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ON SOME APPLICATIONS OF BOCHNER–SCHOENBERG–EBERLEIN TYPE THEOREMS

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1. Introduction

In a preceding paper [5] the authors have applied some Bochner–Schoenberg–Eberlein type theorems for locally convex topological vector spaces to some characterization problems of harmonic analysis on locally compact groups and locally convex vector spaces. It is the objective of the present note to provide some further applications of this method. Section 2 deals with interpolation in Hardy spaces modelled over globally symmetric Riemannian manifolds whereas Section 3 is concerned with the Fourier transformation associated to a differential operator of the Sturm–Liouville type that is also closely related to globally symmetric Riemannian manifolds.

2. Interpolation in Hardy spaces over symmetric manifolds

Let G denote a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. We shall keep to the notations and conventions introduced by Helgason [10], [11]. In particular, let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$ and K the analytic subgroup of G with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Suppose for convenience that the center of G is finite. Then K is a maximal compact subgroup of G and the space G/K of left cosets $\{gK\colon g \in G\}$ equipped with the Riemannian structure induced by the Killing form of $\mathfrak{g}$ is a globally symmetric Riemannian manifold.

Let $\mathfrak{a}_\mathfrak{p}$ be a maximal Abelian subspace in $\mathfrak{p}$ and if $\lambda\colon \mathfrak{a}_\mathfrak{p} \to \boldsymbol{R}$ is a linear form, put

$$\mathfrak{g}^\lambda = \{X \in \mathfrak{g}\colon\ [H, X] = \lambda(H)X \quad \text{for all } H \in \mathfrak{a}_\mathfrak{p}\}.$$

Fix any Weyl chamber $\mathfrak{a}_\mathfrak{p}^+$ in $\mathfrak{a}_\mathfrak{p}$. It is well known that with $\mathfrak{a}_\mathfrak{p}^+$ there is associated an ordering of the restricted roots. Put

$$\mathfrak{n} = \sum_{\lambda>0} \mathfrak{g}^\lambda, \quad \varrho = \frac{1}{2}\sum_{\lambda>0} (\dim \mathfrak{g}^\lambda)\,\lambda$$

and let $A_\mathfrak{p}$, N denote the analytic subgroups of G corresponding to $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{n}$, respectively. Then the Iwasawa decomposition

$$G = KA_\mathfrak{p} N$$

holds by virtue of the analytic diffeomorphism

$$G \ni g \rightsquigarrow k(g)\exp H(g)n(g) \in KA_\mathfrak{p} N.$$

Let $\mathscr{H}(G/K)$ be the complex vector space of harmonic functions on G/K, i.e., the space of all complex-valued functions $f \in \mathscr{C}^\infty_C(G/K)$ such that $Df = 0$ for every G-invariant differential operator D on G/K that annihilates the constants. For each exponent $p \in [\![1, +\infty[\![$ let the complex vector space

$$\mathscr{H}^p(G/K) = \{f \in \mathscr{H}(G/K)\colon \sup_{a\in A_\mathfrak{p}} \int_K |f(kaK)|^p dk < +\infty\}$$

be endowed with the norm (letting $A_\mathfrak{p}^+ = \exp \mathfrak{a}_\mathfrak{p}^+$)

$$f \rightsquigarrow \|f\|_p = \sup_{a\in A_\mathfrak{p}} \Big(\int_K |f(kaK)|^p dk\Big)^{1/p} = \sup_{a\in A_\mathfrak{p}^+} \Big(\int_K |f(kaK)|^p dk\Big)^{1/p}.$$

Moreover, the complex vector space $\mathscr{H}^\infty(G/K) = \mathscr{H}(G/K) \cap \mathscr{C}^b_C(G/K)$ of all bounded harmonic functions on G/K will be equipped with the Čebyšev norm $f \rightsquigarrow \|f\|_\infty = \sup_{gK\in G/K} |f(gK)|$. See Stoll [14].

