

ON LIE ALGEBRAS IN BRAIDED CATEGORIES

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Herrn Prof. Dr. Friedrich Kasch zum 75. Geburtstag gewidmet

Abstract. The category of group-graded modules over an abelian group G is a monoidal category. For any bicharacter of G this category becomes a braided monoidal category. We define the notion of a Lie algebra in this category generalizing the concepts of Lie super and Lie color algebras. Our Lie algebras have n -ary multiplications between various graded components. They possess universal enveloping algebras that are Hopf algebras in the given category. Their biproducts with the group ring are noncommutative noncocommutative Hopf algebras some of them known in the literature. Conversely the primitive elements of a Hopf algebra in the category form a Lie algebra in the above sense.

1. Introduction. With the appearance of quantum groups the study of Hopf algebras has taken an important turn in recent years. Many families of quantum groups are known. Here we will add some new families by means of a construction which might eventually also help to develop a structure theory of quantum groups. In classical structure theory a formal group is decomposed into a smash product of an infinitesimal part with a separable part. The separable part is often a group algebra, whereas the infinitesimal part is often a universal enveloping algebra on which the group algebra operates. The infinitesimal part then is generated by the primitive elements of the formal group.

However, for quantum groups or noncommutative noncocommutative Hopf algebras this kind of decomposition seems to be much more complicated. We will use the following approach.

Let $\chi : G \times G \longrightarrow k^*$ be a bicharacter of the abelian group G . Then the monoidal category of G -graded vector spaces or kG -comodules is a braided monoidal category.

1991 *Mathematics Subject Classification*: Primary 16W30, 17B70; Secondary 16W55, 16S30, 16S40.

Key words and phrases: graded Lie algebra, braided category, braided Hopf algebra, universal enveloping algebra.

The paper is in final form and no version of it will be published elsewhere.

By forming a biproduct with the group algebra kG , each Hopf algebra in this category generates an ordinary Hopf algebra. This process was studied for Hopf algebras with a projection by Radford in [R] (or as the process of bosonization by Majid in [M94a]).

We will give some general techniques for constructing Hopf algebras in the category of G -graded vector spaces. Our special interest lies in Hopf algebras that are generated by primitive elements in the proper sense, that is $\Delta(x) = x \otimes 1 + 1 \otimes x$. In the biproduct these elements become skew primitive elements.

There is a generalized notion of a Lie algebra in symmetric monoidal categories under the name of Lie color algebra (or Lie super algebra). An appropriate definition of a Lie algebra in a *braided* monoidal category should have specific properties. In the classical situation the primitive elements of a Hopf algebra form a Lie algebra. Also the set of derivations of an algebra is a Lie algebra. So a more general definition of a Lie algebra should pass a test with respect to the set primitive elements of a Hopf algebra and with respect to the set of derivations of an algebra. We propose a definition of generalized Lie algebras in the braided monoidal category of G -graded vector spaces. These generalized Lie algebras have n -ary bracket multiplications that are only partially defined, but certain symmetry and Jacobi identities still hold.

Lie color and Lie super algebras as well as ordinary Lie algebras are special cases of our generalized Lie algebras. We will show that the set of primitive elements of a Hopf algebra is such a generalized Lie algebra. Every associative algebra is also a generalized Lie algebra by the same definition of the bracket multiplication. And the set of derivations of an algebra will turn out to be a Lie algebra.

Starting out with a generalized Lie algebra we will construct a universal enveloping algebra that turns out to be a Hopf algebra in the category of G -graded vector spaces. Thus we have obtained a method for constructing new Hopf algebras in this category. By forming biproducts we obtain many old and new ordinary noncommutative noncocommutative Hopf algebras or quantum groups.

Ordinary Hopf algebras over a field of characteristic zero that are generated by their primitive elements are either trivial or infinite dimensional. In fact they are universal enveloping algebras of Lie algebras.

Hopf algebras in braided monoidal categories in characteristic zero that are generated by their primitive elements, however, do not share this property. In certain cases they are similar to universal restricted enveloping algebras of restricted (p-)Lie algebras.

A simple example of such a Hopf algebra is $\mathbf{C}[x]/(x^n)$ with $\Delta(x) = x \otimes 1 + 1 \otimes x$ considered as an object in the braided monoidal category of C_n -graded vector spaces, where C_n is the cyclic group of order n and the braiding is given by a primitive n -th root of unity. It is generated by the primitive element x as an algebra in the given category, but it is finite dimensional. Such a Hopf algebra cannot exist as an ordinary Hopf algebra (in the category of vector spaces).

We will restrict our considerations to group graded vector spaces over a fixed abelian group G . A consequence of one of our main results is that for primitive elements x_1, \dots, x_n

of degree 1 in a C_n -graded Hopf algebra H the expression

$$[x_1, \dots, x_n] := \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is again a primitive element (of degree zero) in H (where S_n is the symmetric group).

By now the reader should be interested to see the definition of a generalized G -graded Lie algebra. Let G be an abelian group with a bicharacter $\chi : G \times G \rightarrow k^*$. For every primitive n -th root of unity ζ we consider certain n -tuples (g_1, \dots, g_n) in G associated with ζ . We will call them ζ -families. Furthermore we construct factors $\rho(\sigma, (g_1, \dots, g_n)) \in k^*$ for each permutation $\sigma \in S_n$ and each ζ -family (g_1, \dots, g_n) . These factors generalize the sign of a permutation. Details will be given in Definition 2.1. A generalized G -graded Lie algebra is a G -graded vector space $P = \bigoplus_{g \in G} P_g$ that has multilinear bracket operations for all ζ and all ζ -families (g_1, \dots, g_n)

$$[\dots] : P_{g_1} \times \dots \times P_{g_n} \rightarrow P_{g_1 + \dots + g_n}$$

satisfying the following generalizations of the symmetry and Jacobi identities

- $[x_1, \dots, x_n] = \rho(\sigma, (g_1, \dots, g_n)) [x_{\sigma(1)}, \dots, x_{\sigma(n)}]$,
- $\sum_{i=1}^{n+1} \zeta^{-i+1} \left(\prod_{j=1}^{i-1} \chi(g_j, g_i) \right) [x_i, [x_1, \dots, \hat{x}_i, \dots, x_{n+1}]] = 0$,
- $[x, [y_1, y_2, \dots, y_n]] = \sum_{i=1}^n \left(\prod_{j=1}^{i-1} \chi(g_j, h) \right) [y_1, \dots, [x, y_i], \dots, y_n]$,

whenever the bracket products are defined. In particular we require $x_i, y_i \in P_{g_i}$ and $x \in P_h$.

The set of primitive elements of a G -graded Hopf algebra is an example for a generalized Lie algebra, and so is any G -graded algebra if we define the bracket operation by

$$[x_1, \dots, x_n] := \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

A Lie super algebra (P_0, P_1) , where P_0 is an ordinary Lie algebra with operations $[\cdot, \cdot] : P_1 \otimes P_1 \rightarrow P_0$ and $[\cdot, \cdot] : P_0 \otimes P_1 \rightarrow P_1$, is an example for this definition. Lie color algebras are also special cases of our concept.

I would like to acknowledge helpful conversations with Edward L. Green, Frank Hanke, Susan Montgomery, Martin Neuchl, Helmut Rohrl, Peter Schauenburg, and Yorck Sommerhäuser.

2. Bicharacters and the bracket multiplication. Throughout let k be an integral domain, let G be an abelian group written additively and let $\chi : G \times G \rightarrow k^*$ be a bicharacter, that is a group homomorphism $\chi : G \otimes_{\mathbf{Z}} G \rightarrow k^*$. Then bicharacters on \mathbf{Z} resp. $\mathbf{Z}/n\mathbf{Z}$ have the form $\chi(i, j) = \zeta^{ij}$.

DEFINITION 2.1. Let $\zeta \in k^*$. An n -tuple (g_1, \dots, g_n) of (not necessarily distinct) elements $g_i \in G$ is a ζ -family of length n , if

$$\chi(g_i, g_j) \chi(g_j, g_i) = \zeta^2$$

for all $i \neq j$.

Observe that a ζ -family (g_1, \dots, g_n) is also a $(-\zeta)$ -family.

If (g_1, \dots, g_n) is a ζ -family and $\sigma \in S_n$ then $\sigma(g_1, \dots, g_n) := (g_{\sigma(1)}, \dots, g_{\sigma(n)})$ is also a ζ -family. Thus the set G_n^ζ of all ζ -families of length n is an S_n -set. Observe that there are ζ -families of varying lengths n .

Let $X \subset G_n^\zeta$ be an S_n -closed subset. We define a map $\rho : S_n \times X \rightarrow k^*$ by

$$\rho(\sigma, (g_1, \dots, g_n)) := \prod_{(i,j) \in R} (\zeta^{-1} \chi(g_{\sigma(j)}, g_{\sigma(i)}))$$

where R is the set of all pairs (i, j) with $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$, i.e. the set of all pairs in σ in reverse position.

