations*

PROPOSITION 2.7. (*cf.* [10]) *Let* A' *be the subalgebra of* A *generated by the elements*

$$\alpha' = \alpha^{N_0}, \ \ \beta' = \beta^{N_0}, \ \ \gamma' = \gamma^{N_0}, \ \ \delta' = \delta^{N_0}.$$

Then A' *is isomorphic to the algebra* A *for the changed parameter* $q' = q^{{N_0}^2}$.

R e m a r k 2.2. The new parameter q' may be equal ± 1.

P r o o f. Using Proposition 1.2. one can see that the elements $\alpha'_k \gamma'^m \beta'^n$, $k \in \mathbf{Z}$, $m, n \in \mathbf{N}$, are linearly independent in the algebra A', where α'_k are defined in analogous way as α_k (see (1.6)).

It suffices to prove that the elements α', β', γ', δ' fulfill the relations (1.5) for the new parameter q'. The first five relations are immediate to prove, the last two will be considered now.

One has the following equation of polynomials

$$\prod_{j=0}^{N_0-1} (1 + q^{2j} x) = 1 + q^{N_0(N_0-1)} x^{N_0} \,. \tag{2.6}$$

Using the mathematical induction one can obtain

$$\alpha^{N_0} \delta^{N_0} = \prod_{j=0}^{N_0-1} (1 + q^{2j} (q\beta\gamma)) \,,$$

which using (2.6) can be written as

$$\alpha^{N_0} \delta^{N_0} = I + q^{N_0^2} \beta^{N_0} \gamma^{N_0} \,.$$

This corresponds to the last but one relation of (1.5). The last relation can be proved in a similar way. **Q.E.D.**

One can easily check the following

PROPOSITION 2.8. *A' is contained in the center of A for q being a root of unity of an odd degree.*

Using the notations introduced in Proposition 2.7., representations v^s, $s = 0, \frac{1}{2}, 1, \ldots$, can be defined in an analogous way as the representations u^s, $s = 0, \frac{1}{2}, 1, \ldots$, (but for the parameter $q' = q^{N_0^2}$, see (1.21)).

According to Theorem 1.14. one has ($N_0 = +\infty$ for $q = \pm 1$)

COROLLARY 2.9. *The representations v^s, $s = 0, \frac{1}{2}, 1, \ldots$ are irreducible. Moreover the following decomposition holds*

$$v^s \mathbin{\bigcirc\!\!\!\!\top} v \approx v^{s-\frac{1}{2}} \oplus v^{s+\frac{1}{2}} \,.$$

PROPOSITION 2.10. (*cf. [2] and Theorem 3.12 of [1]*) *The representations*

$$v^t \mathbin{\bigcirc\!\!\!\!\top} u^s \,, \qquad t = 0, \frac{1}{2}, 1, \ldots \,, \qquad s = 0, \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - \frac{1}{2}$$

are irreducible and nonequivalent.

P r o o f. It suffices to prove linear independence of matrix elements of representations $v^t \mathbin{\bigcirc\!\!\!\!\top} u^s$, $t = 0, \frac{1}{2}, 1, \ldots$, $s = 0, \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - \frac{1}{2}$. But matrix elements of $v^t \mathbin{\bigcirc\!\!\!\!\top} u^s$ belong to $A_{N_0 2t+2s}$ and the numbers $N_0 2t + 2s$ are different for different pairs (t, s). Therefore we need to prove (for given t, s) the linear independence modulo $A_{N_0 2t+2s-1}$ of the matrix elements

$$\left(v^t \mathbin{\bigcirc\!\!\!\!\top} u^s\right)_{ik,jl} = v^t_{ij} u^s_{kl} \,, \qquad i, j = 1, 2, \ldots, 2t \qquad k, l = 1, 2, \ldots, 2s$$

which is equivalent to the linear independence (in our sense) of the elements

$$\alpha_{N_0 a}\beta^{N_0 b}\gamma^{N_0 c}\alpha_k\beta^m\gamma^n\,, \tag{2.7}$$

where $a,k\in\mathbf{Z}$, $b,c,m,n\in\mathbf{N}$ are such that $|a|+b+c=2t$ and $|k|+m+n=2s$ (matrix elements of u^s modulo A_{2s-1} are basis of A_{2s}/A_{2s-1}, cf (1.30), similarly for v^t).

Doing some computations one can see that the linear independence of the elements (2.7) is equivalent to the linear independence of the elements

$$\begin{aligned}
&(\alpha^{N_0})^a(\beta^{N_0})^b(\gamma^{N_0})^c\,\alpha^k\beta^m\gamma^n\,, && && (1)\\
&(\delta^{N_0})^a(\beta^{N_0})^b(\gamma^{N_0})^c\,\delta^k\beta^m\gamma^n\,, && a\geq 1 \text{ or } k\geq 1, && (2)\\
&(\alpha^{N_0})^{a-1}(\beta^{N_0})^b(\gamma^{N_0})^c\,\alpha^{N_0-k}\beta^{m+k}\gamma^{n+k}\,, && a,k\geq 1\,, && (3)\\
&(\delta^{N_0})^{a-1}(\beta^{N_0})^b(\gamma^{N_0})^c\,\delta^{N_0-k}\beta^{m+k}\gamma^{n+k}\,, && a,k\geq 1\,, && (4)
\end{aligned}$$

where $a,b,c,k,m,n\in\mathbf{N}$ are such that $a+b+c=2t$ and $k+m+n=2s$. The above elements are proportional to some elements of the basis (1.8).

It can be easily seen that the elements (1) and (3) are of a different form than the elements (2) and (4).

Moreover the elements may be characterized by the number being the sum of the powers of the elements: β, γ and α (or δ), where we do not take into consideration α^{N_0}, β^{N_0}, γ^{N_0}, δ^{N_0}. The number $2s$ corresponds to the elements (1) and (2), the number N_0+2s corresponds to the elements (3) and (4). Remember that $2s<N_0$.

The elements (1), (2), (3) and (4) have the degree $N_0 2t+2s$, hence they are independent modulo $A_{N_0 2t+2s-1}$ and the proof is finished. **Q.E.D.**

2.3. *All irreducible representations.* Let

$$A_{\text{cent}}=\{a\in A\ :\ \tau(\triangle(a))=\triangle(a)\}\,,$$

where $\tau: A\otimes A\to A\otimes A$ is a linear mapping such that $\tau(a\otimes b)=b\otimes a$ for all $a,b\in A$, $\triangle$ is the comultiplication.

Lemma 2.11.

$$A_{\text{cent}}=\operatorname{span}\left\{\operatorname{Tr}\left(v^t \,\textcircled{\scriptsize T}\, u^s\right)\ :\ t=0,\frac12,1,\ldots,\ \ s=0,\frac12,1,\ldots,\frac{N_0}{2}-\frac12\right\}.$$

Using the above lemma, the fact that traces of all irreducible representations form a linearly independent subset in A_{cent} and Proposition 2.10., one obtains

Theorem 2.12. *The representations*

$$v^t \,\textcircled{\scriptsize T}\, u^s,\qquad t=0,\frac12,1,\ldots,\qquad s=0,\frac12,1,\ldots,\frac{N_0}{2}-\frac12$$

are all nonequivalent irreducible representations of the quantum group (A,u) for q being a root of unity.

