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ON THE CLASSIFICATION OF 3-DIMENSIONAL COLOURED LIE ALGEBRAS

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Abstract. In this paper, complex 3-dimensional Γ-graded ε-skew-symmetric and complex 3-dimensional Γ-graded ε-Lie algebras with either 1-dimensional or zero homogeneous components are classified up to isomorphism.

1. Introduction. Generalised Lie algebras are a natural generalisation of Lie algebras and Lie superalgebras. For the past twenty years, these algebras and their representations have been an object of constant interest in both mathematics and physics (see, e.g., [1], [3], [5], [6], [7], [11], [13], [14], [15], [16], [18], [19], [20], [23], [24], [25], [26], [28], [29], [30] and references there).

It is a problem of fundamental importance, both from theoretical and applied points of view, to obtain a classification of Lie algebras, Lie superalgebras and graded ε-Lie algebras of a given dimension up to isomorphism. Such a classification can be useful not only as an ordering tool, but also as a ground for investigation of representations of these algebras (see the last section).

Hitherto, attention has been restricted to Lie algebras and Lie superalgebras (see, e.g., [2], [4], [9], [10], [12], [17], [21], [27], [32], [33] and references there). Little has been done with regard to the classification of all generalised Lie algebras of a given dimension. The results obtained in this article can be considered as a contribution in that direction.

The present paper contains a classification of complex 3-dimensional Γ-graded ε-skew-symmetric and ε-Lie algebras with either 1-dimensional or zero homogeneous components.

Let us start by looking at two examples of three-dimensional generalised Lie algebras.

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EXAMPLE 1. The graded analogue of $sl(2,\mathbb{C})$ is defined as a complex algebra with three generators e_1, e_2 and e_3 satisfying the commutation relations

$$e_1e_2 + e_2e_1 = e_3, \quad e_1e_3 + e_3e_1 = e_2, \quad e_2e_3 + e_3e_2 = e_1.$$

Let L be a $\mathbb{Z}_2^3$-graded linear space $L = L_{(1,1,0)} \oplus L_{(1,0,1)} \oplus L_{(0,1,1)}$ with basis $e_1 \in L_{(1,1,0)}$, $e_2 \in L_{(1,0,1)}$, $e_3 \in L_{(0,1,1)}$. The homogeneous subspaces of L graded by the elements of $\mathbb{Z}_2^3$ different from $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are zero and so are omitted. The bilinear multiplication $\langle\cdot\,;\,\cdot\rangle : L \times L \mapsto L$ defined for the basis vectors e_1, e_2, e_3 by the formulas

$$\begin{aligned}
\langle e_1\,;\,e_1\rangle &= e_1e_1 - e_1e_1 = 0, & \langle e_1\,;\,e_2\rangle &= e_1e_2 + e_2e_1 = e_3,\\
\langle e_2\,;\,e_2\rangle &= e_2e_2 - e_2e_2 = 0, & \langle e_1\,;\,e_3\rangle &= e_1e_3 + e_3e_1 = e_2,\\
\langle e_3\,;\,e_3\rangle &= e_3e_3 - e_3e_3 = 0, & \langle e_2\,;\,e_3\rangle &= e_2e_3 + e_3e_2 = e_1,
\end{aligned}$$

makes L into a three-dimensional algebra.

EXAMPLE 2. The graded analogue of the Lie algebra of the group of plane motions is defined as a complex algebra with three generators e_1, e_2 and e_3 satisfying the commutation relations $e_1e_2 + e_2e_1 = e_3$, $e_1e_3 + e_3e_1 = e_2$, $e_2e_3 + e_3e_2 = 0$. The linear space L spanned by e_1, e_2, e_3 can be made into a Γ-graded ε-Lie algebra. The grading group Γ and the commutation factor ε are the same as in Example 1 . But the multiplication $\langle\cdot\,;\,\cdot\rangle$ is different and defined by

$$\begin{aligned}
\langle e_1\,;\,e_1\rangle &= e_1e_1 - e_1e_1 = 0, & \langle e_1\,;\,e_2\rangle &= e_1e_2 + e_2e_1 = e_3,\\
\langle e_2\,;\,e_2\rangle &= e_2e_2 - e_2e_2 = 0, & \langle e_1\,;\,e_3\rangle &= e_1e_3 + e_3e_1 = e_2,\\
\langle e_3\,;\,e_3\rangle &= e_3e_3 - e_3e_3 = 0, & \langle e_2\,;\,e_3\rangle &= e_2e_3 + e_3e_2 = 0.
\end{aligned}$$

