EQUIVALENCE, UNCONDITIONALITY
AND CONVERGENCE A.E. OF THE SPLINE BASES IN $L_p$ SPACES

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1. Introduction

In the joint paper by P. Simon, P. Sjölin, and the author [5] it was shown that the
Franklin and Haar bases in $L_p(0, 1)$, $1 < p < \infty$, are equivalent, and a simplified
proof of the result of S. B. Bośkarev [1] on the unconditionality of the Franklin
system in $L_p(0, 1)$ was presented. Moreover, the convergence a.e. of the Fourier
series with respect to the bounded orthonormal set of polygons for functions in
$L_p(0, 1)$, $1 < p < \infty$, was established. We have suggested in [5] the possibility of
extending all the results to spline systems of higher orders. The main goal of this
paper is to carry out this programme. The ideas of the presented proofs are not
in principle new compared with [5]. However, in order to have more or less complete
theory of the spline systems it seems necessary to publish this paper in addition
to the works by Z. Ciesielski and J. Domsta [4], Z. Ciesielski [3], J. Domsta [6],
and S. Ropela [9], [10], [11].

The main results of this paper require some comments. Theorem 3.1 is the
crucial one. It implies in particular the unconditionality of the Haar and Franklin
orthonormal sets in $L_p(0, 1)$ spaces $1 < p < \infty$ (cf. Corollary 3.1). Moreover, as
a consequence we obtain Theorem 3.2 as well. The second non-trivial result is
Theorem 5.1, and it follows essentially from Theorem 3.2 and the highly non-trivial
maximal inequality for Walsh system proved by P. Sjölin in [12]. One would expect
that a result like Theorem 3.1 should give Theorem 4.1. We were able to prove
the equivalence of the spline bases by means of the C. L. Fefferman and E. M. Stein
inequalities [7] only.

2. Preliminaries

The simplest way of defining the spline systems, in which we are interested, is by
the Haar orthonormal functions $\chi_n$, $n = 1, 2, \ldots$, given on $I = (0, 1)$. To do this
let us denote by $D$ the differentiation operator and let us define the following in-
integration operators:
\[
(Gf)(t) = \int_{-1}^{1} f(u) du,
\]
and the usual "scalar product"
\[
(f, g) = \int_{-1}^{1} f(u) g(u) du,
\]
where \(1 \leq p, q \leq \infty \) and \( p^{-1} + q^{-1} = 1 \).

Let \( m \) be an arbitrary but fixed integer, \( m \geq -1 \).

It is clear that the functions \( 1, t, \ldots, t^{m+1}, G^{m+1}X_n(t), n \geq 2 \), are linearly independent over \( I \). The Schmidt orthonormalization procedure applied to this set of functions gives the orthonormal set of splines \( \{ f_n^{m,n}, n \geq -m \} \) of order (of smoothness) \( m \). Following now [3] we define the remaining spline systems as follows. For \( 0 \leq k \leq m+1 \) we set
\[
f_n^{m,k} = P_f^{m-k}, \quad n \geq k - m,
\]
and
\[
g_n^{m,k} = H_f^{m-k}, \quad n \geq k - m.
\]

For our purpose it is more convenient to have more unified notation for the systems \( \{ f_n^{m,k}, n \geq k - m \} \) and \( \{ g_n^{m,k}, n \geq k - m \} \) where \( 0 \leq k \leq m+1 \). To compare these spline systems as bases in \( L_2(I) \) it is good to normalize them suitably in the usual norm. Thus, we define for \( |k| \leq m+1 \), \( n \geq |k|-m \)
\[
h_n^{m,k} = \frac{1}{\|f_n^{m,k}\|_2} f_n^{m,k}\quad \text{for} \quad 0 \leq k \leq m+1,
\]
\[
g_n^{m,k} = \frac{1}{\|f_n^{m,k}\|_2} f_n^{m,k}\quad \text{for} \quad 0 \leq -k \leq m+1,
\]
where
\[
\|f\|_p = \left( \int |f|^p \right)^{1/p}, \quad 1 \leq p \leq \infty.
\]