Note that for $q=\pm1$ the above theorem is also true. In this case one could put $N_0=1$ and $u=v$.

P r o o f. (of Lemma 2.11.). Due to Corollary 2.9., Theorem 2.6. and Theorem 1.14.,

$$
\begin{aligned}
&\operatorname{Tr}(v^t \textcircled{\scriptsize T} u^s) = \operatorname{Tr}(v^t)\operatorname{Tr}(u^s), && s = \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - \frac{1}{2}, \quad t = \frac{1}{2}, 1, \ldots,\\
&\operatorname{Tr}(v^{t+\frac{1}{2}}) = \operatorname{Tr}(v^t)\operatorname{Tr}(v) - \operatorname{Tr}(v^{t-\frac{1}{2}}), && t = \frac{1}{2}, 1, \ldots,\\
&\operatorname{Tr}(v) = \operatorname{Tr}(u^{\frac{N_0}{2}-\frac{1}{2}})\operatorname{Tr}(u) - 2\operatorname{Tr}(u^{\frac{N_0}{2}-1}), \\
&\operatorname{Tr}(u^{s+\frac{1}{2}}) = \operatorname{Tr}(u^s)\operatorname{Tr}(u) - \operatorname{Tr}(u^{s-\frac{1}{2}}), && s = \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - 1,\\
&\operatorname{Tr}(u) = \alpha + \delta .
\end{aligned} \tag{2.8}
$$

We get that the statement of the lemma is equivalent to

$$A_{\text{cent}} = \operatorname{span}\{(\alpha+\delta)^n : n = 0, 1, 2, \ldots\} . \tag{2.9}$$

Let us consider a linear mapping $\psi : A \to A \otimes A$ defined as

$$\psi = \triangle - \tau \circ \triangle , \tag{2.10}$$

The equation (2.9) is equivalent to

$$\ker \psi = \operatorname{span}\{(\alpha+\delta)^n : n = 0, 1, 2, \ldots\} . \tag{2.11}$$

Let $\psi_P = \psi|_{A_P}$ for $P = 0, 1, 2, \ldots$. The equation (2.11) can be replaced by the following

$$\forall\, P = 0, 1, 2, \ldots \qquad \ker \psi_P = \operatorname{span}\{(\alpha+\delta)^n : n = 0, 1, \ldots, P\} .$$

Note that $\dim \ker \psi_P \geq P + 1$ (because $(\alpha+\delta)^n \in \ker \psi_P$). Thus the above equality follows from the inequality

$$\dim \operatorname{Im} \psi_P \geq \dim A_P - (P+1) , \tag{2.12}$$

which we are going to prove now.

The comultiplication $\triangle$ of the Hopf algebra A is given by

$$
\begin{aligned}
\triangle(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma , & \triangle(\delta) &= \gamma \otimes \beta + \delta \otimes \delta ,\\
\triangle(\beta) &= \alpha \otimes \beta + \beta \otimes \delta , & \triangle(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma .
\end{aligned} \tag{2.13}
$$

Let us take into account elements of the form

$$\triangle(\alpha^k \delta^l \beta_m) , \quad k, l \in \mathbf{N} , \ m \in \mathbf{Z} , \ \text{ such that } \ k + l + |m| = p ,$$

for certain $p \in \{0, 1, 2, \ldots, P\}$. There are two cases

1. $m \neq 0$. Using (1.11) and (2.13) one has

$$\triangle(\alpha^k \delta^l \beta^m) = \underline{c\alpha^{k+m}\gamma^l \otimes \alpha^k \beta^{m+l}} + (\text{elements with at least one } \delta) + x ,$$
$$\triangle(\alpha^k \delta^l \gamma^m) = \underline{d\alpha^{k}\gamma^{m+l} \otimes \alpha^{k+m} \beta^{l}} + (\text{elements with at least one } \delta) + y ,$$

where $\quad c, d \in \mathbf{C} , \ c, d \neq 0 , \ x, y \in A_{p-1} \otimes A_p + A_p \otimes A_{p-1}$.

2. $m = 0, \ l \geq 1$.

$$\triangle(\alpha^k \delta^l) = \underline{\alpha^k \gamma^l \otimes \alpha^k \beta^l} + (\text{elements with at least one } \delta) + x ,$$

where $\quad x \in A_{p-1} \otimes A_p + A_p \otimes A_{p-1}$.

One can easily see that the underlined elements and the elements one gets acting with τ on, are linearly independent for $k, l \in \mathbf{N}, \ m \in \mathbf{Z}$ such that $k+l+|m| = p$ ($m \neq 0$

or $l \geq 1$), $p = 0, 1, \ldots, P$ (see the basis (1.10)). Hence the elements $\Psi_P(\alpha^k \delta^l \beta_m)$ are also linearly independent. We have not considered only the elements $\Psi_P(\alpha^k)$, $k = 0, 1, \ldots, P$. Thus we just proved (2.12) as well as the Lemma 2.11. **Q.E.D.**

2.4. *More about irreducible representations.* In this subsection we describe the "diagonal part" of tensor product of any two irreducible representations of $SL_q(2)$.

Using (1.27) one gets that

$$e_{(i)} = \sum_{\substack{i_1, i_2, \ldots, i_{2s} = 1,2 \\ \#\{k : i_k = 2\} = i}} q^{-\#\{\,(m,s)\,:\, m<s,\ i_m > i_s\,\}}\, e_{i_1} \otimes \ldots \otimes e_{i_{2s}}\,, \tag{2.14}$$

$i = 0, 1, \ldots, 2s$, form a basis of K^s, where $\#B$ denotes the number of elements in a set B. Analogous basis elements for v^t are called

$$e'_{(i)}\,, \quad i = 0, 1, \ldots, 2t\,, \tag{2.15}$$

where $e_k,\ k = 1, 2$ are replaced with

$$e'_k = \underbrace{e_k \otimes \ldots \otimes e_k}_{N_0 \text{ times}}, \qquad k = 1, 2\,, \tag{2.16}$$

and q is replaced by $q' = q^{N_0^2}$ in the formula (2.14) .

PROPOSITION 2.13. *The representations $v^t \mathbin{\bigcirc\!\!\!\!\top} u^s$ and $u^s \mathbin{\bigcirc\!\!\!\!\top} v^t$, $t = 0, \frac{1}{2}, 1, \ldots$, $s = 0, \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - \frac{1}{2}$, are equivalent. An invertible interwiner S satisfying*

$$(S \otimes I)(u^s \mathbin{\bigcirc\!\!\!\!\top} v^t) = (v^t \mathbin{\bigcirc\!\!\!\!\top} u^s)(S \otimes I)$$

is (in the bases consisting of tensor products of (2.14) and (2.15)) given by

$$S_{ij,mn} = (q^{N_0})^{2jt + 2is} \delta_{in} \delta_{jm}\,,$$

where $\delta_{ab} = 1$ for $a = b$ and 0 otherwise. Note that $q^{N_0} = \pm 1$.