LEMMA 2.2. *The map ρ satisfies the following relation*

$$\rho(\sigma\tau, (g_1, \dots, g_n)) = \rho(\tau, (g_{\sigma(1)}, \dots, g_{\sigma(n)})) \rho(\sigma, (g_1, \dots, g_n)).$$

Proof. We have to show

$$\begin{aligned} & \prod_{i < j, \sigma\tau(i) > \sigma\tau(j)} (\zeta^{-1} \chi(g_{\sigma\tau(j)}, g_{\sigma\tau(i)})) \\ &= \left(\prod_{k < l, \tau(k) > \tau(l)} (\zeta^{-1} \chi(g_{\sigma\tau(l)}, g_{\sigma\tau(k)})) \right) \left(\prod_{r < s, \sigma(r) > \sigma(s)} (\zeta^{-1} \chi(g_{\sigma(s)}, g_{\sigma(r)})) \right). \end{aligned}$$

We investigate the factors on the left hand side. If $i < j$ with $\sigma\tau(i) < \sigma\tau(j)$ then there is no corresponding factor on the l.h.s. On the r.h.s. there are now two possibilities. If $s := \tau(i) > \tau(j) =: r$ (and thus $\sigma(r) > \sigma(s)$) then the first product contributes a factor $\zeta^{-1} \chi(g_{\sigma\tau(j)}, g_{\sigma\tau(i)})$ for the pair $i < j, \tau(i) > \tau(j)$ and the second product contributes a factor $\zeta^{-1} \chi(g_{\sigma\tau(i)}, g_{\sigma\tau(j)})$ for the pair $r < s, \sigma(r) > \sigma(s)$. These two factors cancel. If $s := \tau(i) < \tau(j) =: r$ then neither pair $i < j, \tau(i) < \tau(j)$ nor $r < s, \sigma(r) < \sigma(s)$ contributes a factor.

However, if $i < j$ with $\sigma\tau(i) > \sigma\tau(j)$ then there is a corresponding factor $\zeta^{-1} \chi(g_{\sigma\tau(j)}, g_{\sigma\tau(i)})$ on the l.h.s. On the r.h.s. there are again two possibilities. If $\tau(i) > \tau(j)$ then the first product contributes a factor $\zeta^{-1} \chi(g_{\sigma\tau(j)}, g_{\sigma\tau(i)})$ for the pair $i < j, \tau(i) > \tau(j)$. The second product does not contribute a factor for the pair $s := \tau(i) > \tau(j) =: r, \sigma(r) < \sigma(s)$. If $s := \tau(i) < \tau(j) =: r$ then the first product does not contribute a factor for the pair $i < j, \tau(i) < \tau(j)$, but the second product contributes a factor $\zeta^{-1} \chi(g_{\sigma\tau(i)}, g_{\sigma\tau(j)})$ for the pair $r < s, \sigma(r) > \sigma(s)$. This argument takes care of all factors on both sides of the formula.

If $g, g' \in G$ have order m resp. m' and if $n = \gcd(m, m')$ then $\chi(g, g')^n = 1$ since $\chi(g, g')^m = \chi(mg, g') = \chi(0, g') = 1$ and $\chi(g, g')^{m'} = \chi(g, m'g') = 1$. Hence $\chi(g, g')$ is a primitive n -th root of unity for all $g, g' \in G$ of finite order, with n chosen suitably.

Apart from this restriction on the choice of $\chi(g, g')$ any combination of values can occur. In particular let ζ be an n -th root of unity. Then there are examples of groups G and elements $g_1, \dots, g_m \in G$ with $\chi(g_i, g_j) = \zeta$ for all $i \neq j$. Take for example $G = C_n \oplus \dots \oplus C_n = \mathbf{Z}g_1 \oplus \dots \oplus \mathbf{Z}g_m$. Then a bicharacter χ is defined as the group homomorphism $\chi : G \otimes G = \bigoplus_{i,j=1}^m \mathbf{Z}g_i \otimes \mathbf{Z}g_j = \bigoplus_{i,j=1}^m \mathbf{Z}(g_i \otimes g_j) \rightarrow k^*$, where the images $\chi(g_i \otimes g_j) = \chi(g_i, g_j)$ are chosen arbitrarily among the n -th roots of unity (including 1).

If (g_1, \dots, g_n) is a ζ -family with at least one element g_i of finite order, then $\chi(g_i, g_j)\chi(g_j, g_i) = \zeta^2$ implies that ζ is a root of unity.

EXAMPLE 2.3. For $g \in G$ we define $|g| := \chi(g, g)$. If g has order m then $|g|$ is a primitive n -th root of unity where n divides the order m of g . Furthermore (g, \dots, g) is a $|g|$ -family in G . Observe also that any pair $(0, g)$ or $(g, 0)$ is a 1-family and a (-1) -family.

The map $\rho : S_n \times X \rightarrow k^*$ reduces to a well known map in the following situation. If $(g_1, \dots, g_n) = (g, \dots, g)$, $|g| = 1$, and $\zeta = -1$ then $\rho(\sigma, (g, \dots, g)) = \text{sgn}(\sigma)$ is the sign of the permutation.

Now we turn to the category of interest for us. By [FM] Remark 3.4 the category \mathcal{M}^{kG} of G -graded k -modules (or the category of kG -comodules) is a braided monoidal category with the tensor product

$$(X \otimes Y)_g = \bigoplus_{h \in G} X_h \otimes Y_{g-h}$$

and the braiding

$$\tau : X \otimes Y \ni x \otimes y \mapsto \chi(\deg(x), \deg(y))y \otimes x \in Y \otimes X$$

where x and y are homogeneous elements of degree $\deg(x), \deg(y) \in G$.

Algebras, coalgebras, bialgebras, and Hopf algebras in the braided monoidal category \mathcal{M}^{kG} will be called (G, χ) -algebras, (G, χ) -coalgebras, (G, χ) -bialgebras, resp. (G, χ) -Hopf algebras.

Let A, B be (G, χ) -algebras in the category \mathcal{M}^{kG} . Then $A \otimes B$ is also a (G, χ) -algebra in \mathcal{M}^{kG} with the multiplication $A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$ (see for example [M94b]).

DEFINITION 2.4. Let A be a (G, χ) -algebra (associative with unit) in \mathcal{M}^{kG} . We define

$$[x_1, \dots, x_n] := \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n))x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(n)}$$

for all $\zeta \in k^*$, all ζ -families (g_1, \dots, g_n) , and all $x_i \in A_{g_i}$.

A special example is $\zeta = -1$, $n = 2$ and $\chi(g_1, g_2) = \chi(g_2, g_1) = 1$. Then

$$[x_1, x_2] = x_1x_2 + \rho((2, 1), (g_1, g_2))x_2x_1 = x_1x_2 - x_2x_1.$$

If $\zeta = -1$, $n = 2$ and $\chi(g_1, g_2) = \chi(g_2, g_1) = -1$, then

$$[x_1, x_2] = x_1x_2 + \rho((2, 1), (g_1, g_2))x_2x_1 = x_1x_2 + x_2x_1.$$

Hence we obtain Lie algebras resp. Lie super algebras as a special case of (2.4). Similarly if $\zeta = -1$, $n = 2$ and (g_1, g_2) is a ζ -family then $\chi(g_1, g_2) = \chi(g_2, g_1)^{-1}$ and

$$[x_1, x_2] = x_1x_2 - \chi(g_1, g_2)x_2x_1,$$

rendering Lie color algebras as another instance of (2.4). For Lie color algebras the assumption $\chi(g_2, g_1) = \chi(g_1, g_2)^{-1}$ for all $g_1, g_2 \in G$ leads to a totally defined multiplication $[\cdot, \cdot] : A \times A \rightarrow A$ since $[x_1, x_2]$ is defined for any pair $x_1, x_2 \in A$ of homogeneous elements. A special new operation is the following bracket product. If $g \in G$ satisfies

$\chi(g, g) = \zeta \neq 1$ then we have the product $[\dots] : \otimes^n A_g \longrightarrow A$

$$[x_1, \dots, x_n] = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

THEOREM 2.5 ((Anti-)Symmetry). *Let $\zeta \in k^*$ and (g_1, \dots, g_n) be a ζ -family. Let $x_i \in A_{g_i}$ and $\sigma \in S_n$. Then*

$$[x_1, \dots, x_n] = \rho(\sigma, (g_1, \dots, g_n))[x_{\sigma(1)}, \dots, x_{\sigma(n)}].$$

Proof. We have

$$\begin{aligned} \rho(\sigma, (g_1, \dots, g_n))[x_{\sigma(1)}, \dots, x_{\sigma(n)}] &= \\ &= \sum_{\tau \in S_n} \rho(\sigma, (g_1, \dots, g_n))\rho(\tau, (g_{\sigma(1)}, \dots, g_{\sigma(n)}))x_{\sigma\tau(1)} \cdots x_{\sigma\tau(n)} \\ &= \sum_{\tau \in S_n} \rho(\sigma\tau, (g_1, \dots, g_n))x_{\sigma\tau(1)} \cdots x_{\sigma\tau(n)}. \end{aligned}$$

In the case of Lie algebras this amounts to

$$[x_1, x_2] = -[x_2, x_1];$$

in the case of Lie super algebras this is

$$[x_1, x_2] = [x_2, x_1]$$

(for $\zeta = -1$, $n = 2$ and $\chi(g_1, g_2) = \chi(g_2, g_1) = -1$); and in the case of Lie color algebras (see [FM] 3.11) this is

$$[x_1, x_2] = -\chi(g_1, g_2)[x_2, x_1].$$

In the following theorem let $(i \dots 1) = (i, i+1, \dots, i-1, i+1, \dots, n+1)$ denote a cycle in S_{n+1} .

THEOREM 2.6 (Jacobi identities).