With the help of the results established in [3] it is not hard to derive for \( |k| \leq m+1 \), \( 1 \leq p \leq \infty \), the following properties:
(1) There is a constant \( C_m \), depending only on \( m \), such that
\[
n^{\frac{1}{2}} |k|^{-1/2} C_m^{-1} \leq \|h_n^{m,k}\|_p \leq C_m n^{1/2} |k|^{-1/2}.
\]
(2) The set \( \{ h_n^{m,k}, j \geq |k|-m \} \) is biorthogonal, i.e. \( (h_n^{m,k}, h_n^{m,k}) = \delta_{j,k} \) for \( j, k \geq |k|-m \).
(3) The system \( \{ h_n^{m,k}, n \geq |k|-m \} \) is a basis in \( L_2(I) \), \( 1 \leq p \leq \infty \).
(4) Let \( n = 2^m + r \), \( 1 \leq r \leq 2^m \), \( t_n = (2^{m+1})^{-1} 2^{m+1} + 1 \), and let \( \|h_n^{m,k}\|_n = |h_n^{m,k} (t_n)| \). Then \( |t_n - 2^{m+1}| = 0(1/n) \) for large \( n \).
(5) There are constants \( C_m \) and \( q_m \), \( 0 < q_m < 1 \), such that
\[
|h_n^{m,k}(t)| \leq C_m n^{\frac{1}{2} |k|-1/2} q_m(t).
\]
holds for \( n \geq 1 \), \( t \in I \).

(6) There is a constant \( C_m \) such that
\[
\sum_{\mu = 0}^{m+1} \|h_n^{m+1,\mu}(t)\|_n \leq C_m 2^{m/2}
\]
holds for \( |k| \leq m+1 \), \( \mu \geq 0 \) and \( t \in I \).

Lemma 2.1 (S. Ropela [10]). Let the integers \( m \) and \( k \) be given such that \( |k| \leq m+1 \). Then \( \{ h_n^{m,k}, n \geq |k|-m \} \) is an unconditional basis, i.e. a Riesz basis in \( L_2(I) \).

Lemma 2.2. Let \( m, m', k, k' \) be given integers such that \( |k| \leq m+1 \) and \( |k'| \leq m'+1 \). Then there is a constant \( C_{m,m',k,k'} \), such that
\[
\sum_{\mu = 0}^{m+1} \sum_{\nu = 0}^{m'+1} |h_n^{m,k}(t) h_n^{m',k'}(s)| \leq C_{m,m',k,k'} 2^{-|t-s|/2}
\]
holds for \( t, s \in I \).

Proof. To each \( n \geq 2 \), \( n = 2^m + r \), \( 1 \leq r \leq 2^m \), there corresponds the dyadic partition
\[
x_n^t = \begin{cases} \frac{i}{2^{m+1}} & \text{for } i \leq 2^m, \\ \frac{i-2^m}{2^{m+1}} & \text{for } i > 2^m. \end{cases}
\]

Now, for given \( t \in I \), \( t > 0 \), let \( \xi(t) \) denote the unique solution of the inequality \( x_n^{\xi(t)} < t \leq x_n^{\xi(t)} \). Then property (5) of \( h_n^{m,k} \) gives
\[
|h_n^{m,k}(t)| \leq C_{m,m',k,k'} 2^{-|t-s|/2}.
\]

Consequently, for some \( C_{m,m',k,k'} \) and \( q_{m,m',k,k'} \) with \( \max(q_m, q_{m',k,k'}) < q_{m,m',k,k'} < 1 \), we have
\[
\sum_{\mu = 0}^{m+1} \sum_{\nu = 0}^{m'+1} |h_n^{m,k}(t) h_n^{m',k'}(s)| \leq C_{m,m',k,k'} 2^m \sum_{\mu = 0}^{m+1} \sum_{\nu = 0}^{m'+1} q_{m,m',k,k'}^{(2^m+1)(2^{m'-1})} \leq C_{m,m',k,k'} 2^m q_{m,m',k,k'}^{2^{m'}}.
\]

On the other hand, there is a constant \( C_{m,m',k,k'} \) such that
\[
\sum_{|t-s| \geq 2^m} 2^{-|t-s|/2} \leq C_{m,m',k,k'} 2^{-|t-s|/2}.
\]

Combining these two inequalities we complete the proof.