P r o o f. One can easily compute the rules (see (1.5), cf Proposition 2.8.)

$$\begin{array}{rclrclrcl}
\alpha^{N_0}\,\delta &=& \delta\,\alpha^{N_0}\,, & \alpha^{N_0}\,\beta &=& q^{N_0}\,\beta\,\alpha^{N_0}\,, & \alpha^{N_0}\,\gamma &=& q^{N_0}\,\gamma\,\alpha^{N_0}\,, \\
\delta^{N_0}\,\alpha &=& \alpha\,\delta^{N_0}\,, & \beta\,\delta^{N_0} &=& q^{N_0}\,\delta^{N_0}\,\beta\,, & \gamma\,\delta^{N_0} &=& q^{N_0}\,\delta^{N_0}\,\gamma\,, \\
\alpha\,\beta^{N_0} &=& q^{N_0}\,\beta^{N_0}\,\alpha\,, & \beta^{N_0}\,\delta &=& q^{N_0}\,\delta\,\beta^{N_0}\,, & \beta^{N_0}\,\gamma &=& \gamma\,\beta^{N_0}\,, \\
\alpha\,\gamma^{N_0} &=& q^{N_0}\,\gamma^{N_0}\,\alpha\,, & \gamma^{N_0}\,\delta &=& q^{N_0}\,\delta\,\gamma^{N_0}\,, & \gamma^{N_0}\,\beta &=& \beta\,\gamma^{N_0}\,.
\end{array} \tag{2.17}$$

One has

$$\begin{aligned} u\, e_1 &= e_1 \otimes \alpha + e_2 \otimes \gamma\,, \\ u\, e_2 &= e_1 \otimes \beta + e_2 \otimes \delta\,. \end{aligned} \tag{2.18}$$

Let $(u^s)_{ij} \in A$ be matrix elements of u^s given in the basis (2.14), i.e. $u^{\mathbin{\bigcirc\!\!\!\!\top} 2s} e_{(j)} = \sum_{i=0}^{2s} (u^s)_{ij}\, e_{(i)}$. Then one has

$$(u^s)_{ij} = \sum_{\substack{k \in \mathbf{Z},\ l, r \in \mathbf{N} \\ l + r = i - j \pmod 2 \\ |k| + l + r = 2s \pmod 2}} a_{klr} \alpha_k \beta^l \gamma^r\,, \quad a_{klr} \in \mathbf{C}\,. \tag{2.19}$$

The only elements that change the quantity of e_2 in $e_{(j)}$ are β and γ (see (2.18)) (this corresponds to the first condition in the above sum). If α "meets" δ they produce $\beta\gamma$ and I (see 1.5) (this corresponds to the second condition).

Similarly, replacing (2.14) by (2.15), one gets that $(v^t)_{mn}$ is a linear combination of $\alpha_{N_0 k'}\beta^{N_0 l'}\gamma^{N_0 r'}$ with $l'+r'=m-n(\text{mod } 2)$, $|k'|+l'+r'=2t(\text{mod } 2)$.

Using (2.14), (2.17) and (2.19), one can obtain

$$(u^s)_{ij}(v^t)_{mn}=(q^{N_0})^{2s(n-m)+2t(j-i)}(v^t)_{mn}(u^s)_{ij}\,.$$

Q.E.D.

R e m a r k 2.3. The equivalence of representations $v^t \textcircled{\top} u^s$ and $u^s \textcircled{\top} v^t$ follows also from (2.8) and Proposition B.3.

Using Proposition 2.13., Corollary 2.4. and Corollary 2.9. one gets

Proposition 2.14.

$$(v^t \textcircled{\top} u^s)\widetilde{\textcircled{\top}}(v^{t'} \textcircled{\top} u^{s'}) \approx \bigoplus_{\substack{r=|t-t'|,\\ \text{step}=1}}^{t+t'} \bigoplus_{\substack{r'=|s-s'|,\\ \text{step}=1}}^{s+s'} v^r \textcircled{\top} \widetilde{u^{r'}}\,,$$

$$t,t'=0,\frac{1}{2},1,\ldots,\quad s,s'=0,\frac{1}{2},1,\ldots,\frac{N_0}{2}-\frac{1}{2}\,,$$

(*for* $s=s'=0$ *we can omit* $\widetilde{\ }$).

Proposition 2.15.

$$\widetilde{u^{tN_0+s}} \approx \left(v^{t-\frac{1}{2}} \textcircled{\top} u^{\frac{N_0}{2}-s-1}\right) \oplus \left(v^t \textcircled{\top} u^s\right)\,,$$

$$\widetilde{u^{tN_0+(\frac{N_0}{2}-\frac{1}{2})}} \approx \left(v^t \textcircled{\top} u^{\frac{N_0}{2}-\frac{1}{2}}\right)\,,$$

$$t=\frac{1}{2},1,\ldots,\quad s=0,\frac{1}{2},1,\ldots,\frac{N_0}{2}-1\,,$$

where the representation $v^t \textcircled{\top} u^s$ *is not a subrepresentation of* u^{tN_0+s} (*this is only the quotient representation*).

P r o o f. Using mathematical induction one can easily prove the decomposition of the thesis (cf Corollary 2.4. and Theorem 2.6.2.). It remains to prove that the representation $v^t \textcircled{\top} u^s$ is not a subrepresentation of u^{tN_0+s}.

Let B be an algebra with unity I generated by two elements a, a^{-1} such that $a\,a^{-1}=a^{-1}\,a=I$. Let F be a homomorphism of A into B such that

$$\mathrm{F}(\alpha)=a\,,\qquad \mathrm{F}(\delta)=a^{-1}\,,\qquad \mathrm{F}(\beta)=0\,,\qquad \mathrm{F}(\gamma)=0\,.$$

One can obtain (cf (2.18))

$$\mathrm{F}(u)\,e_1=e_1\otimes a\,,\quad \mathrm{F}(u)\,e_2=e_2\otimes a^{-1}\,,$$

$$\mathrm{F}(u^s)\,e_{(i)}=e_{(i)}\otimes a^{2s-2i}\,,\quad \mathrm{F}(v^s)\,e'_{(i)}=e'_{(i)}\otimes a^{(2s-2i)N_0}\,,$$

$$s=0,\frac{1}{2},1,\ldots,\quad i=0,1,\ldots,2s\,.$$

Assume that there exists an invariant subspace W of K^{tN_0+s} corresponding to $v^t \bigoplus u^s$. Using F one can prove that $e_{(0)}$ belongs to W. On the other hand (cf (2.14) and (2.18))

$$u^r e_{(0)} = \sum_{i=0}^{2r} e_{(i)} \otimes \alpha^{2r-i}\gamma^i ,$$

where $r = tN_0 + s$ in our case (it suffices to compare the elements multiplying $e_1 \otimes \ldots e_1 \otimes e_2 \otimes \ldots e_2$). The coefficients $\alpha^{2r-i}\gamma^i$ $(i = 0, 1, \ldots, r)$ are linearly independent (see Proposition 1.2.). Thus W is not an invariant subspace. **Q.E.D.**

R e m a r k 2.4. A related fact at the level of universal enveloping algebras is given in Proposition 9.2 of [6].