In order to investigate the interpolation problem in the Hardy spaces $\mathscr{H}^p(G/K)$, $p \in [\![1, +\infty[\![$ (harmonic interpolation), it is necessary to have at hand the Poisson kernel.

Let M denote the centralizer of $\mathfrak{a}_\mathfrak{p}$ in K and K/M the Furstenberg–Moore boundary of G/K. The Poisson kernel is given by

$$P\colon G/K \times K/M \ni (gK, kM) \rightsquigarrow e^{-2\varrho(H(g^{-1}k))}.$$

It should be observed that the function $gK \rightsquigarrow P(gK, kM)$ belongs to $\mathscr{H}(G/K)$ for every point $kM \in K/M$.

Denoting by $\mathrm{Diff}(G/K)$ the set of all differential operators on the globally symmetric Riemannian manifolds G/K, we have the following

THEOREM 1. *Retain the above notations. Let the exponent $p \in [\![1, +\infty[\![$ with dual exponent p', the set $T \subset G/K \times \mathrm{Diff}(G/K)$, and the function $\varphi\colon T \to C$ be given. In order that there shall exist a function $f \in \mathscr{H}^p(G/K)$ such that $\|f\|_p \leqslant M$ and*

$$Df(z) = \varphi(z)$$

for all $(z, D) \in T$, it is necessary and sufficient that the inequality

$$\Big|\sum_{1\leqslant j\leqslant N} c_j \varphi(z_j)\Big| \leqslant M \Big\|\sum_{1\leqslant j\leqslant N} c_j D_j P(z_j, .)\Big\|_{p'}$$

holds for all finite sequences $(c_j)_{1\leqslant j\leqslant N}$ in C and $((z_j, D_j))_{1\leqslant j\leqslant N}$ of pairs of T.

Proof. Let $E = \mathscr{H}^p(G/K)$. If $p > 1$, construct the complex Lebesgue space $L^{p'}_C(K/M)$ with respect to the unique K-invariant normalized Radon measure $\mu \in \mathscr{M}_C(K/M)$ on the Furstenberg–Moore boundary of G/K. Then we put

$$E_0 = \begin{cases} L^{p'}_C(K/M) & \text{if} \quad p > 1, \\ \mathscr{C}_C(K/M) & \text{if} \quad p = 1. \end{cases}$$

For any function $f \in \mathscr{H}^p(G/K)$ let $\nu_f \in \mathscr{M}_C(K/M)$ denote the representing measure of f. It is well known that in the case $p > 1$ the Radon measure ν_f admits a Radon–Nikodým density with respect to the base μ that belongs to $L^p(K/M)$ and that

$$f(gK) = \int_{K/M} P(gK, kM)\, d\nu_f(kM)$$

holds for all points $z = gK \in G/K$. Consequently,

$$Df(gK) = \int_{K/M} DP(gK, kM)\, d\nu_f(kM),$$

for any $D \in \mathrm{Diff}(G/K)$. It follows that the mapping

$$T \ni (z, D) \rightsquigarrow (E \ni f \rightsquigarrow Df(z) \in C)$$

is an embedding of T into $(E, \sigma(E, E_0))'$. An application of Theorem 2 in [5] completes the proof. ■

COROLLARY. *Let $(z_n)_{n\geqslant 1}$ denote a sequence of points in G/K and $(w_n)_{n\geqslant 1}$ a sequence in C. There exists a function $f \in \mathscr{H}^p(G/K)$, $p \in [\![1, +\infty[\![$, such that $\|f\|_p \leqslant M$ and $f(z_n) = w_n$ for all integers $n \geqslant 1$ if and only if the inequality*

$$\Big|\sum_{1\leqslant j\leqslant N} c_j w_j\Big| \leqslant M \Big\|\sum_{1\leqslant j\leqslant N} c_j P(z_j, .)\Big\|_{p'}$$

holds for all finite sequences $(c_j)_{1\leqslant j\leqslant N}$ in C.