1. *Let (g_1, \dots, g_{n+1}) be a ζ -family with ζ a primitive n -th root of unity. Then*

$$\sum_{i=1}^{n+1} \rho((i \dots 1), (g_1, \dots, g_{n+1}))[x_i, [x_1, \dots, \hat{x}_i, \dots, x_{n+1}]] = 0$$

for all $x_i \in A_{g_i}$.

2. *Let (g_1, \dots, g_n) be a ζ -family with ζ a primitive n -th root of unity and let $h \in G$ such that all (h, g_i) are (-1) -families. Then*

$$[x, [y_1, y_2, \dots, y_n]] = \sum_{i=1}^n \left(\prod_{j=1}^{i-1} \chi(g_j, h) \right) [y_1, \dots, [x, y_i], \dots, y_n]$$

for all $x \in A_h$ and $y_i \in A_{g_i}$.

Proof. (1) Let $\tau \in S_n$. Construct $\bar{\tau} \in S_{n+1}$ by $\bar{\tau}(1) = 1$ and $\bar{\tau}(j) = \tau(j-1) + 1$. This defines a bijection between S_n and the set of all $\sigma \in S_{n+1}$ with $\sigma(1) = 1$. Using $h_i := g_{i+1}$ we get

$$\begin{aligned} \rho(\tau, (g_2, \dots, g_{n+1})) &= \rho(\tau, (h_1, \dots, h_n)) \\ &= \prod_{1 \leq i < j \leq n, \tau(i) > \tau(j)} (\zeta^{-1} \chi(h_{\tau(j)}, h_{\tau(i)})) \end{aligned}$$

$$\begin{aligned}
&= \prod_{2 \leq i < j \leq n+1, \tau(i-1) > \tau(j-1)} (\zeta^{-1} \chi(g_{\tau(j-1)+1}, g_{\tau(i-1)+1})) \\
&= \prod_{1 \leq i < j \leq n+1, \bar{\tau}(i) > \bar{\tau}(j)} (\zeta^{-1} \chi(g_{\bar{\tau}(j)}, g_{\bar{\tau}(i)})) \\
&= \rho(\bar{\tau}, (g_1, \dots, g_{n+1}))
\end{aligned}$$

So we get for all $i \in \{1, \dots, n+1\}$

$$\rho(\tau, (g_1, \dots, g_{i-1}, \hat{g}_i, g_{i+1}, \dots, g_{n+1})) = \rho(\bar{\tau}, (g_i, g_1, \dots, g_{i-1}, \hat{g}_i, g_{i+1}, \dots, g_{n+1})).$$

Now let $\sigma = \sigma_i \in S_{n+1}$ with $i := \sigma(1)$. Let $\bar{\tau} := (1 \dots i)\sigma_i$. Then $\bar{\tau}(1) = 1$, so $\bar{\tau}$ comes from some $\tau \in S_n$. Furthermore $\sigma_i = (i \dots 1)\bar{\tau}$. By Lemma 2.2 we get

$$\begin{aligned}
\rho(\sigma_i, (g_1, \dots, g_{n+1})) &= \rho(\bar{\tau}, (g_i, g_1, \dots, \hat{g}_i, \dots, g_{n+1}))\rho((i \dots 1), (g_1, \dots, g_{n+1})) \\
&= \rho(\tau, (g_1, \dots, \hat{g}_i, \dots, g_{n+1}))\rho((i \dots 1), (g_1, \dots, g_{n+1})).
\end{aligned}$$

Given $\tau \in S_n$ we define $\tilde{\tau}$ by $\tilde{\tau}(n+1) := n+1$ and $\tilde{\tau}(j) := \tau(j)$ else. This defines a bijection between S_n and the set of all $\sigma \in S_{n+1}$ with $\sigma(n+1) = n+1$. Then we get

$$\rho(\tau, (g_1, \dots, g_n)) = \rho(\tilde{\tau}, (g_1, \dots, g_{n+1}))$$

and

$$\rho(\tau, (g_1, \dots, \hat{g}_i, \dots, g_{n+1})) = \rho(\tilde{\tau}, (g_1, \dots, \hat{g}_i, \dots, g_{n+1}, g_i)).$$

Now let $\sigma = \sigma^i \in S_{n+1}$ with $i := \sigma(n+1)$. Let $\tilde{\tau} := (n+1 \dots i)\sigma^i$. Then $\tilde{\tau}(n+1) = n+1$, so $\tilde{\tau}$ comes from some $\tau \in S_n$. Furthermore $\sigma^i = (i \dots n+1)\tilde{\tau}$ and we get

$$\rho(\sigma^i, (g_1, \dots, g_{n+1})) = \rho(\tau, (g_1, \dots, \hat{g}_i, \dots, g_{n+1}))\rho((i \dots n+1), (g_1, \dots, g_{n+1})).$$

To prove the Jacobi identity we observe that since ζ is a primitive n -th root of unity

$$\begin{aligned}
\chi(g_i, g_1 + \dots + \hat{g}_i + \dots + g_{n+1})\chi(g_1 + \dots + \hat{g}_i + \dots + g_{n+1}, g_i) \\
= \prod_{j \neq i} \chi(g_i, g_j)\chi(g_j, g_i) = \zeta^{2n} = 1,
\end{aligned}$$

hence we have a (-1) -family $(g_i, g_1 + \dots + \hat{g}_i + \dots + g_{n+1})$ and

$$\begin{aligned}
[x_i, [x_1, \dots, \hat{x}_i, \dots, x_{n+1}]] &= \\
&= x_i[x_1, \dots, \hat{x}_i, \dots, x_{n+1}] - \left(\prod_{j \neq i} \chi(g_i, g_j) \right) [x_1, \dots, \hat{x}_i, \dots, x_{n+1}]x_i.
\end{aligned}$$

Furthermore we use $\zeta^n = 1$ to get

$$\begin{aligned}
\rho((i \dots 1), (g_1, \dots, g_{n+1})) \prod_{j \neq i} \chi(g_i, g_j) &= \\
&= \left(\prod_{j < i} (\zeta^{-1} \chi(g_j, g_i)) \right) \left(\prod_{j \neq i} (\zeta^{-1} \chi(g_i, g_j)) \right) \\
&= \prod_{j > i} (\zeta^{-1} \chi(g_i, g_j)) = \rho((i \dots n+1), (g_1, \dots, g_{n+1})).
\end{aligned}$$

With these notations we can now evaluate

$$\sum_{i=1}^{n+1} \rho((i \dots 1), (g_1, \dots, g_{n+1})) [x_i, [x_1, \dots, \hat{x}_i, \dots, x_{n+1}]] =$$

$$\begin{aligned}
&= \sum_{i=1}^{n+1} \rho((i \dots 1), (g_1, \dots, g_{n+1})) x_i [x_1, \dots, \hat{x}_i, \dots, x_{n+1}] \\
&\quad - \rho((i \dots 1), (g_1, \dots, g_{n+1})) \left(\prod_{j \neq i} \chi(g_i, g_j) \right) [x_1, \dots, \hat{x}_i, \dots, x_{n+1}] x_i \\
&= \sum_{i=1}^{n+1} \sum_{\tau \in S_n} \rho((i \dots 1), (g_1, \dots, g_{n+1})) \rho(\tau, (g_1, \dots, \hat{g}_i, \dots, g_{n+1})) x_{\sigma_i(1)} \cdots x_{\sigma_i(n+1)} \\
&\quad - \rho((i \dots n+1), (g_1, \dots, g_{n+1})) \rho(\tau, (g_1, \dots, \hat{g}_i, \dots, g_{n+1})) x_{\sigma^i(1)} \cdots x_{\sigma^i(n+1)} \\
&= \sum_{\sigma \in S_{n+1}} \rho(\sigma, (g_1, \dots, g_{n+1})) x_{\sigma(1)} \cdots x_{\sigma(n+1)} - \rho(\sigma, (g_1, \dots, g_{n+1})) x_{\sigma(1)} \cdots x_{\sigma(n+1)} \\
&= 0.
\end{aligned}$$

(2) Since $\chi(h, g_i)\chi(g_i, h) = (-1)^2$ and $\chi(g_i, g_j)\chi(g_j, g_i) = \zeta^2$ we have $\chi(h, g_1 + \dots + g_n)\chi(g_1 + \dots + g_n, h) = 1$ and $\chi(g_i, h + g_j)\chi(h + g_j, g_i) = \zeta^2$ so that all terms are defined.