One can prove

$$S_M(\sigma_k - \mathrm{id}) = 0 \quad \text{for } k = 1, \ldots, M-1,\ M \in \mathbf{N}$$

in a similar way as Proposition 1.3. Let $i_1, i_2, \ldots, i_M = 1, 2$ be such that $\#\{k : i_k = 2\} = m$. Using (1.13) one gets

$$S_M\ e_{i_1} \otimes \ldots \otimes e_{i_M} = q^{-\#\{\ (r,t)\ :\ r<t,\ \ i_r > i_t\ \}}\ S_M e_1 \otimes \ldots \otimes e_1 \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{m \text{ times}} .$$

On the other hand (cf Corollary 1.4. and the proof of Proposition 1.10.)

$$S_M e_1 \otimes \ldots \otimes e_1 \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{m \text{ times}} = \mathrm{Fact}_{\frac{1}{q}}(M-m)\ \mathrm{Fact}_{\frac{1}{q}}(m)\ e_{(m)} .$$

Thus $\mathrm{Im}\, S_M = \mathrm{span}\,\{e_{(m)}\ :\ M - N_0 < m < N_0\}$. Therefore $\mathrm{Im}\, S_{N_0+2s}$ is a carrier vector space of $u^{\frac{N_0}{2}-s-1}$ in Proposition 2.15. for $t = \frac{1}{2},\ \ 0 \le s \le \frac{N_0}{2} - 1$. Moreover, $S_M = 0$ for $M \ge 2N_0 - 1$.

2.5. *Haar measure.*

THEOREM 2.16. *The quantum group $SL_q(2)$ does not have the Haar functional.*

P r o o f. Assume that the Haar functional h does exist. Let us take into consideration the representation $\widetilde{u^{tN_0+s}}$ of Proposition 2.15. for $t = \frac{1}{2},\ \ s = \frac{N_0}{2} - 1$. One has an explicit form

$$\widetilde{u^{N_0-1}} \approx I \oplus (v \bigoplus u^{\frac{N_0}{2}-1}) ,$$

where I is a subrepresentation of u^{N_0-1}. Applying h to (A.3) one obtains $(h \otimes h)\triangle = h$, hence $P = (\mathrm{id} \otimes h)u^{N_0-1}$ is a projection. But

$$\begin{aligned} \mathrm{Tr}\, P &= h\mathrm{Tr}\, u^{N_0-1} = h\mathrm{Tr}\, \widetilde{u^{N_0-1}} \\ &= \mathrm{Tr}\,(\mathrm{id} \otimes h) \left[I \oplus \left(v \bigoplus u^{\frac{N_0}{2}-1} \right) \right] \\ &= \mathrm{Tr}\,(1 \oplus 0 \oplus \ldots \oplus 0) = 1 , \end{aligned}$$

and $P \approx 1 \oplus 0 \oplus \ldots \oplus 0$. Applying (A.3) to u^{N_0-1}, one obtains $Pu^{N_0-1} = u^{N_0-1}P = PI$ and

$$u^{N_0-1} \approx I \oplus \left(v \bigoplus u^{\frac{N_0}{2}-1} \right)$$

in contradiction with Proposition 2.15. **Q.E.D.**

2.6. *Enveloping algebra.* In this subsection we describe representations of $U_q sl(2)$ corresponding to the irreducible representations of $SL_q(2)$.

Let us recall (cf [5], [9]) that quantum universal enveloping algebra $U = U_q sl(2)$ is the algebra with identity I generated by $q^{\frac{1}{2}H}$, $q^{-\frac{1}{2}H}$, J^+, J^-, satisfying

$$\begin{gathered} q^{\frac{1}{2}H}\, q^{-\frac{1}{2}H} = q^{-\frac{1}{2}H}\, q^{\frac{1}{2}H} = I\,, \\ q^{\frac{1}{2}H}\, J^{\pm}\, q^{-\frac{1}{2}H} = q^{\pm 1}\, J^{\pm}\,, \\ \left[J^+, J^-\right] = \frac{(q^{\frac{1}{2}H})^2 - (q^{-\frac{1}{2}H})^2}{q - q^{-1}} \end{gathered}$$

with Hopf algebra structure given by

$$\begin{gathered} \triangle\, q^{\pm\frac{1}{2}H} = q^{\pm\frac{1}{2}H} \otimes q^{\pm\frac{1}{2}H}\,, \\ \triangle\, J^{\pm} = q^{\frac{1}{2}H} \otimes J^{\pm} + J^{\pm} \otimes q^{-\frac{1}{2}H}\,. \end{gathered}$$

There exists (cf [8], [4]) unique bilinear pairing $U \times A \ni (l, a) \to \langle l\ , a\rangle \in \mathbf{C}$ satisfying

$$\begin{gathered} \langle l_1 l_2\ , a\rangle = \langle l_1 \otimes l_2\ , \triangle a\rangle\,, \\ \langle l\ , a_1 a_2\rangle = \langle \triangle l\ , a_1 \otimes a_2\rangle\,, \\ l, l_1, l_2 \in U\,, \quad a, a_1, a_2 \in A\,, \end{gathered}$$

$$\begin{gathered} \langle q^{\frac{1}{2}H}\ , \alpha\rangle = q^{\frac{1}{2}}\,, \qquad \langle q^{\frac{1}{2}H}\ , \delta\rangle = q^{-\frac{1}{2}}\,, \\ \langle q^{-\frac{1}{2}H}\ , \alpha\rangle = q^{-\frac{1}{2}}\,, \ \langle q^{-\frac{1}{2}H}\ , \delta\rangle = q^{\frac{1}{2}}\,, \\ \langle q^{\frac{1}{2}H}\ , I\rangle = 1\,, \qquad \langle q^{-\frac{1}{2}H}\ , I\rangle = 1\,, \\ \langle J^-\ , \gamma\rangle = q^{-\frac{1}{2}}\,, \qquad \langle J^+\ , \beta\rangle = q^{\frac{1}{2}}\,, \end{gathered}$$

$q^{\frac{1}{2}H}, q^{-\frac{1}{2}H}, J^+, J^-$ vanish at $\alpha, \beta, \gamma, \delta, I$ in all other cases (nonzero $l^{\pm}_{ij}$ of [8] are given by

$$\begin{gathered} l^+_{11}(a) = l^-_{22}(a) = \langle q^{\frac{1}{2}H}\ , a\rangle\,, \quad l^+_{12}(a) = (q - q^{-1})\,\langle J^-\ , a\rangle\,, \\ l^+_{22}(a) = l^-_{11}(a) = \langle q^{-\frac{1}{2}H}\ , a\rangle\,, \ l^-_{21}(a) = -(q - q^{-1})\,\langle J^+\ , a\rangle\,, \end{gathered}$$

$a \in A$. $X_{\pm}$, K of [4] correspond to $q^{\mp\frac{1}{2}}J^{\pm}$, $q^{\frac{1}{2}H}$). We shall write $l(a)$ instead of $\langle l\ , a\rangle$.