Of course, Theorem 1 is applicable to the case of harmonic functions on bounded symmetric domains in the complex vector space C^n. In this connection also see Hahn–Mitchell [9].

3. Applications to a differential operator of the Sturm–Liouville type

Let us consider the ordinary differential operator of the following form:

$$L = -\frac{1}{A}\frac{d}{dx}\Big(A\frac{d}{dx}\Big) \qquad (x > 0),$$

where the real-valued coefficient function A is defined on the real half-line R_+ and satisfies the following conditions (see Chébli [1]–[4]):

(1) $A(0) = 0$, $A(x) > 0$ for $x > 0$. There exist numbers $\varepsilon > 0$, $\alpha > 0$ and an odd real-valued function $B \in \mathscr{C}^\infty(R)$ such that

$$\frac{A'(x)}{A(x)} = \frac{\alpha}{x} + B(x)$$

holds for all $x \in]\!]0, \varepsilon[\![$ ("Condition of regularity").

(2) A is a nondecreasing monotonic function such that $\lim_{x\to\infty} A(x) = +\infty$; $\frac{A'}{A}$ is a nonincreasing monotonic function and $\varrho = \lim_{x\to\infty} \frac{1}{2}\frac{A'(x)}{A(x)}$ exists ("Condition of convexity").

EXAMPLES. (i) Denote by $I_0(\boldsymbol{R}^n)$ the neutral connected component of the group of all isometries of the real Euclidean space $\boldsymbol{R}^n$ $(n \geqslant 2)$. Then $I_0(\boldsymbol{R}^n)/\mathrm{SO}(n) = \boldsymbol{R}^n$ is a symmetric Riemannian manifold of Euclidean type. The radial part of the Laplace operator of $\boldsymbol{R}^n$ is given by

$$\Delta_0 = \left(\frac{d}{dr}\right)^2 + \frac{(n-1)}{r}\frac{d}{dr}.$$

If $A\colon r \rightsquigarrow r^{n-1}$ $(r > 0)$, then $L = -\Delta_0$ takes the form indicated above.

(ii) Let $M = G/K$ be a symmetric Riemannian manifold of noncompact type and rank one. The the function $A\colon x \rightsquigarrow (\sinh x)^p . (\sin 2x)^q$ yields for suitable exponents $p, q > 0$ the radial part of the Laplace–Beltrami operator on M and satisfies conditions (1), (2) supra. See Flensted–Jensen [7] and Flensted–Jensen and Koornwinder [8]

The differential operator L is formally self-adjoint in the complex Hilbert space $L^2(\boldsymbol{R}_+, A dx)$. Let

$$\mathrm{dom}(L) = \left\{u \in L^2(\boldsymbol{R}_+, A dx)\colon Lu \in L^2(\boldsymbol{R}_+, A dx),\ \lim_{x\to 0} A(x)\frac{du}{dx} = 0\right\}$$

be the domain of L. Then the operator $(L, \mathrm{dom}(L))$ is self-adjoint, its spectrum is contained in $\boldsymbol{R}_+$ and its continuous part is the interval $[\![\varrho^2, +\infty[\![$. Moreover, any solution Φ of the differential equation

$$L\Phi - s . \Phi = 0 \quad (s > 0),$$

which satisfies the conditions

$$\Phi(0, s) = 1,$$

$$\frac{d}{dx}\Phi(0, s) = 0,$$

verifies the estimate

$$|\Phi(x, s)| \leqslant c_\varrho(1+x)e^{-\varrho x} \quad (s \in [\![\varrho^2, +\infty[\![) \tag{1}$$

with a suitable constant $c_\varrho > 0$ (Chébli [3]). The Fourier–Stieltjes transformation $\mathscr{F}_L$ associated with the operator $(L, \mathrm{dom}(L))$ admits the form

$$\mathscr{F}_L\colon \mathscr{M}^1(\boldsymbol{R}_+) \ni \mu \rightsquigarrow \Big(\boldsymbol{R}_+ \ni s \rightsquigarrow \int_{\boldsymbol{R}_+} \Phi(x, s)\,d\mu(x)\Big).$$