Let $\sigma \in S_n$ with $\sigma(j) = i$. Then we have

$$\begin{aligned}
&\rho(\sigma, (g_1, \dots, g_{i-1}, h + g_i, g_{i+1}, \dots, g_n)) = \\
&= \left(\prod_{k < l, \sigma(k) > \sigma(l)} (\zeta^{-1} \chi(g_{\sigma(l)}, g_{\sigma(k)})) \right) \left(\prod_{j < l, i > \sigma(l)} \chi(g_{\sigma(l)}, h) \right) \left(\prod_{k < j, \sigma(k) > i} \chi(h, g_{\sigma(k)}) \right) \\
&= \rho(\sigma, (g_1, \dots, g_n)) \left(\prod_{\sigma^{-1}(i) < l, i > \sigma(l)} \chi(g_{\sigma(l)}, h) \right) \left(\prod_{k < \sigma^{-1}(i), \sigma(k) > i} \chi(h, g_{\sigma(k)}) \right).
\end{aligned}$$

We abbreviate $z_k := u_k := y_k$ for $k \neq i$, $z_i := xy_i$, and $u_i := y_i x$. Then we get

$$\begin{aligned}
&\sum_{i=1}^n \left(\prod_{r=1}^{i-1} \chi(h, g_r) \right) [y_1, \dots, [x, y_i], \dots, y_n] \\
&= \sum_{i=1}^n \left(\prod_{r=1}^{i-1} \chi(h, g_r) \right) [y_1, \dots, xy_i - \chi(h, g_i) y_i x, \dots, y_n] \\
&= \sum_{i=1}^n \sum_{\sigma \in S_n} \left(\left(\prod_{r=1}^{i-1} \chi(h, g_r) \right) \rho(\sigma, (g_1, \dots, h + g_i, \dots, g_n)) z_{\sigma(1)} \cdots z_{\sigma(n)} \right. \\
&\quad \left. - \left(\prod_{r=1}^{i-1} \chi(h, g_r) \right) \chi(h, g_i) \rho(\sigma, (g_1, \dots, h + g_i, \dots, g_n)) u_{\sigma(1)} \cdots u_{\sigma(n)} \right) \\
&= \sum_{\sigma \in S_n} \sum_{j=1}^n \left(\prod_{r=1}^{\sigma(j)-1} \chi(h, g_r) \right) \rho(\sigma, (g_1, \dots, g_n)) \left(\prod_{j < l, \sigma(j) > \sigma(l)} \chi(g_{\sigma(l)}, h) \right) \cdot \\
&\quad \cdot \left(\prod_{l < j, \sigma(l) > \sigma(j)} \chi(h, g_{\sigma(l)}) \right) y_{\sigma(1)} \cdots xy_{\sigma(j)} \cdots y_{\sigma(n)} \\
&\quad - \sum_{\sigma \in S_n} \sum_{j=1}^n \left(\prod_{r=1}^{\sigma(j)-1} \chi(h, g_r) \right) \chi(h, g_{\sigma(j)}) \rho(\sigma, (g_1, \dots, g_n)) \left(\prod_{j < l, \sigma(j) > \sigma(l)} \chi(g_{\sigma(l)}, h) \right) \cdot \\
&\quad \cdot \left(\prod_{l < j, \sigma(l) > \sigma(j)} \chi(h, g_{\sigma(l)}) \right) y_{\sigma(1)} \cdots y_{\sigma(j)} x \cdots y_{\sigma(n)}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) x y_{\sigma(1)} \cdots y_{\sigma(n)} \\
&\quad - \sum_{\sigma \in S_n} \left(\prod_{r=1}^n \chi(h, g_r) \right) \rho(\sigma, (g_1, \dots, g_n)) y_{\sigma(1)} \cdots y_{\sigma(n)} x \\
&= [x, [y_1, \dots, y_n]].
\end{aligned}$$

if we can show that the coefficients reduce appropriately. For the first term ($j = 1$) of the first sum and the last term ($j = n$) of the second sum this is easy to see. The other terms cancel for all $\sigma \in S_n$ and all j . In fact, let $q := \sigma(j)$ and $p = \sigma(j-1)$. We have to show

$$\begin{aligned}
&\left(\prod_{r=1}^{\sigma(j)-1} \chi(h, g_r) \right) \left(\prod_{j < l, \sigma(j) > \sigma(l)} \chi(g_{\sigma(l)}, h) \right) \left(\prod_{l < j, \sigma(l) > \sigma(j)} \chi(h, g_{\sigma(l)}) \right) \\
&= \left(\prod_{r=1}^{\sigma(j-1)-1} \chi(h, g_r) \right) \chi(h, g_{\sigma(j-1)}) \left(\prod_{j-1 < l, \sigma(j-1) > \sigma(l)} \chi(g_{\sigma(l)}, h) \right) \\
&\quad \cdot \left(\prod_{l < j-1, \sigma(l) > \sigma(j-1)} \chi(h, g_{\sigma(l)}) \right)
\end{aligned}$$

We change parameters by $\tau = \sigma^{-1}$ with $\tau(p) + 1 = \tau(q)$ and have to show

$$\begin{aligned}
&\left(\prod_{r=1}^{q-1} \chi(h, g_r) \right) \left(\prod_{q > l, \tau(q) < \tau(l)} \chi(g_l, h) \right) \left(\prod_{l > q, \tau(l) < \tau(q)} \chi(h, g_l) \right) \\
&= \left(\prod_{r=1}^p \chi(h, g_r) \right) \left(\prod_{p > l, \tau(p) < \tau(l)} \chi(g_l, h) \right) \left(\prod_{l > p, \tau(l) < \tau(p)} \chi(h, g_l) \right).
\end{aligned}$$

This can be easily checked if one considers the cases $p < q$ and $p > q$ separately.

3. Primitive elements. We come to the main technical theorem of this paper which has applications to primitive elements in Hopf algebras.

THEOREM 3.1. *Let A be a (G, χ) -algebra in \mathcal{M}^{kG} . Then the following hold in $A \otimes A$*

$$[x_1 \otimes 1 + 1 \otimes x_1, \dots, x_n \otimes 1 + 1 \otimes x_n] = [x_1, \dots, x_n] \otimes 1 + 1 \otimes [x_1, \dots, x_n]$$

for all primitive n -th roots of unity ζ , all ζ -families (g_1, \dots, g_n) , and all $x_i \in A_{g_i}$.

Proof. We have to evaluate

$$\begin{aligned}
&[x_1 \otimes 1 + 1 \otimes x_1, \dots, x_n \otimes 1 + 1 \otimes x_n] \\
&= \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) (x_{\sigma(1)} \otimes 1 + 1 \otimes x_{\sigma(1)}) \cdots (x_{\sigma(n)} \otimes 1 + 1 \otimes x_{\sigma(n)}).
\end{aligned}$$

Observe that

$$\begin{aligned}
(x_i \otimes 1)(x_j \otimes 1) &= (x_i x_j \otimes 1), \\
(x_i \otimes 1)(1 \otimes x_j) &= (x_i \otimes x_j), \\
(1 \otimes x_i)(1 \otimes x_j) &= (1 \otimes x_i x_j), \\
(1 \otimes x_i)(x_j \otimes 1) &= \chi(g_i, g_j)(x_j \otimes x_i).
\end{aligned}$$

We collect all terms of the form $c_{\sigma,i} \cdot x_{\sigma(1)} \dots x_{\sigma(i)} \otimes x_{\sigma(i+1)} \dots x_{\sigma(n)}$ with $\sigma \in S_n$ and want to show that they are zero for all $1 < i < n$.

By Theorem 2.5 $[x_1 \otimes 1 + 1 \otimes x_1, \dots, x_n \otimes 1 + 1 \otimes x_n]$ and $[x_{\sigma(1)} \otimes 1 + 1 \otimes x_{\sigma(1)}, \dots, x_{\sigma(n)} \otimes 1 + 1 \otimes x_{\sigma(n)}]$ have (generically) proportional terms, in particular $c_{\sigma,i} \cdot x_{\sigma(1)} \dots x_{\sigma(i)} \otimes x_{\sigma(i+1)} \dots x_{\sigma(n)} = \rho(\sigma, (g_1, \dots, g_n)) c_{1,i} \cdot x_{\sigma(1)} \dots x_{\sigma(i)} \otimes x_{\sigma(i+1)} \dots x_{\sigma(n)}$. So we may just consider the case $\sigma = 1$ or the term $c_{1,i} \cdot x_1 \dots x_i \otimes x_{i+1} \dots x_n$.

The term $c_{1,n} \cdot x_1 \dots x_n \otimes 1$ occurs only in the product $(x_1 \otimes 1 + 1 \otimes x_1)(x_2 \otimes 1 + 1 \otimes x_2) \dots (x_n \otimes 1 + 1 \otimes x_n)$ with the factor $c_{1,n} = 1$. The same holds for the term $c_{1,1} \cdot 1 \otimes x_1 \dots x_n$. Now we consider exclusively terms $c_{1,i} \cdot x_1 \dots x_i \otimes x_{i+1} \dots x_n$ with $0 < i < n$.

We study which terms of the expansion of

$$\sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) (x_{\sigma(1)} \otimes 1 + 1 \otimes x_{\sigma(1)}) \dots (x_{\sigma(n)} \otimes 1 + 1 \otimes x_{\sigma(n)})$$

contribute to $c_{1,i} \cdot x_1 \dots x_i \otimes x_{i+1} \dots x_n$. This will be those products of factors $x_1 \otimes 1, \dots, x_i \otimes 1$ and $1 \otimes x_{i+1}, \dots, 1 \otimes x_n$ where the terms $x_1 \otimes 1, \dots, x_i \otimes 1$ occur in the given natural order possibly interrupted by factors from the second set and similarly the terms $1 \otimes x_{i+1}, \dots, 1 \otimes x_n$ occur in the given natural order possibly interrupted by factors from the first set. Such a product will occur in the expansion of

$$\rho(\sigma, (g_1, \dots, g_n)) (x_{\sigma(1)} \otimes 1 + 1 \otimes x_{\sigma(1)}) \dots (x_{\sigma(n)} \otimes 1 + 1 \otimes x_{\sigma(n)})$$

whenever $\sigma \in S_n$ is a shuffle of $\{1, \dots, i\}$ with $\{i+1, \dots, n\}$, i.e. if $1 \leq j < k \leq i$ or $i+1 \leq j < k \leq n$ then $\sigma^{-1}(j) < \sigma^{-1}(k)$.