For any representation w of the quantum group we introduce a unital representation Π_w of the algebra U by

$$[\Pi_w(l)]_{ij} = l(w_{ij}) = [(\mathrm{id} \otimes l)w]_{ij}\ , \quad i, j = 1, 2, \ldots, \dim w\,.$$

We are going to describe Π_w for any irreducible representation w of $SL_q(2)$. Let us fix $s = 0, \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - \frac{1}{2}$. One can obtain

$$\triangle^{(2s)} q^{\pm\frac{1}{2}H} = \underbrace{q^{\pm\frac{1}{2}H} \otimes \ldots \otimes q^{\pm\frac{1}{2}H}}_{2s \text{ times}}, \tag{2.20}$$

$$\triangle^{(2s)} J^{\pm} = \sum_{m=1}^{2s} \underbrace{q^{\frac{1}{2}H} \otimes \ldots \otimes q^{\frac{1}{2}H}}_{m-1 \text{ times}} \otimes J^{\pm} \otimes \underbrace{q^{-\frac{1}{2}H} \otimes \ldots \otimes q^{-\frac{1}{2}H}}_{2s-m \text{ times}}\,. \tag{2.21}$$

Using (2.20) one can show that

$$(\mathrm{id} \otimes q^{\pm\frac{1}{2}H})\, u^s\, e_{(j)} = q^{\pm(s-j)}\, e_{(j)}\,, \quad j = 0, 1, \ldots, 2s\,. \tag{2.22}$$

Using (2.21) one has

$$
\begin{aligned}
(\mathrm{id}\otimes J^-)\, u^{\textcircled{\scriptsize T} 2s} = \sum_{m=1}^{2s} & \underbrace{\begin{pmatrix} q^{\frac12} & 0 \\ 0 & q^{-\frac12}\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} q^{\frac12} & 0 \\ 0 & q^{-\frac12}\end{pmatrix}}_{m-1 \text{ times}} \otimes \\
& \begin{pmatrix} 0 & 0 \\ q^{-\frac12} & 0\end{pmatrix} \otimes \underbrace{\begin{pmatrix} q^{-\frac12} & 0 \\ 0 & q^{\frac12}\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} q^{-\frac12} & 0 \\ 0 & q^{\frac12}\end{pmatrix}}_{2s-m \text{ times}} .
\end{aligned} \tag{2.23}
$$

On the other hand

$$
(\mathrm{id}\otimes J^-)\, u^s\, e_{(j)} = \sum_{m=0}^{2s} J^-(u^s_{mj})\, e_{(m)} . \tag{2.24}
$$

Thus $J^-(u^s_{mj})$ equals to the coefficient multiplying $e_1\otimes\ldots\otimes e_1\otimes e_2\otimes\ldots\otimes e_2$. Considering (2.23), that coefficient can be nonzero only for $m=j+1$, $j<2s$ (cf (2.14)). In that case it is obtained from the following explicitly written part of $e_{(j)}$

$$
e_{(j)} = \sum_{r=0}^{m-1} q^{-r} \underbrace{e_1\otimes\ldots\otimes e_1}_{2s-m \text{ times}} \otimes \underbrace{\overbrace{e_2\otimes\ldots\otimes e_2}^{r \text{ times}} \otimes e_1\otimes e_2\otimes\ldots\otimes e_2}_{m \text{ times}} + \ldots
$$

Doing some computations,

$$
J^-(u^s_{j+1,j}) = q^{s-1}\frac{1-q^{-2(j+1)}}{1-q^{-2}} ,
$$

$j<2s$. Doing similar computations for J^+ and using (2.22) one obtains

$$
\begin{aligned}
\Pi_{u^s}(q^{\pm\frac12 H})\, e_{(j)} &= q^{\pm(s-j)}\, e_{(j)} ,\\
\Pi_{u^s}(J^-)\, e_{(j)} &= q^{s-j-1}[j+1]\, e_{(j+1)} ,\\
\Pi_{u^s}(J^+)\, e_{(j)} &= q^{j-s}[2s-j+1]\, e_{(j-1)} ,
\end{aligned}
$$

$j=0,1,\ldots,2s$, where $[l]=\frac{q^l-q^{-l}}{q-q^{-1}}$, $e_{(2s+1)}=e_{(-1)}=0$. Using (2.20) and (2.21), one has

$$
\begin{aligned}
(\mathrm{id}\otimes q^{\pm\frac12 H})\, v\, e'_1 &= q^{\pm\frac{N_0}{2}}\, e'_1 ,\\
(\mathrm{id}\otimes q^{\pm\frac12 H})\, v\, e'_2 &= q^{\mp\frac{N_0}{2}}\, e'_2 ,\\
(\mathrm{id}\otimes J^{\pm})\, v\, e'_i &= 0 , \quad i=1,2 .
\end{aligned}
$$

Therefore (cf (2.14)—(2.16))

$$
\begin{aligned}
\Pi_{v^t}(q^{\pm\frac12 H})\, e'_{(i)} &= q^{\pm N_0(t-i)}\, e'_{(i)} ,\\
\Pi_{v^t}(J^{\pm})\, e'_{(i)} &= 0 , \quad i=0,1,\ldots,2t ,
\end{aligned}
$$

$t=0,\frac12,1,\ldots$. Thus for $w=v^t \textcircled{\scriptsize T} u^s$ we have

$$
\begin{aligned}
\Pi_w(q^{\pm\frac12 H})\, e'_{(i)}\otimes e_{(j)} &= q^{\pm[N_0(t-i)+s-j]}\, e'_{(i)}\otimes e_{(j)} ,\\
\Pi_w(J^-)\, e'_{(i)}\otimes e_{(j)} &= q^{N_0(t-i)+s-j-1}[j+1]\, e'_{(i)}\otimes e_{(j+1)} ,\\
\Pi_w(J^+)\, e'_{(i)}\otimes e_{(j)} &= q^{N_0(t-i)+j-s}[2s-j+1]\, e'_{(i)}\otimes e_{(j-1)} ,
\end{aligned}
$$

$$i = 0, 1, \ldots, 2t\,, \quad j = 0, 1, \ldots, 2s\,, \quad t = 0, \frac{1}{2}, 1, \ldots\,, \quad s = 0, \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - \frac{1}{2}\,.$$

Hence $\Pi_w \approx \pi_{-t} \oplus \pi_{-t+1} \oplus \ldots \oplus \pi_t$, where π_r, $r \in \mathbf{Z}/2$, is the $(2s+1)$-dimensional representation of U described by (1) in [9], with $\omega = q^{N_0 r} \in \{1, -1, i, -i\}$ (for $s = \frac{N_0}{2} - \frac{1}{2}$ we take (2) for even roots or (4) for odd roots instead of (1), $\mu = 2(s + N_0 r)$, $\alpha = \beta = 0$).