It is easy to check that the bilinear multiplications (brackets) in Examples 1 and 2 satisfy the following three axioms:

Γ-grading axiom:

$$\langle L_\alpha\,,\,L_\beta\rangle \subseteq L_{\alpha+\beta} \quad \text{for any } \alpha, \beta \in \Gamma \tag{1}$$

ε-skew-symmetry condition:

$$\langle A, B\rangle = -\varepsilon(\deg(A), \deg(B))\langle B, A\rangle \tag{2}$$

ε-Jacobi identity:

$$\begin{aligned}
&\varepsilon(\deg(C), \deg(A))\langle A, \langle B, C\rangle\rangle + \varepsilon(\deg(B), \deg(C))\langle C, \langle A, B\rangle\rangle +\\
&\varepsilon(\deg(A), \deg(B))\langle B, \langle C, A\rangle\rangle = 0
\end{aligned} \tag{3}$$

for nonzero homogeneous A, B and C. Here $\deg(A) = \alpha \in \Gamma$ if $A \in L_\alpha$.

In Examples 1 and 2, Γ is the group $\mathbb{Z}_2^3$, and $\varepsilon : \Gamma \times \Gamma \mapsto \mathbb{C}$ is defined by the matrix

$$[\varepsilon(i,j)] = \begin{pmatrix} 1 & -1 & -1\\ -1 & 1 & -1\\ -1 & -1 & 1 \end{pmatrix}.$$

Elements of this matrix specify in the natural way values of ε on the set $\{(1,1,0), (1,0,1), (0,1,1)\} \times \{(1,1,0), (1,0,1), (0,1,1)\} \subset \mathbb{Z}_2^3 \times \mathbb{Z}_2^3$. (The elements $(1,1,0), (1,0,1), (0,1,1)$ are ordered and numbered by the numbers 1, 2, 3 respectively.) The values of ε on other elements from $\mathbb{Z}_2^3 \times \mathbb{Z}_2^3$ do not affect the multiplication $\langle \cdot\,;\,\cdot\rangle$. It is easily seen that the function ε can be extended to the whole $\Gamma \times \Gamma$ so that

$$\varepsilon(g+h, f) = \varepsilon(g, f)\cdot\varepsilon(h, f), \quad \varepsilon(f, g+h) = \varepsilon(f, g)\cdot\varepsilon(f, h), \tag{4}$$

$$\varepsilon(g, h)\cdot\varepsilon(h, g) = 1, \tag{5}$$

for $f, g, h \in \Gamma$.

For an arbitrary abelian group Γ a function ε obeying (4) and (5) is called a *commutation factor* on Γ. A Γ-graded linear space $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ equipped with a bilinear multiplication $\langle \cdot\,;\,\cdot\rangle : L \times L \mapsto L$ obeying the Γ-grading axiom and the ε-skew-symmetry condition is called a Γ*-graded* ε*-skew-symmetric algebra.* If moreover the ε-Jacobi identity is satisfied, then L is called a Γ*-graded* ε*-Lie algebra.* Γ-graded ε-Lie algebras are also called *coloured Lie algebras* or *generalised Lie algebras.* For an arbitrary generalised Lie algebra of dimension n the multiplication is given by the relations $\langle e_i, e_j\rangle = \sum_{k=1}^n a_{i,j}^k e_k$, where the basis $\{e_i\}_{i=1}^n$ consists of homogeneous elements. The coefficients $a_{i,j}^k$ are called structure constants. Because of bilinearity of $\langle\cdot,\cdot\rangle$, they form a tensor called a structure tensor. The conditions (1), (2) and (3) imply additional restrictions on the structure tensor. *A generalised Lie algebra is completely defined by a choice of commutation factor and structure constants.* After a change of basis one gets new structure constants ${a_{i,j}^l}' = c_i^t \cdot c_j^s \cdot \tilde{c}_k^l \cdot a_{t,s}^k = \frac{1}{detC}\cdot c_i^t \cdot c_j^s \cdot \Delta_l^k \cdot a_{t,s}^k$ where $C^{-1} = ||\tilde{c}_k^l||_{l,k=1}^n$ is the inverse of the matrix C of the change of bases in L and $\Delta_l^k = (-1)^{k+l} m_l^k$, m_l^k being a complementary minor for the element c_k^l.