THEOREM 2. *Suppose $\varrho > 0$. Let $S \subset]\!]\varrho^2, +\infty[\![$ be a set having at least one cluster point and let the function $f\colon S \to \boldsymbol{C}$ be given. In order that there exists a unique measure $\nu \in \mathscr{M}^1(\boldsymbol{R}_+)$ such that $f = \mathscr{F}_L\nu$ on S and $||\nu|| \leqslant M$ it is necessary and sufficient that the inequality*

$$\Big|\sum_{1\leqslant k\leqslant N} c_k f(x_k)\Big| \leqslant M \Big\|\sum_{1\leqslant k\leqslant N} c_k \Phi(\,.\,, x_k)\Big\|_\infty$$

holds for all finite sequences $(c_k)_{1\leqslant k\leqslant N}$ in $\boldsymbol{C}$ and $(x_k)_{1\leqslant k\leqslant N}$ in S.

Proof. Let $E = \mathscr{C}_0(\boldsymbol{R}_+)$ be the vector space of all continuous complex-valued functions on $\boldsymbol{R}_+$ vanishing at infinity. Equip E with the topology of uniform convergence. Then its dual is the space $E' = \mathscr{M}^1(\boldsymbol{R}_+)$ of all bounded complex Radon measures on $\boldsymbol{R}_+$. Choose $A_0 = \{\Phi(\,.\,, s)\colon s \in S\}$ and let $\varphi\colon A_0 \ni \Phi(\,.\,, s) \rightsquigarrow f(s) \in \boldsymbol{C}$, $(s \in S)$. An application of Theorem 1 of [5] which is possible by (1) yields the existence of a measure $\nu \in \mathscr{M}^1(\boldsymbol{R}_+)$ such that $f(s) = \mathscr{F}_L\nu(s)$ for $s \in S$ and $||\nu|| \leqslant M$. Since $]\!]\varrho^2, +\infty[\![\ni s \rightsquigarrow \Phi(x, s)$ is an analytic function for any $x \in \boldsymbol{R}_+$, it follows by the Lebesgue–Fubini theorem and by Cauchy's formula that the function

$$s \rightsquigarrow \mathscr{F}_L\nu(s)$$

is analytic on the open interval $]\!]\varrho^2, +\infty[\![$. Hence the uniqueness of ν. ∎

COROLLARY. *In particular, the function f can be extended holomorphically into the subset $\Sigma = \{s = \gamma + i\beta \in \boldsymbol{C}\colon \beta^2 < 4\varrho^2\gamma\}$ of the complex plane $\boldsymbol{C}$.*

Define the translation operators (Chébli [1]–[4])

$$T^y\colon L^2(\boldsymbol{R}_+, A dx) \ni f \rightsquigarrow T^y f \in L^2(\boldsymbol{R}_+, A dx)$$

according to

$$\mathscr{F}_L(T^y f)\colon s \rightsquigarrow \Phi(y, s)\mathscr{F}_L f(s) \quad (y \in \boldsymbol{R}_+).$$

Then $||T^y|| \leqslant 1$ for all points $y \in \boldsymbol{R}_+$. Let $\mathscr{D}(\boldsymbol{R})$ be the Schwartz space of complex-valued test functions on $\boldsymbol{R}$ and $\mathscr{D}_0(\boldsymbol{R})$ its subspace of all even test functions $\psi \in \mathscr{D}(\boldsymbol{R})$. Then we have

$$T^y f(x) = \int_{|x-y|}^{|x+y|} f(z)\,d\mu_{x,y}(z) \quad (x, y \in \boldsymbol{R}_+)$$