2.7. *Enveloping algebra according to Lusztig.* Let us introduce the generators of [6] as follows:

$$E = q^{\frac{1}{2}} q^{\frac{1}{2}H} J^+\,, \quad F = q^{-\frac{1}{2}} J^- q^{-\frac{1}{2}H}\,, \quad K = \left(q^{\frac{1}{2}H}\right)^2\,. \tag{2.25}$$

One can compute:

$$\begin{gathered} KK^{-1} = K^{-1}K = 1\,, \\ KEK^{-1} = q^2 E\,, \quad KFK^{-1} = q^{-2} F\,, \\ EF - FE = \frac{K - K^{-1}}{q - q{-}1}\,, \\ \triangle E = K \otimes E + E \otimes I\,, \\ \triangle F = F \otimes K^{-1} + I \otimes F\,, \\ \triangle K = K \otimes K\,. \end{gathered}$$

Let U_L be the Hopf algebra over $\mathbf{C}(q)$ generated by E, F, $K^{\pm 1}$ satisfying the above relations, where q is an indeterminate (v in [6]) commuting with every element of U_L.

We also have a bilinear pairing $U_L \times A \to \mathbf{C}(q^{\frac{1}{2}})$:

$$\begin{gathered} \langle K\,, \alpha\rangle = q\,, \qquad \langle K\,, \delta\rangle = q^{-1}\,, \\ \langle K^{-1}\,, \alpha\rangle = q^{-1}\,, \quad \langle K^{-1}\,, \delta\rangle = q\,, \\ \langle E\,, \beta\rangle = q^{\frac{3}{2}}\,, \qquad \langle F\,, \gamma\rangle = q^{-\frac{3}{2}}\,, \\ \langle K\,, I\rangle = \langle K^{-1}\,, I\rangle = 1 \end{gathered}$$

and zero for the rest of combinations. According to [6] $U_{\mathcal{A}}$ is $\mathbf{C}[q, q^{-1}]$ algebra generated by:

$$K\,, \quad K^{-1}\,, \quad E\,, \quad F\,, \quad E^{(M)} = \frac{E^M}{[M]!}\,, \quad F^{(M)} = \frac{F^M}{[M]!}\,,$$

where $M = 2, 3, \ldots$ and $[M]! = \prod_{j=1}^{M} \frac{q^j - q^{-j}}{q - q^{-1}} = q^{-\frac{M^2}{2} + \frac{M}{2}} \operatorname{Fact}_q(M) \in \mathbf{C}[q, q^{-1}]$. We think about A also as $\mathbf{C}[q, q^{-1}]$ algebra.

Using the mathematical induction w.r.t. n, l, m one can derive the formula (a corrected version of (13) of [4]) for a bilinear pairing $U_{\mathcal{A}} \times A \to \mathbf{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$:

$$\langle E^k K^t F^i\,, \alpha_n \beta^l \gamma^m\rangle$$
$$= \begin{cases} [k]!\,[i]!\,q^{\frac{3}{2}(k-i) - t(k+i) + n(t+k)} \delta_{kl} \delta_{im} & , \ n \le 0 \\[2ex] [k]!\,[i]!\,q^{\frac{3}{2}(k-i) - t(k+i) + n(t+k) - (k-l)^2} \\ \quad \cdot \dfrac{\operatorname{Fact}_{\frac{1}{q}}(n)}{\operatorname{Fact}_{\frac{1}{q}}(k-l) \operatorname{Fact}_{\frac{1}{q}}(n-k+l)} \\ \quad \cdot \left(\delta_{k-l,0} + \delta_{k-l,1} + \ldots + \delta_{k-l,n}\right) \delta_{k-l,i-m} & , \ n > 0 \end{cases} \tag{2.26}$$

In the same way as in section 2.6 we get:

$$\begin{aligned}\Pi_{u^s}(K^{\pm 1})\, e_{(j)} &= q^{\pm 2(s-j)}\, e_{(j)}\,,\\ \Pi_{u^s}(E)\, e_{(j)} &= q^{\frac{3}{2}}[2s-j+1]\, e_{(j-1)}\,,\\ \Pi_{u^s}(F)\, e_{(j)} &= q^{-\frac{3}{2}}[j+1]\, e_{(j+1)}\,,\end{aligned}$$

$j = 0, 1, \ldots, 2s$, $s = 0, \frac{1}{2}, 1, \ldots, \frac{M}{2} - \frac{1}{2}$, where $e_{(2s+1)} = e_{(-1)} = 0$. Matrix $v = \begin{pmatrix} \alpha^M & \beta^M \\ \gamma^M & \delta^M \end{pmatrix}$ (see (2.4)) is not a representation at the moment, since q is an indeterminate. Using (2.26) we get:

$$\begin{aligned}\langle E\,, v\rangle = \langle F\,, v\rangle &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\\ \langle K^{\pm 1}\,, v\rangle &= \begin{pmatrix} q^{\pm M} & 0 \\ 0 & q^{\mp M} \end{pmatrix}.\end{aligned}$$

Hence

$$\begin{aligned}\Pi_{v^t}(K^{\pm 1})\, e'_{(i)} &= q^{\pm 2M(t-i)}\, e'_{(i)}\,,\\ \Pi_{v^t}(E)\, e'_{(i)} &= 0\,,\\ \Pi_{v^t}(F)\, e'_{(i)} &= 0\,,\end{aligned}$$

$t = 0, \frac{1}{2}, 1, \ldots$, $i = 0, 1, \ldots, 2t$. Thus for $w = v^t \textcircled{\scriptsize T} u^s$ we have

$$\begin{aligned}\Pi_w(K^{\pm 1})\, e'_{(i)} \otimes e_{(j)} &= \langle K^{\pm 1}\,, v^t\rangle\langle K^{\pm 1}\,, u^s\rangle\, e'_{(i)} \otimes e_{(j)}\\ &= q^{\pm 2((s-j)+M(t-i))}\, e'_{(i)} \otimes e_{(j)}\,,\\ \Pi_w(E)\, e'_{(i)} \otimes e_{(j)} &= \langle K\,, v^t\rangle\langle E\,, u^s\rangle\, e'_{(i)} \otimes e_{(j)}\\ &= q^{\frac{3}{2}+2M(t-i)}[2s-j+1]\, e'_{(i)} \otimes e_{(j-1)}\,,\\ \Pi_w(F)\, e'_{(i)} \otimes e_{(j)} &= \langle I\,, v^t\rangle\langle F\,, u^s\rangle\, e'_{(i)} \otimes e_{(j)}\\ &= q^{-\frac{3}{2}}[j+1]\, e'_{(i)} \otimes e_{(j+1)}\,,\end{aligned}$$

$$i = 0, 1, \ldots, 2t\,, \quad j = 0, 1, \ldots, 2s\,, \quad t = 0, \frac{1}{2}, 1, \ldots\,, \quad s = 0, \frac{1}{2}, 1, \ldots, \frac{M}{2} - \frac{1}{2}\,.$$