R e m a r k 1. We saw that the linear spaces spanned by the generators e_1, e_2 and e_3 in Examples 1 and 2 can be made into a Γ-graded ε-Lie algebras. The graded analogue of $sl(2,\mathbb{C})$ and the graded analogue of the Lie algebra of the group of plane motions are their universal enveloping algebras.

Attention! The following three assumptions will be made:

1. The matrices of commutation factors contain values corresponding only to non-zero homogeneous subspaces, as in Examples 1 and 2. The information about zero components is useless.

2. All commutation factors are assumed to be *injective.* Injectivity means that the matrix of a commutation factor contains no equal rows (and hence no equal columns). This can always be achieved by joining the homogeneous subspaces corresponding to equal rows into one homogeneous subspace. Note that such procedure does not influence the defining commutation relations.

3. All commutation factors are assumed to take values 1 or -1.

Isomorphism of generalised skew-symmetric or generalised Lie algebras is defined as follows.

Definition 1. A generalised skew-symmetric (Lie) algebra L_1 and a generalised skew-symmetric (Lie) algebra L_2 with injective commutation factors are said to be *isomorphic* if there exists a map $\sigma : L_1 \mapsto L_2$ such that:

1) σ is an ordinary isomorphism of algebras L_1 and L_2;
2) homogeneous elements are mapped into the homogeneous ones;
3) σ preserves the commutation factor.

2. The classification of Γ-graded ε-skew-symmetric algebras. In Table 1, we present in lexicographic order all 3-dimensional injective commutation factors.

Proposition 1. *There are* 44 *injective commutation factors of dimension* 3.

Table 1

$$\varepsilon_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \varepsilon_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \varepsilon_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\varepsilon_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \varepsilon_5 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \varepsilon_6 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$\varepsilon_7 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \varepsilon_8 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \quad \varepsilon_9 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\varepsilon_{10} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{11} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \varepsilon_{12} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\varepsilon_{13} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{14} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \varepsilon_{15} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\varepsilon_{16} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{17} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \quad \varepsilon_{18} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\varepsilon_{19} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{20} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \varepsilon_{21} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$\varepsilon_{22} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad \varepsilon_{23} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad \varepsilon_{24} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\varepsilon_{25} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{26} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \varepsilon_{27} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\varepsilon_{28} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \varepsilon_{29} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \varepsilon_{30} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\varepsilon_{31} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{32} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \varepsilon_{33} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\varepsilon_{34} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{35} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \varepsilon_{36} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\varepsilon_{37} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \varepsilon_{38} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{39} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\varepsilon_{40} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \varepsilon_{41} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \quad \varepsilon_{42} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\varepsilon_{43} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \varepsilon_{44} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

The matrix of any isomorphism can be decomposed into product of a substitution matrix and a diagonal matrix, since all homogeneous subspaces are one-dimensional. So, we have an action of the substitution group S_3 on the set $ICF(3)$ of injective 3-dimensional commutation factors. We will denote this action by $T^{S_3}_{ICF(3)}$. The orbits of the action $T^{S_3}_{ICF(3)}$ are tabulated in Table 2.

Table 2

S_3 / $Orb(T^{S_3}_{ICF(3)})$	(1)	(1; 2)	(1; 3)	(2; 3)	(1; 2; 3)	(1; 3; 2)
ν_1	ε_1	ε_5	ε_{11}	ε_1	ε_5	ε_{11}
ν_2	ε_2	ε_6	ε_{35}	ε_4	ε_{29}	ε_{14}
ν_3	ε_3	ε_{23}	ε_{26}	ε_3	ε_{23}	ε_{26}
ν_4	ε_7	ε_{24}	ε_7	ε_{12}	ε_{12}	ε_{24}
ν_5	ε_8	ε_{25}	ε_{32}	ε_{15}	ε_{36}	ε_{28}
ν_6	ε_9	ε_{30}	ε_{20}	ε_{13}	ε_{17}	ε_{37}
ν_7	ε_{10}	ε_{31}	ε_{43}	ε_{16}	ε_{41}	ε_{39}
ν_8	ε_{18}	ε_{18}	ε_{18}	ε_{18}	ε_{18}	ε_{18}
ν_9	ε_{19}	ε_{19}	ε_{42}	ε_{22}	ε_{42}	ε_{22}
ν_{10}	ε_{21}	ε_{38}	ε_{33}	ε_{21}	ε_{38}	ε_{33}
ν_{11}	ε_{27}	ε_{27}	ε_{27}	ε_{27}	ε_{27}	ε_{27}
ν_{12}	ε_{34}	ε_{34}	ε_{44}	ε_{40}	ε_{44}	ε_{40}