with $\mu_{x,y} \in \mathscr{M}^1(\boldsymbol{R})$, $\mathrm{Supp}(\mu_{x,y}) \subseteq [\![|x-y|, |x+y|]\!]$ and $\mu_{x,y} = \mu_{y,x}$ (Chébli [3]). The translation operators $(T^y)_{y\geqslant 0}$ enable us to define the following convolution product

$$f * g\colon x \rightsquigarrow \int_{\boldsymbol{R}_+} T^x f(y) . g(y) A(y)\,dy$$

on $\mathscr{D}_0(\boldsymbol{R})$. Then $L^1(\boldsymbol{R}_+, A dx)$ becomes a commutative semisimple convolution algebra over $\boldsymbol{C}$ with spectrum $X(L^1(\boldsymbol{R}_+, A dx)) = \{\Phi(\,.\,, s)\colon s \in \bar{\Sigma}\}$. Moreover, denote by $\mathrm{Mult}(L^1(\boldsymbol{R}_+, A dx))$ the commutative Banach algebra of all multipliers on $L^1(\boldsymbol{R}_+, A dx)$, i.e., of all continuous linear mappings $T\colon L^1(\boldsymbol{R}_+, A dx) \to L^1(\boldsymbol{R}_+, A dx)$ such that

$$T(f * g) = (Tf) * g$$

holds for all pairs f, g of $L^1(\boldsymbol{R}_+, A dx)$.

THEOREM 3. *Let $T\colon L^1(\boldsymbol{R}_+, A\,dx) \to L^1(\boldsymbol{R}_+, A\,dx)$ be a continuous linear mapping. The following statements are pairwise equivalent:*

(i) $T \in \operatorname{Mult}(L^1(\boldsymbol{R}_+, A\,dx))$;

(ii) *There exists a measure $\nu \in \mathscr{M}^1(\boldsymbol{R}_+)$ such that*

$$T\colon L^1(\boldsymbol{R}_+, A\,dx) \ni g \rightsquigarrow \nu * g;$$

(iii) $T^x \circ Tg = T \circ T^x g$ *(Wendel condition) for all $x \in \boldsymbol{R}_+$, $g \in L^1(\boldsymbol{R}_+, A\,dx)$;*

(iv) *There exists a bounded and continuous function $f\colon \overline{\Sigma} \to \boldsymbol{C}$ with*

$$f \cdot \mathscr{F}_L(L^1(\boldsymbol{R}_+, A\,dx)) \subseteq \mathscr{F}_L(L^1(\boldsymbol{R}_+, A\,dx)).$$

Proof. (i) $\Rightarrow$ (ii): Fix $T \in \operatorname{Mult}(L^1(\boldsymbol{R}_+, A\,dx))$. Let $\{u_n\colon n \in \boldsymbol{N}\}$ be an approximate unit for $L^1(\boldsymbol{R}_+, A\,dx)$ such that $\|u_n\|_1 = 1$ for all $n \in \boldsymbol{N}$. Define the linear form

$$\mu_n\colon \mathscr{C}_0(\boldsymbol{R}_+) \ni g \rightsquigarrow \int_{\boldsymbol{R}_+} g \cdot (Tu_n) A\,dx \in \boldsymbol{C}.$$

It follows $\|\mu_n\| \leqslant \|T\|$ for all $n \in \boldsymbol{N}$. Hence by Alaoglu's theorem $\{\mu_n\colon n \in \boldsymbol{N}\}$ is a relatively compact subset of $\mathscr{M}^1(\boldsymbol{R}_+)$ with respect to the vague topology. Let the subsequence $(\mu_{n_k})_{k\in\boldsymbol{N}}$ converge vaguely to the measure $\nu \in \mathscr{M}^1(\boldsymbol{R}_+)$. It suffices to prove that

$$Tg = \nu * g$$

holds for all functions $g \in \mathscr{D}_0(\boldsymbol{R})$. In order to do this we shall establish the identity