To evaluate such a product we have to interchange factors according to the rule given above, namely,

$$(1 \otimes x_m)(x_l \otimes 1) = \chi(g_m, g_l)(x_l \otimes x_m) = \chi(g_m, g_l)(x_l \otimes 1)(1 \otimes x_m) \text{ for } 1 \leq l \leq i < m \leq n.$$

This rule has to be applied for every pair in the product (and in σ) in reverse position $j < k$ and $m := \sigma(j) > \sigma(k) = l$ thus producing a factor $\chi(g_{\sigma(j)}, g_{\sigma(k)})$. So the total contribution of

$$\rho(\sigma, (g_1, \dots, g_n)) (x_{\sigma(1)} \otimes 1 + 1 \otimes x_{\sigma(1)}) \dots (x_{\sigma(n)} \otimes 1 + 1 \otimes x_{\sigma(n)})$$

to $c_{1,i} \cdot x_1 \dots x_i \otimes x_{i+1} \dots x_n$ is

$$\begin{aligned} \rho(\sigma, (g_1, \dots, g_n)) \prod_{j < k, \sigma(j) > \sigma(k)} \chi(g_{\sigma(j)}, g_{\sigma(k)}) \\ = \prod_{j < k, \sigma(j) > \sigma(k)} (\zeta^{-1} \chi(g_{\sigma(k)}, g_{\sigma(j)})) \chi(g_{\sigma(j)}, g_{\sigma(k)}) = \zeta^t \end{aligned}$$

where t is the number of pairs in reverse position in σ . Here we have used that (g_1, \dots, g_n) is a ζ -family.

To determine the number t of pairs in reverse position for a fixed σ we observe that the ordering of $1, \dots, i$ and of $i+1, \dots, n$ must be preserved in the product. We count how many steps the factor $1 \otimes x_{i+1}$ has been moved to the left or how many terms from $x_1 \otimes 1, \dots, x_i \otimes 1$ in the product are to the right of $1 \otimes x_{i+1}$ and call this number λ_1 . Observe that $0 \leq \lambda_1 \leq i$. Similarly λ_2 denotes the number of terms from $x_1 \otimes 1, \dots, x_i \otimes 1$

in the product that are to the right of $1 \otimes x_{i+2}$. We have $0 \leq \lambda_2 \leq \lambda_1$. In a similar way we continue to define the numbers λ_j with $0 \leq \lambda_{n-i} \leq \dots \leq \lambda_2 \leq \lambda_1 \leq i$. The evaluation of the selected product then gives a term

$$\zeta^{\lambda_1 + \dots + \lambda_{n-i}} x_1 \dots x_i \otimes x_{i+1} \dots x_n.$$

To get the number of terms $\zeta^t x_1 \dots x_i \otimes x_{i+1} \dots x_n$ in $[x_1 \otimes 1 + 1 \otimes x_1, \dots, x_n \otimes 1 + 1 \otimes x_n]$ we have to count all possibilities to represent $t = \lambda_1 + \dots + \lambda_{n-i}$ with $0 \leq \lambda_{n-i} \leq \dots \leq \lambda_2 \leq \lambda_1 \leq i$ or the number $p(i, n-i, t)$ of partitions of t into at most $n-i$ parts each $\leq i$. Thus we can now determine the factor $c_{1,i}$ for $c_{1,i} \cdot x_1 \dots x_i \otimes x_{i+1} \dots x_n$ in the expansion of $[x_1 \otimes 1 + 1 \otimes x_1, \dots, x_n \otimes 1 + 1 \otimes x_n]$ as

$$c_{1,i} = \sum_{t \geq 0} p(i, n-i, t) \zeta^t.$$

By a theorem of Sylvester ([A] Theorem 3.1) we have

$$\sum_{t \geq 0} p(i, n-i, t) q^t = \frac{(1 - q^n)(1 - q^{n-1}) \dots (1 - q^{n-i+1})}{(1 - q^i)(1 - q^{i-1}) \dots (1 - q)}$$

hence $c_{1,i} = 0$ for $1 < i < n$ since ζ is a *primitive* n -th root of unity (see also [T] p. 2632).

So we have shown

$$\begin{aligned} [x_1 \otimes 1 + 1 \otimes x_1, \dots, x_n \otimes 1 + 1 \otimes x_n] &= \\ &= \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) x_{\sigma(1)} \dots x_{\sigma(n)} \otimes 1 + 1 \otimes x_{\sigma(1)} \dots x_{\sigma(n)} \\ &= [x_1, \dots, x_n] \otimes 1 + 1 \otimes [x_1, \dots, x_n]. \end{aligned}$$

Now we will study Hopf algebras in the category \mathcal{M}^{kG} and their primitive ⁽¹⁾ elements. An element $x \in H$ is *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. The set of primitive elements of an ordinary Hopf algebra forms a Lie algebra. This is not true for Hopf algebras in \mathcal{M}^{kG} .

Let $P(H)$ denote the set of primitive elements of H and $P_g(H)$ the set of primitive elements of degree g in H . Then

$$P(H) = \bigoplus_{g \in G} P_g(H)$$

since for any $x = \sum_{g \in G} x_g$ we have

$$\sum_{g \in G} \Delta(x_g) = \Delta(x) = x \otimes 1 + 1 \otimes x = \left(\sum_{g \in G} x_g \right) \otimes 1 + 1 \otimes \left(\sum_{g \in G} x_g \right),$$

so by comparing homogeneous terms we get $\Delta(x_g) = x_g \otimes 1 + 1 \otimes x_g$. Thus the homogeneous components x_g of x are again primitive hence in $P_g(H)$ which shows $P(H) \subseteq \bigoplus_{g \in G} P_g(H)$. The converse inclusion is trivial.

THEOREM 3.2. *Let H be a (G, χ) -Hopf algebra in \mathcal{M}^{kG} . Then for all primitive n -th roots of unity $\zeta \neq 1$ and all ζ -families (g_1, \dots, g_n) the following is a linear map*

$$[\dots] : P_{g_1}(H) \otimes \dots \otimes P_{g_n}(H) \longrightarrow P_{g_1 + \dots + g_n}(H)$$

⁽¹⁾ This notion of primitivity is not related to the notion of primitivity for roots of unity.

$$[x_1, \dots, x_n] := \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Proof. It is clear that the degree of each product $x_{\sigma(1)} \cdots x_{\sigma(n)}$ is $g_1 + \dots + g_n$. So we only have to show that $[x_1, \dots, x_n]$ is primitive. But that is a consequence of Theorem 3.1.

Observe that we have special multiplications

$$[\dots] : P_g(H) \otimes \dots \otimes P_g(H) \longrightarrow P_{ng}(H)$$

for all $g \in G$ with $|g| \neq 1$ a primitive n -th root of unity.

DEFINITION 3.3. Let A be an algebra in \mathcal{M}^{kG} . A *derivation* from A to A of degree $g \in G$ is a family of linear maps $(d_h : A_h \longrightarrow A_{h+g} | h \in G)$ such that

$$d(ab) = d(a)b + \chi(g, h)ad(b)$$

for all $a \in A_h, b \in A_{h'}$, all $h, h' \in G$.

It is clear that all derivations from A to A of all degrees form an object $\text{Der}(A)$ in \mathcal{M}^{kG} and that there is an operation $\text{Der}(A) \otimes A \longrightarrow A$.

COROLLARY 3.4. *Der(A) is a (G, χ) -Lie algebra.*

Proof. Let m denote the multiplication of A . An endomorphism $x : A \longrightarrow A$ of degree $g \in G$, i.e. $x_h : A_h \longrightarrow A_{h+g}$ for all $h \in G$ is a derivation iff $m(x \otimes 1 + 1 \otimes x) = xm$ where $(x \otimes y)(a \otimes b) = \chi(\deg(y), \deg(a))x(a) \otimes y(b)$ for homogeneous elements a and b in A .

To show that $\text{Der}(A)$ is a Lie algebra it suffices to show that it is closed under Lie multiplication since it is a subobject of $\text{End}(A)$, the inner endomorphism object of A which is known to be an algebra in the category \mathcal{M}^{kG} . Let ζ be a primitive n -th root of unity and let (g_1, \dots, g_n) be a ζ -family. Then

$$\begin{aligned} & m([x_1, \dots, x_n] \otimes 1 + 1 \otimes [x_1, \dots, x_n]) \\ &= m[x_1 \otimes 1 + 1 \otimes x_1, \dots, x_n \otimes 1 + 1 \otimes x_n] \\ &= m \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) (x_{\sigma(1)} \otimes 1 + 1 \otimes x_{\sigma(1)}) \cdots (x_{\sigma(n)} \otimes 1 + 1 \otimes x_{\sigma(n)}) \\ &= \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) m(x_{\sigma(1)} \otimes 1 + 1 \otimes x_{\sigma(1)}) \cdots (x_{\sigma(n)} \otimes 1 + 1 \otimes x_{\sigma(n)}) \\ &= \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) x_{\sigma(1)} \cdots x_{\sigma(n)} \cdot m \\ &= [x_1, \dots, x_n]m \end{aligned}$$

for all derivations x_1, \dots, x_n of degrees g_1, \dots, g_n respectively. Hence $[x_1, \dots, x_n]$ again is a derivation.