It determines an equivalence relation such that for every orbit ν of the action there corresponds a class $W(\nu)$ of graded ε-skew-symmetric algebras with commutation factors from this orbit.

The classes $W(\nu)$ are disjoint. Every algebra belongs to one of these classes. Moreover, there are no isomorphic algebras belonging to different classes, since if L and L' are isomorphic, then ε and ε' belong to the same orbit. This means that isomorphic algebras belong to the same class. From this it follows that the problem of classification of all algebras under consideration may be reduced to classifying separately algebras of each class $W(\nu)$.

The set of all orbits $Orb(T^{S_3}_{ICF(3)})$ can be decomposed as $Orb(T^{S_3}_{ICF(3)}) = Orb_1 \cup Orb_2 \cup Orb_3$, where $Orb_1 = \{\nu_8, \nu_{10}\}$, $Orb_2 = \{\nu_4, \nu_5, \nu_6, \nu_7, \nu_9, \nu_{11}, \nu_{12}\}$, $Orb_3 = \{\nu_1, \nu_2, \nu_3\}$. We will use the following notations:

$$\begin{aligned} P_1 &= W(\nu_8) \cup W(\nu_{10}) \\ P_2 &= W(\nu_4) \cup W(\nu_5) \cup W(\nu_6) \cup W(\nu_7) \cup W(\nu_9) \cup W(\nu_{11}) \cup W(\nu_{12}) \\ P_3 &= W(\nu_1) \cup W(\nu_2) \cup W(\nu_3). \end{aligned}$$

Then $P_i \cap P_j = \emptyset, i \neq j$, every algebra belongs to one of the classes P_i, and there are no isomorphic algebras belonging to different classes. The classes P_1, P_2 and P_3 exhaust all possible variants of grading sets, i.e. sets of elements labelling non-zero homogeneous subspaces. It is easily seen that

$$
\begin{aligned}
&(\ P_1\ ) \Leftrightarrow (\ \forall\ i \in \{1,2,3\}:\ \omega_i \neq \omega_0, \omega_1 + \omega_2 = \omega_3)\\
&(\ P_2\ ) \Leftrightarrow (\ \forall\ i \in \{1,2,3\}:\ \omega_i \neq \omega_0, \omega_1 + \omega_2 \neq \omega_3)\\
&(\ P_3\ ) \Leftrightarrow \left(\begin{array}{l} \exists\ i \in \{1,2,3\}: \\ \omega_i = \omega_0, \omega_k + \omega_j \neq \omega_i, i \neq j \neq k \neq i, \{j,k\} \subseteq \{1,2,3\} \end{array} \right),
\end{aligned}
$$

where ω_1, ω_2, ω_3 are elements grading the non-zero homogeneous subspaces, and ω_0 is the identity of the grading group. In grading sets for the class P_3 there is one marked element, namely, the identity of the grading group. We may always assume that the homogeneous subspace graded by the identity element is placed first (before the other two subspaces), since the other possibilities can be reduced to this one by an appropriate change of basis.

The classification table 3 is divided into three parts corresponding to the classes P_1, P_2, P_3. The matrices C in Table 3 describe the change of basis transforming the general commutation relations defining the algebras of a class P_i to the commutation relations in the canonical algebras from this class. Both the general commutation relations and the canonical ones are given in the table.

Now we will formulate and prove a classification theorem for 3-dimensional Γ-graded ε-skew-symmetric algebras with one-dimensional or zero homogeneous components. The proof of the classification is partly contained in the classification table itself. However, the calculations leading to the classification are not included in the article because of the lack of space.