$$\int_{\boldsymbol{R}_+} f \cdot (Tg) A\,dx = \int_{\boldsymbol{R}_+} f \cdot (\nu * g) A\,dx \tag{2}$$

for all $f \in \mathscr{D}_0(\boldsymbol{R})$. Let $\varepsilon > 0$ be given. There exists $N(\varepsilon) \in \boldsymbol{N}$ such that for $n \geqslant N(\varepsilon)$,

$$\begin{aligned}
&\Big|\int_{\boldsymbol{R}_+} f \cdot T(u_n * g) A\,dx - \int_{\boldsymbol{R}_+} f(\nu * g) A\,dx\Big| \\
&\quad = \Big|\int_{\boldsymbol{R}_+} f(x)\Big(\int_{\boldsymbol{R}_+} T^x(Tu_n)(z) \cdot g(z) A(z)\,dz\Big) A(x)\,dx - \\
&\qquad\qquad - \int_{\boldsymbol{R}_+} f(x)\Big(\int_{\boldsymbol{R}_+} T^x g(y)\,d\nu(y)\Big) A(x)\,dx\Big| \\
&\quad = \Big|\int_{\boldsymbol{R}_+} f(x)\Big(\int_{\boldsymbol{R}_+\times\boldsymbol{R}_+} Tu_n(y)\,d\mu_{z,x}(y) g(z) A(z)\,dz\Big) A(x)\,dx - \\
&\qquad\qquad - \int_{\boldsymbol{R}_+} f(x)\Big(\int_{\boldsymbol{R}_+\times\boldsymbol{R}_+} g(z)\,d\mu_{x,y}(z)\,d\nu(y)\Big) A(x)\,dx\Big| \\
&\quad = \Big|\int_{\boldsymbol{R}_+} Tu_n(y) \cdot h(y) A(y)\,dy - \int_{\boldsymbol{R}_+} h(y)\,d\nu(y)\Big| < \frac{\varepsilon}{2},
\end{aligned}$$

where h is defined in the following way:

$$\mathscr{D}_0(\boldsymbol{R}) \ni h\colon y \rightsquigarrow \int_{\boldsymbol{R}_+\times\boldsymbol{R}_+} f(x) g(z)\,d\mu_{y,x}(z) A(x)\,dx.$$

Observe that we have used the Lebesgue–Fubini theorem and the symmetry properties of the measures $d\mu_{x,y}(z) A(y)\,dy$ (Chébli [3]). Likewise the number $N(\varepsilon)$ may be chosen such that

$$\Big|\int_{\boldsymbol{R}_+} f \cdot T(u_n * g) \cdot A\,dx - \int_{\boldsymbol{R}_+} f \cdot Tg \cdot A\,dx\Big| \leqslant \|f\|_\infty \cdot \|T\| \cdot \|u_n * g - g\|_1 < \frac{\varepsilon}{2}$$

holds for $n \geqslant N(\varepsilon)$.

Consequently, we obtain

$$\Big|\int_{\boldsymbol{R}_+} f \cdot (\nu * g) A\,dx - \int_{\boldsymbol{R}_+} f \cdot (Tg) A\,dx\Big| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the statement (2) follows.

(ii) $\Rightarrow$ (i): trivial.

(ii) $\Rightarrow$ (iii): Observe that $T^x \in \operatorname{Mult}(L^1(\boldsymbol{R}_+, A\,dx))$ holds for all $x \in \boldsymbol{R}_+$. Consequently, there exists a measure $\nu_x \in \mathscr{M}^1(\boldsymbol{R}_+)$ such that $T^x g = \nu_x * g$ holds for all $g \in L^1(\boldsymbol{R}_+, A\,dx)$. Hence

$$\begin{aligned}
T(T^x g) = T(\nu_x * g) &= \nu * (\nu_x * g) = (\nu * \nu_x) * g \\
&= (\nu_x * \nu) * g = \nu_x * (\nu * g) = T^x(Tg)
\end{aligned}$$

for all $g \in L^1(\boldsymbol{R}_+, A\,dx)$.