Using mathematical induction we can prove that $\langle E^k\,, u^{\textcircled{\scriptsize T} 2s}\rangle = 0$ for $2s < k$. In particular we have

$$\langle E^{(M)}\,, u^s\rangle = 0 \quad \text{for} \quad s < \frac{M}{2}\,.$$

In analogous way we get

$$\langle F^{(M)}\,, u^s\rangle = 0 \quad \text{for} \quad s < \frac{M}{2}\,.$$

Using formula $(a+b)^M = \sum_{r=0}^{M} \frac{\mathrm{Fact}_q(M)}{\mathrm{Fact}_q(r)\mathrm{Fact}_q(M-r)} a^r b^{M-r}$, where $ba = q^2 ab$, we get

$$\triangle(E^{(M)}) = \left(\sum_{r=1}^{M-1} \frac{q^{r(M-r)}}{[r]![M-r]!} E^r K^{M-r} \otimes E^{M-r}\right) + K^M \otimes E^{(M)} + E^{(M)} \otimes I\,. \tag{2.27}$$

In virtue of (2.26)

$$\langle E^{(M)}\ ,v\rangle = \begin{pmatrix} 0 & q^{\frac{3}{2}M} \\ 0 & 0 \end{pmatrix}, \qquad \langle F^{(M)}\ ,v\rangle = \begin{pmatrix} 0 & 0 \\ q^{-\frac{3}{2}M} & 0 \end{pmatrix}.$$

Using the formula $\langle E^r K^{M-r}\ ,v\rangle = 0$ for $r = 1,\ldots,M-1$, (2.27) and doing some computations, we get

$$\langle E^{(M)}\ ,v^{\textcircled{\scriptsize T}\,2t}\rangle = \sum_{m=1}^{2t} \underbrace{\begin{pmatrix} q^{M^2} & 0 \\ 0 & q^{-M^2} \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} q^{M^2} & 0 \\ 0 & q^{-M^2} \end{pmatrix}}_{m-1 \text{ times}} \otimes$$

$$\begin{pmatrix} 0 & q^{\frac{3}{2}M} \\ 0 & 0 \end{pmatrix} \otimes \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{2t-m \text{ times}}.$$

Let us now specialize q to be a root of unity of degree $N \geq 3$ (cf [6]). Let $M = N_0$. Doing similar comparison of coefficients as in (2.23) and (2.24), we get

$$\Pi_{v^t}(E^{(N_0)})\, e'_{(i)} = q^{\frac{3}{2}N_0 - iN_0^2}\,(2t-i+1)\, e'_{(i-1)}\,, \quad i = 0,1,\ldots,2t\,,$$
$$\Pi_{v^t}(F^{(N_0)})\, e'_{(i)} = q^{-\frac{3}{2}N_0 + iN_0^2}\,(i+1)\, e'_{(i+1)}\,, \quad i = 0,1,\ldots,2t\,,$$

where $t = 0, \frac{1}{2}, 1, \ldots$. Thus for $w = v^t \textcircled{\scriptsize T} u^s$ we get (cf (2.27)):

$$\begin{aligned} \Pi_w(E^{(N_0)})\, e'_{(i)} \otimes e_{(j)} &= \langle E^{(N_0)}\ ,v^t\rangle\langle I\ ,u^s\rangle\, e'_{(i)} \otimes e_{(j)} \\ &= q^{\frac{3}{2}N_0 - iN_0^2}(2t-i+1)\, e'_{(i-1)} \otimes e_{(j)}\,, \\ \Pi_w(F^{(N_0)})\, e'_{(i)} \otimes e_{(j)} &= \langle F^{(N_0)}\ ,v^t\rangle\langle K^{-N_0}\ ,u^s\rangle\, e'_{(i)} \otimes e_{(j)} \\ &= q^{-\frac{3}{2}N_0 + iN_0^2 - 2sN_0}(i+1)\, e'_{(i+1)} \otimes e_{(j)}\,, \end{aligned}$$

$$i = 0,1,\ldots,2t\,, \quad j = 0,1,\ldots,2s\,, \quad t = 0, \frac{1}{2}, 1, \ldots\,, \quad s = 0, \frac{1}{2}, 1, \ldots, \frac{N_0}{2} - \frac{1}{2}\,.$$

Comparing with representations of [6] (in the case of N odd as in [6]) we infer that w is isomorphic to the representation $L_q(N2s + 2t)$. Irreducibility of the obtained representations is related to the Lemma 6.1 of [3].

A. Basic concepts. Here we recall the basic facts concerning quantum groups and their representations (see [13]).

Let $(A, \triangle)$ be a Hopf algebra. We set $\triangle^{(2)} = \triangle$,

$$\triangle^{(n)} = (\triangle \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id})\triangle^{(n-1)}\,, \quad n = 3,4,\ldots\,.$$

Let K be a finite dimensional vector space over $\mathbf{C}$ and $e_1, \ldots, e_d \in K$ its basis. Then $B(K) \otimes A \cong B(\mathbf{C}^d) \otimes A \cong M_d(A)$.

One can define a linear mapping

$$\textcircled{\scriptsize T} : \mathrm{M}_N(A) \times \mathrm{M}_N(A) \longrightarrow \mathrm{M}_N(A \otimes A)$$

by

$$(v \textcircled{\scriptsize T} w)_{ij} = \sum_k v_{ik} \otimes w_{kj}\,, \quad i,j = 1,2,\ldots,N\,.$$

An element $v \in B(K) \otimes A$ is called *a representation* (*in the carrier vector space* K) *of a quantum group corresponding to a Hopf algebra* $(A, \triangle, \kappa, e)$, iff

$$(\mathrm{id} \otimes \triangle)\, v = v \,\textcircled{\scriptsize$\perp$}\, v\,, \tag{A.1}$$

$$(\mathrm{id} \otimes e)\, v = \mathrm{id}\,. \tag{A.2}$$

One can use the shortcut *representation* having in mind the above definition.

Let v and w be representations, K_v and K_w be its carrier vector spaces.

An operator $S \in B(K_v, K_w)$ such that

$$(S \otimes I)\, v = w\, (S \otimes I)$$

is called *a morphism* (*intertwiner*) of representations v and w.

A set of morphisms intertwining v with w is denoted by $\mathrm{Mor}(v, w)$.