THEOREM 2. *Any* 3*-dimensional* Γ*-graded* ε*-skew-symmetric algebra with one-dimensional or zero homogeneous components is isomorphic, in the sense of Definition* 1*, to one of the algebras specified in Table* 3*. The algebras from different entries of the table or with different values of the parameter* μ *are non-isomorphic.*

Table 3

$\boldsymbol{P}_1:\quad \langle e_1,e_1\rangle = 0,\ \langle e_2,e_2\rangle = 0,\ \langle e_3,e_3\rangle = 0$ $\langle e_1,e_2\rangle = \gamma e_3,\ \langle e_1,e_3\rangle = \beta e_2,\ \langle e_2,e_3\rangle = \alpha e_1.$	
1) $\alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0,$ $\langle e_1,e_1\rangle = 0,\ \langle e_2,e_2\rangle = 0,\ \langle e_3,e_3\rangle = 0,$ $\langle e_1,e_2\rangle = e_3,\ \langle e_1,e_3\rangle = e_2,\ \langle e_2,e_3\rangle = e_1$ $\varepsilon:\ \ \varepsilon_{18},\ \varepsilon_{21}$	$C = \begin{pmatrix} \frac{1}{\sqrt{\beta}\sqrt{\gamma}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}\sqrt{\gamma}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\alpha}\sqrt{\beta}} \end{pmatrix}$
2) $\alpha \neq 0,\ \beta \neq 0,\ \gamma = 0,$ $\langle e_1,e_1\rangle = 0,\ \langle e_2,e_2\rangle = 0,\ \langle e_3,e_3\rangle = 0,$ $\langle e_1,e_2\rangle = 0,\ \langle e_1,e_3\rangle = e_2,\ \langle e_2,e_3\rangle = e_1$ $\varepsilon:\ \ \varepsilon_{21}$	$C = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta}\sqrt{\alpha}} \end{pmatrix}$
3) $\alpha = 0,\ \beta \neq 0,\ \gamma \neq 0,$ $\langle e_1,e_1\rangle = 0,\ \langle e_2,e_2\rangle = 0,\ \langle e_3,e_3\rangle = 0,$ $\langle e_1,e_2\rangle = e_3,\ \langle e_1,e_3\rangle = e_2,\ \langle e_2,e_3\rangle = 0$ $\varepsilon:\ \ \varepsilon_{18},\ \varepsilon_{21}$	$C = \begin{pmatrix} \frac{1}{\sqrt{\beta}\sqrt{\gamma}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\gamma}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta}} \end{pmatrix}$

Table 3 (cont.)

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4) $\alpha \neq 0,\ \beta = 0,\ \gamma = 0,$

$\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = 0,$

$\langle e_1, e_2\rangle = 0,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = e_1$

$\varepsilon:\ \ \varepsilon_{21}$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\alpha}} \end{pmatrix}$$

5) $\alpha = 0,\ \beta = 0,\ \gamma \neq 0,$

$\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = 0,$

$\langle e_1, e_2\rangle = e_3,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$

$\varepsilon:\ \ \varepsilon_{18},\ \varepsilon_{21}$

$$C = \begin{pmatrix} \frac{1}{\sqrt{\gamma}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\gamma}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6) $\alpha = 0,\ \beta = 0,\ \gamma = 0,$

$\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = 0,$

$\langle e_1, e_2\rangle = 0,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$

$\varepsilon:\ \ \varepsilon_{18},\ \varepsilon_{21}$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boldsymbol{P}_2:$ $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = 0$

$\langle e_1, e_2\rangle = 0,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$

$\varepsilon:\ \ \varepsilon_7,\ \varepsilon_8,\ \varepsilon_9,\ \varepsilon_{10},\ \varepsilon_{19},\ \varepsilon_{27},\ \varepsilon_{34}$

$\boldsymbol{P}_3:$ $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = \alpha e_1,\ \langle e_3, e_3\rangle = \beta e_1$

$\langle e_1, e_2\rangle = \gamma e_2,\ \langle e_1, e_3\rangle = te_3,\ \langle e_2, e_3\rangle = 0.$

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1) $\alpha \neq 0, \beta \neq 0, \gamma \neq 0, t \neq 0,$