4. Lie algebras and universal enveloping algebras

DEFINITION 4.1. An object $P = \bigoplus P_g \in \mathcal{M}^{kG}$ together with operations

$$[\dots] : P_{g_1} \otimes \dots \otimes P_{g_n} \longrightarrow P_{g_1 + \dots + g_n}$$

for all $n \in \mathbb{N}$, all primitive n -th roots of unity ζ , and all ζ -families (g_1, \dots, g_n) , is called a (G, χ) -Lie algebra if the following identities hold:

1. for all primitive n -th roots of unity ζ , all ζ -families (g_1, \dots, g_n) , all $\sigma \in S_n$, and all $x_i \in P_{g_i}$

$$[x_1, \dots, x_n] = \rho(\sigma, (g_1, \dots, g_n))[x_{\sigma(1)}, \dots, x_{\sigma(n)}],$$

2. for all primitive n -th roots of unity ζ , all ζ -families (g_1, \dots, g_{n+1}) , and for all $x_i \in P_{g_i}$

$$\sum_{i=1}^{n+1} \rho((i \dots 1), (g_1, \dots, g_{n+1}))[x_i, [x_1, \dots, \hat{x}_i, \dots, x_{n+1}]] = 0,$$

3. for all primitive n -th roots of unity ζ , all ζ -families (g_1, \dots, g_n) , all $h \in G$ such that all (h, g_i) are (-1) -families, all $y_i \in P_{g_i}$, and all $x \in P_h$

$$[x, [y_1, y_2, \dots, y_n]] = \sum_{i=1}^n \left(\prod_{j=1}^{i-1} \chi(g_j, h) \right) [y_1, \dots, [x, y_i], \dots, y_n].$$

The (G, χ) -Lie algebras form a category in a straightforward way and the construction given in section 3 defines a functor $P : (G, \chi)\text{-Hopf} \rightarrow (G, \chi)\text{-Lie}$ as can be easily verified.

Theorems 3.2, 2.5, and 2.6 show that this notion of generalized Lie algebras passes its test on the set of primitive elements of a Hopf algebra:

COROLLARY 4.2. *Let H be a (G, χ) -Hopf algebra in \mathcal{M}^{kG} . Then the set of primitive elements $P(H)$ forms a (G, χ) -Lie algebra.*

LEMMA 4.3. *Let A be a (G, χ) -algebra in \mathcal{M}^{kG} . Then A carries the structure of a (G, χ) -Lie algebra A^L with the operations*

$$[x_1, \dots, x_n] := \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(n)}$$

for all (roots of unity) $\zeta \in k^*$, all ζ -families (g_1, \dots, g_n) , and all $x_i \in A_{g_i}$.

Proof. This is a rephrasing of Theorems 2.5 and 2.6.

This lemma defines a functor $(G, \chi)\text{-Alg} \ni A \mapsto A^L \in (G, \chi)\text{-Lie}$.

THEOREM 4.4. *Let P be a (G, χ) -Lie algebra. Then there is a universal associative enveloping algebra $U(P)$ in \mathcal{M}^{kG} .*

Proof. We define $U(P) := T(P)/I$ where $T(P)$ is the tensor algebra and $I := ([x_1, \dots, x_n] - \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(n)})$ is the ideal generated by all terms formed for all (roots of unity) $\zeta \in k^*$, all ζ -families (g_1, \dots, g_n) , and all $x_i \in P_{g_i}$.

The tensor algebra is constructed in \mathcal{M}^{kG} with the natural grading. Since the relations are G -homogeneous, the algebra $U(P)$ is also in \mathcal{M}^{kG} .

By the universal property of the tensor algebra any (G, χ) -Lie homomorphism $f : P \rightarrow A^L$ extends uniquely to a (G, χ) -algebra homomorphism $g : U(P) \rightarrow A$, so that $U(P)$ is the universal associative algebra generated by P .

COROLLARY 4.5. *The functor ${}^{-L} : (G, \chi)\text{-Alg} \rightarrow (G, \chi)\text{-Lie}$ has the left adjoint functor $U : (G, \chi)\text{-Lie} \rightarrow (G, \chi)\text{-Alg}$.*

Observe that the identities for (G, χ) -Lie algebras play no special role in the construction of the universal enveloping algebra. Its construction depends only on the given operations, which must be homogeneous so that the residue class algebra is G -graded again.

If we consider the set $P(H)$ of primitive elements of an arbitrary (G, χ) -Hopf algebra H then there are many homogeneous partially defined common operations in all $P(H)$. We used the most basic ones in our generalization of Lie algebras. We would like to know if the bracket-operations and identities given in the definition of a (G, χ) -Lie algebra are a generating set for all possible operations and relations for the G -graded module $P(H)$ for every Hopf algebra in \mathcal{M}^{kG} , every abelian group G , every bicharacter χ and every integral domain k . This should be true for the identities if all maps from (G, χ) -Lie algebras into their universal enveloping algebras are injective. To prove this, some kind of generalization of the Poincaré-Birkhoff-Witt theorem is needed.

Now we show how to construct Hopf algebras in the braided category of G -graded vector spaces using universal enveloping algebras..

THEOREM 4.6. *Let P be a (G, χ) -Lie algebra. Then the universal enveloping algebra $U(P)$ is a (G, χ) -Hopf algebra.*

Proof. We define a homomorphism $\delta : P \rightarrow (U(P) \otimes U(P))^L$ in \mathcal{M}^{kG} by $\delta(x) := \bar{x} \otimes 1 + 1 \otimes \bar{x}$ where \bar{x} is the canonical image of $x \in P$ in $U(P)$. Then we have

$$\begin{aligned} & [\delta(x_1), \dots, \delta(x_n)] \\ &= [\bar{x}_1 \otimes 1 + 1 \otimes \bar{x}_1, \dots, \bar{x}_n \otimes 1 + 1 \otimes \bar{x}_n] \\ &= [\bar{x}_1, \dots, \bar{x}_n] \otimes 1 + 1 \otimes [\bar{x}_1, \dots, \bar{x}_n] \quad (\text{by Theorem 3.1}) \\ &= [\bar{x}_1, \dots, \bar{x}_n] \otimes 1 + 1 \otimes [x_1, \dots, x_n] \quad (\text{by definition of } U(P)) \\ &= \delta([x_1, \dots, x_n]). \end{aligned}$$

Hence $\delta : P \rightarrow (U(P) \otimes U(P))^L$ is a (G, χ) -Lie homomorphism that factors uniquely through $\Delta : U(P) \rightarrow U(P) \otimes U(P)$, an algebra homomorphism. So for all $x \in P$ we have $\Delta(\bar{x}) = \bar{x} \otimes 1 + 1 \otimes \bar{x}$.

Since $(\Delta \otimes 1)\delta(x) = \bar{x} \otimes 1 \otimes 1 + 1 \otimes \bar{x} \otimes 1 + 1 \otimes 1 \otimes \bar{x} = (1 \otimes \Delta)\delta(x)$ and since $(\Delta \otimes 1)\delta = (1 \otimes \Delta)\delta$ is a (G, χ) -Lie homomorphism, it factors through a unique algebra homomorphism $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ so that $\Delta : U(P) \rightarrow U(P) \otimes U(P)$ is coassociative.

The counit $\varepsilon : U(P) \rightarrow k$ is defined by the zero morphism $0 : P \rightarrow k$. Thus $U(P)$ is a bialgebra.

The definition of the antipode is somewhat more complicated. We consider $U(P)^{op}$ the opposite algebra of $U(P)$ where the multiplication in $U(P)^{op}$ is defined by $x_2 \circ x_1 = \chi(g_2, g_1)x_1x_2$ for homogeneous elements $x_i \in U(P)_{g_i}$ hence $x_1x_2 = \chi(g_2, g_1)^{-1}x_2 \circ x_1$.

Define a homomorphism $S : P \rightarrow U(P)^{op}$ by $S(x) := -\bar{x}$. To show that this is a (G, χ) -Lie homomorphism let ζ be a primitive n -th root of unity, (g_1, \dots, g_n) be a ζ -family, and $x_i \in P_{g_i}$. We first consider the permutation $\tau \in S_n$ with $\tau(i) = n + 1 - i$. Then $\tau^2 = 1$ and

$$\rho(\tau, (g_1, \dots, g_n)) = \prod_{i < j} (\zeta^{-1} \chi(g_{n+1-j}, g_{n+1-i})) = \prod_{i < j} (\zeta \chi(g_j, g_i)^{-1})$$

$$= \zeta^{n(n-1)/2} \prod_{i < j} \chi(g_j, g_i)^{-1} = (-1)^{n-1} \prod_{i < j} \chi(g_j, g_i)^{-1}.$$

Using Lemma 2.2 we get for all $\sigma \in S_n$

$$\begin{aligned} & \rho(\sigma, (g_1, \dots, g_n)) \prod_{i < j} \chi(g_{\sigma(j)}, g_{\sigma(i)})^{-1} \\ &= (-1)^{n-1} \rho(\sigma, (g_1, \dots, g_n)) \rho(\tau, (g_{\sigma(1)}, \dots, g_{\sigma(n)})) \\ &= (-1)^{n-1} \rho(\sigma\tau, (g_1, \dots, g_n)). \end{aligned}$$

Thus we have

$$\begin{aligned} S([x_1, \dots, x_n]) &= -\overline{[x_1, \dots, x_n]} = -[\bar{x}_1, \dots, \bar{x}_n] \\ &= -\sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) \bar{x}_{\sigma(1)} \dots \bar{x}_{\sigma(n)} \\ &= -\sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) \left(\prod_{i < j} \chi(g_{\sigma(j)}, g_{\sigma(i)})^{-1} \right) \bar{x}_{\sigma(n)} \circ \dots \circ \bar{x}_{\sigma(1)} \\ &= (-1)^n \sum_{\sigma \in S_n} \rho(\sigma\tau, (g_1, \dots, g_n)) \bar{x}_{\sigma(n)} \circ \dots \circ \bar{x}_{\sigma(1)} \\ &= (-1)^n \sum_{\sigma \in S_n} \rho(\sigma, (g_1, \dots, g_n)) \bar{x}_{\sigma(1)} \circ \dots \circ \bar{x}_{\sigma(n)} \\ &= [-\bar{x}_1, \dots, -\bar{x}_n]^{op} = [S(x_1), \dots, S(x_n)]^{op}. \end{aligned}$$

Thus S can be extended to an algebra homomorphism $S : U(P) \rightarrow U(P)^{op}$.