Representations v and w are called *equivalent* iff $Mor(v, w)$ contains an invertible element. Then we write $v \approx w$.

One can define a linear mapping

$$\textcircled{\scriptsize$\top$} : \mathrm{M}_N(A) \times \mathrm{M}_{N'}(A) \longrightarrow \mathrm{M}_{NN'}(A)$$

by

$$(v \,\textcircled{\scriptsize$\top$}\, w)_{ik,jl} = v_{ij} w_{kl}\,, \quad i, j = 1, 2, \ldots, N\,, \quad k, l = 1, 2, \ldots, N'\,,$$

where $v \in \mathrm{M}_N(A)$, $w \in \mathrm{M}_{N'}(A)$.

One can check that if v and w are representations, then $v \,\textcircled{\scriptsize$\top$}\, w$ is also a representation. We denote $v^{\textcircled{\tiny$\top$} k} = v \,\textcircled{\scriptsize$\top$} \ldots \textcircled{\scriptsize$\top$}\, v$ (k times).

Definition A.1. A functional $\mathrm{h} \in A'$ is called *a Haar functional of a quantum group* $(A, \triangle)$ if $\mathrm{h}(I) = 1$ and

$$\forall\, a \in A \qquad (\mathrm{h} \otimes \mathrm{id})\triangle a = (\mathrm{id} \otimes \mathrm{h})\triangle a = \mathrm{h}(a)\, I\,. \tag{A.3}$$

One has $\mathrm{h}(u) = 0$ for any irreducible representation u different from I.

B. Quotient representations. Here we investigate the quotient representations and the operation $\widetilde{\quad}$.

Let $u \in B(K) \otimes A$ be given by

$$u = \sum_{r=1}^{R} m_r \otimes u_r\,, \tag{B.1}$$

where $K = \mathbf{C}^N$ and $m_1, m_2, \ldots, m_R \in B(K)$ are linearly independent as well as $u_1, u_2, \ldots, u_R \in A$. One can introduce a linear mapping $\hat{u} : K \to K \otimes A$ given by

$$\forall\, x \in K \qquad \hat{u}\, x = \sum_{r=1}^{R} m_r\, x \otimes u_r\,. \tag{B.2}$$

This is the formula (2.7) in [13]. $\hat{u}$ corresponds to u, because $B(K, K \otimes A)$ and $B(K) \otimes A$ are canonically isomorphic.

PROPOSITION B.1. *Let $(A,\triangle)$ be a quantum group with a counit e and $u \in M_N(A)$. Then the following two statements are equivalent*:

1. u is a representation of $(A,\triangle)$.

2. For each $x \in K$ $i)$ $(\mathrm{id}\otimes\triangle)\hat{u}\,x = (\hat{u}\otimes\mathrm{id})\hat{u}\,x$,
$ii)$ $(\mathrm{id}\otimes e)\hat{u}\,x = x$.

Proof. The equivalence of (A.1) and i) is proved in [13]. The equivalence of (A.2) and ii) can be proved in a similar way. **Q.E.D.**

Let u be a representation and K be its carrier space. We say (as in [13]) that L is u-invariant subspace if and only if

$$\widehat{u}(L) \subset L\otimes A\,.$$

Then the element $u_{|_L} \in B(L)\otimes A$ corresponding to the restriction $\widehat{u}_{|_L} : L \to L\otimes A$ is a representation acting on L.

The embedding $L \to K$ intertwines $u_{|_L}$ with u, $u_{|_L}$ is called a subrepresentation of u (cf [13]).

DEFINITION B.1. Let u be a representation, K its carrier vector space, $L\subset K$ u-invariant space and let $v = u_{|_L}$. Let $[x] \in {}^{K}\!/_{L}$ be an element of the quotient space and $x \in K$ a representative of the class. The representation w given by

$$\forall\, x \in K \qquad \hat{w}\,[x] = \sum_{r=1}^{R}\,[m_r\,x]\otimes u_r\,, \tag{B.3}$$

is called *a quotient representation* and is denoted by ${}^{u}\!/_{v}$.

Doing some easy computations one can prove that the quotient representation w is uniquely determined and that it is in fact a representation (use Proposition B.1).

Let us define a mapping

$$\widetilde{\quad} \;:\; \big(\text{representations}\big) \longrightarrow \begin{pmatrix}\text{completely reducible}\\ \text{representations}\end{pmatrix} \tag{B.4}$$

as follows

1. If u is an irreducible representation, then $\widetilde{u} = u$.
2. If v is a subrepresentation of u, then we set $\widetilde{u} = \widetilde{v}\oplus\widetilde{({}^{u}\!/_{v})}$.

PROPOSITION B.2. *Let u be a representation of a quantum group $(A,\triangle)$. Then the representation $\widetilde{u}$ is uniquely determined up to an equivalence.*

Proof. We first notice

$$\mathrm{Tr}\,\widetilde{u} = \mathrm{Tr}\,u\,. \tag{B.5}$$

Then let

$$\widetilde{u} \approx v_1\oplus\ldots\oplus v_k \tag{B.6}$$

be a decomposition of $\widetilde{u}$ into irreducible components. Hence

$$\mathrm{Tr}\,u = \mathrm{Tr}\,v_1 + \ldots + \mathrm{Tr}\,v_k$$

and ($\operatorname{Tr} v_i$ are linearly independent if v_i are nonequivalent irreducible representations) (B.6) is uniquely determined. **Q.E.D.**

Let us introduce the notation

$$\widetilde{\textcircled{T}} = \widetilde{\ } \circ \textcircled{T} . \tag{B.7}$$

PROPOSITION B.3. *Let* u, v *be representations of a quantum group, such that* $\operatorname{Tr} u \operatorname{Tr} v = \operatorname{Tr} v \operatorname{Tr} u$ *and* $v \textcircled{T} u$ *is irreducible. Then* $v \textcircled{T} u \approx u \textcircled{T} v$.

Proof. One has

$$\operatorname{Tr}(u \widetilde{\textcircled{T}} v) = \operatorname{Tr}(u \textcircled{T} v) = \operatorname{Tr} u \operatorname{Tr} v = \operatorname{Tr} v \operatorname{Tr} u = \operatorname{Tr}(v \textcircled{T} u) = \operatorname{Tr}(v \widetilde{\textcircled{T}} u),$$

which means that $u \widetilde{\textcircled{T}} v \approx v \widetilde{\textcircled{T}} u$. The representation $v \widetilde{\textcircled{T}} u \approx v \textcircled{T} u$ is irreducible, therefore the representation $u \widetilde{\textcircled{T}} v$ is irreducible. Thus one has $u \widetilde{\textcircled{T}} v \approx u \textcircled{T} v$. **Q.E.D.**

One has $u \widetilde{\textcircled{T}} v \approx \widetilde{u} \widetilde{\textcircled{T}} \widetilde{v}$ for any two representations u, v (trace of both sides is the same).

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