$\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = e_1,\ \langle e_3, e_3\rangle = e_1,$

$\langle e_1, e_2\rangle = \mu e_2,\ \langle e_1, e_3\rangle = e_3,\ \langle e_2, e_3\rangle = 0$

$\varepsilon:\ \ \varepsilon_3$ $\qquad \mu \in \mathbb{C} \setminus \{0\}$

$$C = \begin{pmatrix} \frac{1}{t} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}\sqrt{t}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta}\sqrt{t}} \end{pmatrix}$$

2) $\alpha \neq 0, \beta \neq 0, \gamma \neq 0, t = 0,$

$\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = e_1,\ \langle e_3, e_3\rangle = e_1,$

$\langle e_1, e_2\rangle = e_2,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$

$\varepsilon:\ \ \varepsilon_3$

$$C = \begin{pmatrix} \frac{1}{\gamma} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}\sqrt{\gamma}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta}\sqrt{\gamma}} \end{pmatrix}$$

3) $\alpha = 0, \beta \neq 0, \gamma \neq 0, t \neq 0,$

$\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = e_1,$

$\langle e_1, e_2\rangle = \mu e_2,\ \langle e_1, e_3\rangle = e_3,\ \langle e_2, e_3\rangle = 0$

$\varepsilon:\ \ \varepsilon_2, \varepsilon_3$ $\qquad \mu \in \mathbb{C} \setminus \{0\}$

$$C = \begin{pmatrix} \frac{1}{t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta}\sqrt{t}} \end{pmatrix}$$

4) $\alpha \neq 0, \beta \neq 0, \gamma = 0, t \neq 0,$

$\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = e_1,\ \langle e_3, e_3\rangle = e_1,$

$\langle e_1, e_2\rangle = 0,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$

$\varepsilon:\ \ \varepsilon_3$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

Table 3 (cont.)

5) $\alpha = 0, \beta \neq 0, \gamma \neq 0, t = 0$, $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = e_1$, $\langle e_1, e_2\rangle = e_2,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$ $\varepsilon:\ \ \varepsilon_2, \varepsilon_3$	$C = \begin{pmatrix} \frac{1}{\gamma}0 & 0 \\ 01 & 0 \\ 00\frac{1}{\sqrt{\beta}} & \frac{1}{\sqrt{\gamma}} \end{pmatrix}$
6) $\alpha = 0, \beta = 0, \gamma \neq 0, t \neq 0$, $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = 0$, $\langle e_1, e_2\rangle = \mu e_2,\ \langle e_1, e_3\rangle = e_3,\ \langle e_2, e_3\rangle = 0$ $\varepsilon:\ \ \varepsilon_1, \varepsilon_2, \varepsilon_3$ $\qquad \mu \in \mathbb{C} \setminus \{0\}$	$C = \begin{pmatrix} \frac{1}{t}00 \\ 010 \\ 001 \end{pmatrix}$
7) $\alpha = 0, \beta \neq 0, \gamma = 0, t \neq 0$, $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = e_1$, $\langle e_1, e_2\rangle = 0,\ \langle e_1, e_3\rangle = e_3,\ \langle e_2, e_3\rangle = 0$ $\varepsilon:\ \ \varepsilon_2, \varepsilon_3$	$C = \begin{pmatrix} \frac{1}{t}0 & 0 \\ 01 & 0 \\ 00\frac{1}{\sqrt{\beta}\sqrt{t}} \end{pmatrix}$
8) $\alpha = 0, \beta \neq 0, \gamma = 0, t = 0$, $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = e_1$, $\langle e_1, e_2\rangle = 0,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$ $\varepsilon:\ \ \varepsilon_2, \varepsilon_3$	$C = \begin{pmatrix} 10 & 0 \\ 01 & 0 \\ 00\frac{1}{\sqrt{\beta}} \end{pmatrix}$
9) $\alpha = 0, \beta = 0, \gamma \neq 0, t = 0$, $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = 0$, $\langle e_1, e_2\rangle = e_2,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$ $\varepsilon:\ \ \varepsilon_1, \varepsilon_2, \varepsilon_3$	$C = \begin{pmatrix} \frac{1}{\gamma}00 \\ 010 \\ 001 \end{pmatrix}$
10) $\alpha = 0, \beta = 0, \gamma = 0, t \neq 0$, $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = 0$, $\langle e_1, e_2\rangle = 0,\ \langle e_1, e_3\rangle = e_3,\ \langle e_2, e_3\rangle = 0$ $\varepsilon:\ \ \varepsilon_2$	$C = \begin{pmatrix} \frac{1}{t}00 \\ 010 \\ 001 \end{pmatrix}$
11) $\alpha = 0,\ \beta = 0,\ \gamma = 0,\ t = 0$, $\langle e_1, e_1\rangle = 0,\ \langle e_2, e_2\rangle = 0,\ \langle e_3, e_3\rangle = 0$, $\langle e_1, e_2\rangle = 0,\ \langle e_1, e_3\rangle = 0,\ \langle e_2, e_3\rangle = 0$ $\varepsilon:\ \ \varepsilon_1, \varepsilon_2, \varepsilon_3$	$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. The classification of Γ-graded ε-Lie algebras. In this section we will classify all 3-dimensional Γ-graded ε-Lie algebras with one-dimensional homogeneous components. Using the classification of graded ε-skew-symmetric algebras presented in Theorem 2, the graded ε-Lie algebras can be classified by checking which of the ε-skew-symmetric algebras satisfy the ε-Jacobi identity.