We show now that S is the antipode for the bialgebra $U(P)$. Let $x \in P_h$, $x_i \in P_{g_i}$. We observe first that

$$S(\bar{x}\bar{x}_1 \dots \bar{x}_n) = S(\bar{x}) \circ S(\bar{x}_1 \dots \bar{x}_n) = \chi(h, g_1 + \dots + g_n) S(\bar{x}_1 \dots \bar{x}_n) S(\bar{x})$$

since S is a (G, χ) -algebra homomorphism. We now prove $\nabla(1 \otimes S)\Delta(a) = \varepsilon(a)$ for homogeneous terms $a = 1$, $a = \bar{x}$ resp. $a = \bar{x}\bar{x}_1 \dots \bar{x}_n$ in $U(P)$ by induction.

$$\nabla(1 \otimes S)\Delta(1) = 1S(1) = 1 = \varepsilon(1),$$

$$\nabla(1 \otimes S)\Delta(\bar{x}) = \bar{x} + S(\bar{x}) = 0 = \varepsilon(\bar{x}).$$

Since we can write $\Delta(a) = \sum_i a_{i,1} \otimes a_{i,2}$ with each of the terms $a_{i,j}$ homogeneous and $\deg(a_{i,1}) + \deg(a_{i,2}) = \deg(a)$ ($U(P)$ is a kG -comodule coalgebra) we get

$$\begin{aligned} \nabla(1 \otimes S)\Delta(xa) &= \nabla(1 \otimes S)(x \otimes 1 + 1 \otimes x) \left(\sum a_{i,1} \otimes a_{i,2} \right) \\ &= \nabla(1 \otimes S) \left(\sum xa_{i,1} \otimes a_{i,2} + \sum \chi(h, \deg(a_{i,1})) a_{i,1} \otimes xa_{i,2} \right) \\ &= \sum xa_{i,1} S(a_{i,2}) + \sum \chi(h, \deg(a_{i,1})) a_{i,1} S(xa_{i,2}) \\ &= 0 + \sum \chi(h, \deg(a_{i,1})) \chi(h, \deg(a_{i,2})) a_{i,1} S(a_{i,2}) S(x) \\ &= \chi(h, \deg(a)) \sum a_{i,1} S(a_{i,2}) S(x) \\ &= 0 = \varepsilon(xa). \end{aligned}$$

Analogously we get $\nabla(S \otimes 1)\Delta = \varepsilon$, so that S is an antipode and $U(P)$ is a (G, χ) -Hopf algebra.

We come to an interesting consequence for a universal enveloping algebra. There is a new multiplication on $U(P)$ defined by $a \cdot b := \chi(h, g)\chi(g, h)ab$ for a, b homogeneous with $\deg(a)=h, \deg(b)=g$. This algebra is $U(P)^{op\ op}$ which in general will be different from $U(P)$. The map $P \ni x \mapsto \bar{x} \in U(P)^{op\ op}$ is a Lie homomorphism (similar to the proof that S was a Lie homomorphism). Thus it induces a homomorphism $U(P) \rightarrow U(P)^{op\ op}$. Since $U(P)^{op\ op}$ is also generated by P it is easy to see, that it is also a universal enveloping algebra hence $U(P)$ and $U(P)^{op\ op}$ are isomorphic under the given homomorphism.

COROLLARY 4.7. *The functor $U : (G, \chi)\text{-Lie} \rightarrow (G, \chi)\text{-Hopf}$ is left adjoint to the functor $P : (G, \chi)\text{-Hopf} \rightarrow (G, \chi)\text{-Lie}$.*

Now we give some examples of braided monoidal categories of G -graded modules together with the associated (G, χ) -Lie structures induced by associative algebras. Let k be an algebraically closed field of characteristic zero.

EXAMPLE 4.8. (1) Let $G = \{0\}$. Then we have $\chi(0, 0) = 1$ and hence $\chi(0, 0)\chi(0, 0) = 1 = (-1)^2$ so that $(0, 0)$ is a (-1) -family. Thus we have a bracket multiplication

$$[\cdot, \cdot] : P_0 \otimes P_0 \rightarrow P_0,$$

which for associative algebras A is defined by

$$[x, y] = xy - yx.$$

This is the example of ordinary Lie algebras. The Lie identities of Definition 4.1 reduce to

$$[x_1, x_2] = \rho((2, 1), (0, 0))[x_2, x_1] = -[x_2, x_1],$$

$$\begin{aligned} \sum_{i=1}^3 \rho((i \dots 1), (0, 0, 0))[x_i, [x_1, \dots, \hat{x}_i, \dots, x_3]] &= \\ &= [x_1, [x_2, x_3]] + \rho((2, 1), (0, 0, 0))[x_2, [x_1, x_3]] + \rho((3, 2, 1), (0, 0, 0))[x_3, [x_1, x_2]] \\ &= [x_1, [x_2, x_3]] - [x_2, [x_1, x_3]] + [x_3, [x_1, x_2]] = 0, \end{aligned}$$

and

$$[x, [y_1, y_2]] = [[x, y_1], y_2] + \chi(0, 0)[y_1, [x, y_2]] = [[x, y_1], y_2] + [y_1, [x, y_2]].$$

(2) Let $G = \mathbf{Z}/2\mathbf{Z} = \{0, 1\}$ with the bicharacter $\chi(i, j) = (-1)^{ij}$. We have $\chi(0, i) = 1$ and hence $\chi(0, 1)\chi(1, 0) = 1 = (-1)^2$ so the following are (-1) -families: $(0, 0)$, $(0, 1)$, and $(1, 0)$ that induce bracket multiplications

$$\begin{aligned} [\cdot, \cdot] : P_0 \otimes P_0 &\rightarrow P_0, \\ [\cdot, \cdot] : P_0 \otimes P_1 &\rightarrow P_1, \\ [\cdot, \cdot] : P_1 \otimes P_0 &\rightarrow P_1, \end{aligned}$$

which for associative algebras A are defined by

$$[x, y] = xy - yx.$$

Furthermore we have $\chi(1, 1) = -1$ and hence $\chi(1, 1)\chi(1, 1) = 1 = (-1)^2$ so $(1, 1)$ is another (-1) -family that induces a bracket multiplication

$$[\cdot, \cdot] : P_1 \otimes P_1 \rightarrow P_0,$$

with

$$[x, y] = xy + \rho((2, 1), (1, 1))yx = xy + yx.$$

This is the example of Lie super algebras. The Lie identities of Definition 4.1 reduce to

$$\begin{aligned} [x_1, x_2] &= \rho((2, 1), (\deg(x_1), \deg(x_2)))[x_2, x_1] = -(-1)^{\deg(x_1)\deg(x_2)}[x_2, x_1], \\ [x_1, [x_2, x_3]] &- (-1)^{\deg(x_1)\deg(x_2)}[x_2, [x_1, x_3]] \\ &+ (-1)^{(\deg(x_1)+\deg(x_2))\deg(x_3)}[x_3, [x_1, x_2]] = 0, \\ [x, [y_1, y_2]] &= [[x, y_1], y_2] + (-1)^{\deg(x)\deg(y_1)}[y_1, [x, y_2]]. \end{aligned}$$

(3) Let G be an arbitrary abelian group with a bicharacter χ such that $\chi(g_1, g_2) = \chi(g_2, g_1)^{-1}$ for all $g_1, g_2 \in G$. Then we have $\chi(g_1, g_2)\chi(g_2, g_1) = 1 = (-1)^2$. This defines (-1) -families (g_1, g_2) together with bracket operations

$$\begin{aligned} [\cdot, \cdot] : P_{g_1} \otimes P_{g_2} &\longrightarrow P_{g_1+g_2}, \\ [x, y] &= xy + \rho((2, 1), (g_1, g_2))yx = xy - \chi(g_1, g_2)yx. \end{aligned}$$

This is the example of Lie color algebras for an abelian group. The Lie identities of Definition 4.1 reduce to

$$\begin{aligned} [x_1, x_2] &= \rho((2, 1), (\deg(x_1), \deg(x_2)))[x_2, x_1] = -\chi(\deg(x_1), \deg(x_2))[x_2, x_1], \\ [x_1, [x_2, x_3]] &- \chi(\deg(x_1), \deg(x_2))[x_2, [x_1, x_3]] \\ &+ \chi(\deg(x_1) + \deg(x_2), \deg(x_3))[x_3, [x_1, x_2]] = 0, \\ [x, [y_1, y_2]] &= [[x, y_1], y_2] + \chi(\deg(x), \deg(y_1))[y_1, [x, y_2]]. \end{aligned}$$