(iii) $\Rightarrow$ (i): Let $f, g \in L^1(\boldsymbol{R}_+, A\,dx)$ and $h \in L^\infty(\boldsymbol{R}_+, A\,dx)$ be arbitrary functions. Then

$$\begin{aligned}
\int_{\boldsymbol{R}_+} T(f * g) \cdot h \cdot A\,dx &= \int_{\boldsymbol{R}_+} (f * g) \cdot {}^tT(h) A\,dx \\
&= \int_{\boldsymbol{R}_+}\Big(\int_{\boldsymbol{R}_+} T^x f(y) \cdot g(y) A(y)\,dy\Big) {}^tTh(x) A(x)\,dx \\
&= \int_{\boldsymbol{R}_+}\Big(\int_{\boldsymbol{R}_+} T^x f(y) {}^tTh(x) A(x)\,dx\Big) g(y) A(y)\,dy \\
&= \int_{\boldsymbol{R}_+}\Big(\int_{\boldsymbol{R}_+} T^y f(x) {}^tTh(x) A(x)\,dx\Big) g(y) A(y)\,dy \\
&= \int_{\boldsymbol{R}_+}\Big(\int_{\boldsymbol{R}_+} T(T^y f)(x) h(x) A(x)\,dx\Big) g(y) A(y)\,dy \\
&= \int_{\boldsymbol{R}_+}\Big(\int_{\boldsymbol{R}_+} T^y(Tf)(x) h(x) A(x)\,dx\Big) g(y) A(y)\,dy \\
&= \int_{\boldsymbol{R}_+}\Big(\int_{\boldsymbol{R}_+} T^x(Tf)(y) g(y) A(y)\,dy\Big) h(x) A(x)\,dx \\
&= \int_{\boldsymbol{R}_+} (Tf * g)(x) \cdot h(x) A(x)\,dx
\end{aligned}$$

by the Lebesgue–Fubini theorem and the symmetry of the measure $\mu_{x,y}$ in (x, y) (cf. Chébli [3]).

(i) $\Leftrightarrow$ (iv): See the general results for semisimple commutative Banach algebras proved by Larsen in Chapter 1 of [12], [13]. ∎

COROLLARY. *Suppose $\varrho > 0$. Let $S \subset]\varrho^2, +\infty[$ be a set having at least one cluster point and let the function $f\colon S \to C$ be given. If the inequality*

$$\Big|\sum_{1\leqslant k\leqslant N} c_k f(x_k)\Big| \leqslant M \Big\|\sum_{1\leqslant k\leqslant N} c_k \Phi(\,.\,, x_k)\Big\|_\infty$$

holds for all finite sequences $(c_k)_{1\leqslant k\leqslant N}$ in C and $(x_k)_{1\leqslant k\leqslant N}$ in S, then f may be extended to a unique function $f\colon \bar{\Sigma} \to C$ that satisfies condition (iv) *of Theorem* 3 *supra.*

Proof. The conclusion follows by Theorems 2 and 3. ■

4. Concluding remarks

The fact that in Section 2 the Furstenberg–Moore boundary K/M of G/K is compact and that in Section 3 the eigenfunctions $\Phi(\,.\,, s)$ ($s \geqslant \varrho^2$) of the differential operator L are decreasing at infinity for $\varrho > 0$ according to (1) implies the direct applicability of the interpolation theorems proved in [5]. On the other hand, we have indicated in Section 3 that the Laplace–Beltrami operator of symmetric Riemannian manifolds G/K of noncompact type and rank one may be chosen for L. Therefore, these circumstances suggest an investigation of our situation in the higher rank case and in the Euclidean type case also. But then the difficulty arises that for arbitrary symmetric Riemannian manifolds no decreasing properties of the zonal spherical functions $\Phi(\,.\,, s)$ are known. These problems among others will be considered in the succeeding paper [6].

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