Because of the bilinearity of bracket multiplication, the left part of the ε-Jacobi identity is a 3-linear form and so is completely defined by its values on a basis. Therefore, it suffices to check the ε-Jacobi identity for the basis elements.

Let $\{e_i\}_{i=1}^3$ be a basis of L, such that $e_i \in L_{\omega_i},\ i = 1, 2, 3$.

For basis elements $e_{i_1}, e_{i_2}, e_{i_3}$ the ε-Jacobi identity can be written in the form

$$\varepsilon(i_3,i_1)\langle e_{i_1},\langle e_{i_2},e_{i_3}\rangle\rangle+\varepsilon(i_2,i_3)\langle e_{i_3},\langle e_{i_1},e_{i_2}\rangle\rangle+$$
$$\varepsilon(i_1,i_2)\langle e_{i_2},\langle e_{i_3},e_{i_1}\rangle\rangle=0,$$

where $\{i_1,i_2,i_3\}\in\{1,2,3\}$ and $\varepsilon(i,j)=\varepsilon(\omega_i,\omega_j)$.

Table 4 presents values of the left part of the ε-Jacobi identity for the basis elements in each of the cases P_1, P_2, P_3.

Table 4

	P_1	P_2	P_3
$(1;1;1)$	0	0	0
$(1;1;2)$	$\beta\cdot\gamma\cdot(\varepsilon(2,1)-\varepsilon(3,1))e_2$	0	0
$(1;2;2)$	$\alpha\cdot\gamma(\varepsilon(2,2)-1)e_1$	0	$\alpha\cdot\gamma(\varepsilon(2,2)-1)e_1$
$(2;2;2)$	0	0	$-3\alpha\cdot\gamma\cdot\varepsilon(2,2)e_2$
$(1;1;3)$	$\beta\cdot\gamma\cdot(\varepsilon(3,1)-\varepsilon(2,1))e_3$	0	0
$(1;2;3)$	0	0	0
$(1;3;2)$	0	0	0
$(1;3;3)$	$\alpha\cdot\beta\cdot(\varepsilon(3,2)-\varepsilon(3,1))e_1$	0	$\beta\cdot t\cdot(\varepsilon(3,3)-1)e_1$
$(2;2;3)$	$\alpha\cdot\gamma\cdot(1-\varepsilon(2,2))e_3$	0	$-\alpha\cdot t\varepsilon(2,3)e_3$
$(2;3;3)$	$\alpha\cdot\beta\cdot(\varepsilon(3,1)-\varepsilon(3,2))e_2$	0	$-\beta\cdot\gamma\cdot\varepsilon(3,2)e_2$
$(3;3;3)$	0	0	$-3\beta\cdot t\cdot\varepsilon(3,3)e_3$

The set of triples $\{1,2,3\}^3$ splits into orbits under the action of the permutation $s=(1;2;3)$. Each orbit consist of 3 elements and determines the ordered triple of basis elements for which the ε-Jacobi identity is written down. It is enough to know the left part of ε-Jacobi identity for one element of each orbit. The first column of the Table is a list of triples of indices (one from each orbit) for which we calculate the left part of the ε-Jacobi identity. In the second, third and fourth column, the left parts of the ε-Jacobi identity for the algebras from the classes P_1, P_2, P_3 respectively, are given for each triple from the first column. Note that in the second column the parameters α, β, γ are the structure constants for the algebras from the class P_1, but in the fourth column α, β, γ are the structure constants for the algebras from the class P_3.