(4) Let $G = \mathbf{Z}/3\mathbf{Z} = \{0, 1, 2\}$ with the bicharacter $\chi(i, j) = \zeta^{ij}$ where ζ is a primitive 3rd root of unity. Then $\chi(0, i)\chi(i, 0) = 1 = (-1)^2$ so that we get (-1) -families $(0, i)$ with

$$\begin{aligned} [\cdot, \cdot] : P_0 \otimes P_i &\longrightarrow P_i, \\ [\cdot, \cdot] : P_i \otimes P_0 &\longrightarrow P_i, \\ [x, y] &= xy - yx. \end{aligned}$$

Furthermore we have $\chi(i, j)\chi(j, i) = (\zeta^{ij})^2$ with a primitive 3rd root of unity ζ^{ij} for all $i, j \neq 0$. This gives ζ -families $(1, 1, 1)$ and $(2, 2, 2)$ and no ζ^2 -family. The associated Lie structure is

$$[\cdot, \cdot] : P_i \otimes P_i \otimes P_i \longrightarrow P_0$$

with

$$[x_1, x_2, x_3] = \sum_{\sigma \in S_3} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$$

since $\rho(\sigma, (i, i, i)) = \prod(\zeta^{-1}\chi(i, i)) = 1$ for all $\sigma \in S_3$. The Lie identities of Definition 4.1 for these ternary brackets reduce to

$$\begin{aligned} [x_1, x_2, x_3] &= [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}], \\ [x_1, [x_2, x_3, x_4]] &+ [x_2, [x_1, x_3, x_4]] + [x_3, [x_1, x_2, x_4]] + [x_4, [x_1, x_2, x_3]] = 0, \end{aligned}$$

for $x_1, x_2, x_3, x_4 \in P_i$, $i = 1$ or $i = 2$. Here we also find a first example where the two Jacobi identities mean different things. The second Jacobi identity for $x \in P_0$ and $y_1, y_2, y_3 \in P_i$, $i = 1$ or $i = 2$ reduces to

$$[x, [y_1, y_2, y_3]] = [[x, y_1], y_2, y_3] + [y_1, [x, y_2], y_3] + [y_1, y_2, [x, y_3]].$$

Furthermore there are $(-\zeta)$ -families $(1, 1, 1, 1, 1, 1)$ and $(2, 2, 2, 2, 2, 2)$ for the primitive 6-th root of unity $-\zeta$ which give maps $\otimes^6 P_1 \rightarrow P_0$ and $\otimes^6 P_2 \rightarrow P_0$. The associated Lie multiplication of an associative algebra is

$$[x_1, \dots, x_6] = \sum_{\sigma \in S_6} \text{sgn}(\sigma) x_{\sigma(1)} \dots x_{\sigma(6)}$$

since $\rho(\sigma, (i, i, i, i, i, i)) = \prod(-\zeta^{-1}\chi(i, i)) = \text{sgn}(\sigma)$ for all $\sigma \in S_6$.

(5) Let $G = C_n \times \dots \times C_n$ (r -times) with generators t_1, \dots, t_r . Since later we will take the biproduct of a (G, χ) -Hopf algebra with kG , we will write G multiplicatively in this example. Let χ be given by $\chi(t_i, t_j) = \zeta$, a primitive n -th root of unity. Then (t_1, \dots, t_1) is a ζ -family. A (G, χ) -Lie algebra will have a bracket operation

$$[\cdot, \cdot] : \otimes^n P_{t_1} \longrightarrow P_0$$

with

$$[x_1, \dots, x_n] = \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)}.$$

(6) The last example of this kind comes with $G = \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ and generators g_1, g_2 each of order 3. Define χ by $\chi(g_1, g_1) := \zeta$, a primitive 3rd root of unity, $\chi(g_1, g_2) := \zeta^2$, $\chi(g_2, g_1) = 1$, and $\chi(g_2, g_2) = \zeta$. Then there are several ζ -families, among others (g_1, g_1, g_2) and (g_1, g_2, g_2) . They define brackets

$$\begin{aligned} [\cdot, \cdot] &: P_{g_1} \otimes P_{g_1} \otimes P_{g_2} \longrightarrow P_{2g_1+g_2}, \\ [\cdot, \cdot] &: P_{g_1} \otimes P_{g_2} \otimes P_{g_2} \longrightarrow P_{g_1+2g_2}, \end{aligned}$$

with

$$\begin{aligned} [x_1, x_2, x_3] &= \sum_{\sigma \in S_3} \rho(\sigma, (g_1, g_1, g_2)) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \\ &= x_1 x_2 x_3 + x_2 x_1 x_3 + \zeta x_1 x_3 x_2 + \zeta x_2 x_3 x_1 + \zeta^2 x_3 x_1 x_2 + \zeta^2 x_3 x_2 x_1 \end{aligned}$$

resp.

$$\begin{aligned} [x_1, x_2, x_3] &= \sum_{\sigma \in S_3} \rho(\sigma, (g_1, g_1, g_2)) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \\ &= x_1 x_2 x_3 + x_1 x_3 x_2 + \zeta x_2 x_1 x_3 + \zeta x_3 x_1 x_2 + \zeta^2 x_2 x_3 x_1 + \zeta^2 x_3 x_2 x_1. \end{aligned}$$

We close with some examples of (G, χ) -Lie algebras and of (G, χ) -Hopf algebras generated by them. We also examine some of the biproduct Hopf algebras one always obtains from (G, χ) -Hopf algebras. If H is a (G, χ) -Hopf algebra then $H \otimes kG$ is a Hopf algebra by

$$\begin{aligned} (x \otimes g)(y \otimes h) &= \chi(g, \text{deg}(y)) xy \otimes gh, \\ \Delta(1 \otimes g) &= (1 \otimes g) \otimes (1 \otimes g), \end{aligned}$$

and

$$\Delta(x \otimes 1) = \sum_{a+b=c} (y_{a,i} \otimes b) \otimes (z_{b,i} \otimes 1),$$

where $\Delta(x) = \sum_{a+b=c} \sum_i y_{a,i} \otimes z_{b,i}$ with $\deg(x) = c$, $\deg(y_{a,i}) = a$, and $\deg(z_{b,i}) = b$ (see [FM] Corollary 3.5).

Most of the (G, χ) -Hopf algebras are infinite-dimensional. In fact, the only finite-dimensional Hopf algebras generated by their primitives, we know, are given in the following example under (1) and (2).

EXAMPLE 4.9. (1) $P = P_1 = kx$ with $[x, x] = 0$ defines a “commutative” (C_2, χ) -Lie algebra with χ as in Example 4.8. (2). It generates the (C_2, χ) -Hopf algebra $H = k[x]/(x^2)$, the universal enveloping algebra of P . The biproduct $H \star kC_2$ is the well known smallest noncommutative noncocommutative Hopf algebra ([FM] Example 3.9).

(2) Let (G, χ) be as in Example 4.8 (5). $P = P_{t_1} = kx$ with $[x, \dots, x] = 0$ defines a “commutative” (G, χ) -Lie algebra with $\chi(t_1, t_1) = \zeta$, a primitive n -th root of unity. It generates the (G, χ) -Hopf algebra $H = k[x]/(x^n)$, given in the introduction (for $r = 1$). The biproduct $U(P) \star kG$ is the free algebra generated by the elements x, t_1, \dots, t_r subject to the following relations $t_i^n = 1$, $t_i t_j = t_j t_i$, and $x^n = 0$. There is one more set of relations that is obtained from the multiplicative rule for the biproduct $(1 \otimes t_i)(x \otimes 1) = \chi(t_i, t_1)x \otimes t_i$ or – identifying $x \otimes 1$ with x and $1 \otimes t_i$ with t_i – the relations $t_i x = \zeta x t_i$. Hence the biproducts form the family of noncommutative noncocommutative Hopf algebras given in [T].

(3) $P = \bigoplus_{i=0}^2 P_i$, $P_0 = kz$, $P_1 = kx \oplus ky$, $P_2 = 0$ with $([,] : \otimes^6 P_1 \rightarrow 1) = 0$ and $[x, x, x] = [y, y, y] = [x, y, y] = 0$, $[x, x, y] = z$, and $[z, x] = [z, y] = 0$ define a (C_3, χ) -Lie algebra with $\chi(\bar{1}, \bar{1}) = \zeta$, a primitive 3rd root of unity. It generates the (C_3, χ) -Hopf algebra $H = k\langle x, y \rangle / (x^3, y^3, xy^2 + yxy + y^2x)$ (we have $z = 2(x^2y + xyx + yx^2)$), the universal enveloping algebra of P . The biproduct $U(P) \star kC_3$ has generators x, y, t with the relations $x^3 = y^3 = 0$, $t^3 = 1$, $xy^2 + yxy + y^2x = 0$, $xt = \zeta tx$, and $yt = \zeta ty$.

(4) Let $G = C_3$, and χ be as before. Then $P = \bigoplus_{i=0}^2 P_i$, $P_0 = ky$, $P_1 = kx$, $P_2 = 0$ with $[x, x, x, x, x, x] = 0$, $[x, x, x] = y$, and $[y, x] = 0$ defines a (C_3, χ) -Lie algebra. It generates the (C_3, χ) -Hopf algebra $H = k[x, y]/(y - 6x^3) = k[x]$, the universal enveloping algebra of P .

(5) For G, χ as before let $P_0 = ky$, $P_1 = kx$, and $P_2 = 0$ with $[x, x, x, x, x, x] = 0$, $[x, x, x] = 0$ and $[y, x] = x$. Then P is a (C_3, χ) -Lie algebra. It generates the (C_3, χ) -Hopf algebra $H = k\langle x, y \rangle / (x^3, x - yx + xy)$, the universal enveloping algebra of P .

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