From Table 4, the necessary and sufficient conditions for the ε-Jacobi identity to be satisfied can be obtained in each case P_i by putting the corresponding left parts of the ε-Jacobi identity equal to 0.

As a result, we get the following classification Theorem.

THEOREM 3. *The classification of 3-dimensional injective generalised ε-Lie algebras with one-dimensional or zero homogeneous components is provided by the following list of (canonical) algebras:*

$$(P_1,\varepsilon_{18},1),(P_1,\varepsilon_{18},3),(P_1,\varepsilon_{18},5),(P_1,\varepsilon_{18},6);$$
$$(P_1,\varepsilon_{21},3),(P_1,\varepsilon_{21},4),(P_1,\varepsilon_{21},5),(P_1,\varepsilon_{21},6);$$
$$(P_2,\varepsilon_7),(P_2,\varepsilon_8),(P_2,\varepsilon_9),(P_2,\varepsilon_{10}),(P_2,\varepsilon_{19}),(P_2,\varepsilon_{27}),(P_2,\varepsilon_{34});$$
$$(P_3,\varepsilon_1,6),(P_3,\varepsilon_1,9),(P_3,\varepsilon_1,11);$$
$$(P_3,\varepsilon_2,6),(P_3,\varepsilon_2,8),(P_3,\varepsilon_2,9),(P_3,\varepsilon_2,10),(P_3,\varepsilon_2,11);$$
$$(P_3,\varepsilon_3,6),(P_3,\varepsilon_3,8),(P_3,\varepsilon_3,9),(P_3,\varepsilon_3,11).$$

Here (P_i, ε_j, k) denotes an algebra from the subclass k of the class P_i with commutation factor ε_j (see Table 3*).*

Remark 2. Theorem 3 gives a classification of complex graded ε-Lie algebras of dimension 3 with 1-dimensional homogeneous components. The ε-Jacobi identity provides us with the Poincaré-Birkhoff-Witt theorem (see [25], [28]). It means that the graded ε-Lie algebras can be considered as algebras with quadratic permutation relations, and it is important to study, for such algebras, the involutions (real structures) and their $\star$-representations by bounded and unbounded operators in a Hilbert space.

4. Some interesting problems. This section is devoted to some interesting problems connected with the results of this paper.

1) The first problem is the classification of all graded ε-Lie algebras up to isomorphism. It is not yet solved, as it is still open for Lie algebras and superalgebras. Some progress towards a solution of this problem for Lie algebras was achieved by a study of their deformations and contractions. Therefore, it is very likely that similar methods can be applied successfully for graded ε-Lie algebras.

2) After complex graded ε-Lie algebras were classified, the problem of classifying their real structures (involutions) naturally arises. Moreover, if real graded ε-Lie algebras were classified, then it would be interesting to study the correspondence between these real algebras and the real structures of the complex ε-Lie algebras.

3) Once all involutions for graded ε-Lie algebras are found, we can begin on the problem of classifying, up to unitary equivalence, $\star$-representations of these real structures by bounded and unbounded operators in a Hilbert space. The main part of this problem is the classification of their irreducible $\star$-representations. In [8], [22], all the irreducible representations of the real structures of the algebra $(P_1, \varepsilon_{18}, 1)$ were classified. In [31] the irreducible representations of the algebra $(P_1, \varepsilon_{18}, 3)$ with the identity involution were classified. The methods developed in [22], [31] could be applied to other algebras from the classification in Theorem 3 or to more general generalised Lie algebras.

4) Finally, let us note that the classification presented in this paper contains infinite families of algebras, depending on some continuous parameter. It would be interesting to investigate how representations of these algebras and the representations of their real structures behave under a change of parameter. In particular, do the representations change continuously when the parameter tends to the limit points of the parameter set?

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