The Walsh system \( \{ w_n, n \geq 1 \} \) is a bounded orthonormal set related to the Haar system \( \{ X_n, n \geq 1 \} \) as follows:
\[
w_n = X_1,
\]
and
\[
w_{2^m+\nu} = \sum_{\lambda=1}^{2^m} A_{2^m}^{(\nu)} X_{2^m+\nu}, \quad 1 \leq \nu \leq 2^m,
\]
\[
X_{2^m+\nu} = \sum_{\lambda=1}^{2^m} A_{2^m}^{(\nu)} w_{2^m+\nu}, \quad 1 \leq \lambda \leq 2^m,
\]
where \( A_{2^m}^{(\nu)} = (w_{2^m+\nu}, X_{2^m+\nu}), \quad A_{2^m}^{(\nu)} = A_{2^m}^{(\nu)} \).
Starting with the system \( \{ h^{(m,n)} , n \geq |k| - m \} \) instead of the Haar system, we define \( w^{(m,n)} , n \geq |k| - m \), in the same way as \( w_n \), i.e. \( w^{(m,n)} = h^{(m,n)} \) for \( n = |k| - m, \ldots, 1 \), and (cf. [2] and [9])

\[
\sum_{l=1}^{2^n} A^{(m,n)}_{l+1} h^{(m,n)}_{l+1} , \quad 1 \leq n < 2^n .
\]

It now follows that

\[
h^{(m,n)} = \sum_{l=1}^{2^n} A^{(m,n)}_{l+1} h^{(m,n)}_{l+1} , \quad 1 \leq l < 2^n .
\]

Since \( A^{(m,n)}_{l+1} = \pm 2^{-n/2} \), we obtain by property (6) of \( \{ h^{(m,n)} \} \) that for some constant \( C_n \)

\[
|w^{(m,n)}(t)| \leq C_n , \quad n \geq |k| - m , \quad t \in I.
\]

Thus, \( \{ w^{(m,n)} , n \geq |k| - m \} \), \( k \leq m + 1 \), is a set of splines of order \( m - k \) uniformly bounded on \( I \). According to property (2) we find that the set \( \{ w^{(m,n)} , w^{(m-k,i)} , i, j \geq |k| - m \} \) is biorthogonal whenever \( |k| \leq m + 1 \).

**Lemma 2.3** (S. Ropela [9]). Let the integers \( m \) and \( k \) be given such that \( |k| \leq m + 1 \) and let \( 1 < p < \infty \). Then \( \{ w^{(m,n)} , n \geq |k| - m \} \) is a basis in \( L_p(I) \).

It seems to be a good place to identify the spline systems for particular choice of the parameters \( m \) and \( k \) with some of the known systems:

- \( \{ h^{(m,n)} \} \) (the Haar system),
- \( \{ w^{(m,n)} \} \) (the Walsh system),
- \( \{ h^{(m,n)} \} \) (the Franklin system),
- \( \{ w^{(m,n)} \} \) (the Franklin system for this type see [2]).

For the proof of the equivalence of the spline bases we need an inequality of C. L. Fefferman and E. M. Stein [7]. To state it we recall the definition of the maximal function. If \( f \in L_p(I) \), then the **maximal function** is defined as

\[
(Mf)(t) = \sup_{a \in I} \frac{1}{|a|} \int_a^t |f| .
\]

where the supremum is taken over all intervals \( a \subset I \) such that \( t \in a \).

**Lemma 2.4** (C. L. Fefferman and E. M. Stein [7]). Let \( 1 < p < \infty \) and let \( g_1, g_2, \ldots \) be a sequence of functions in \( L_p(I) \) with the property that

\[
\sum_{i=1}^{\infty} \left( \sum_{n=1}^{\infty} |g_n(x_i)|^p \right)^{1/p} \leq C_p \sum_{n=1}^{\infty} |g_n(x_i)|^p .
\]

Then there is a constant \( C_p \) depending on \( p \) only such that

\[
\left\| \sum_{n=1}^{\infty} (Mg_n(x))^p \right\|_p \leq C_p \left\| \sum_{n=1}^{\infty} |g_n(x)|^p \right\|_p .
\]

In the proof of the maximal inequality for the basis \( \{ w^{(m,n)} , n \geq |k| - m \} \) an important role is played by

**Lemma 2.5.** Let \( m \) and \( k \), \( |k| \leq m + 1 \), be given, and let

\[
H^{(m,n)}_f = \sum_{j=-m}^{m} (f, h^{(m-n,j)}) h^{(m,n)},
\]

Then there is a constant \( C_m \) such that

\[
H^{(m,n)}_f \leq C_m M f , \quad f \in L_p(I),
\]

where

\[
H^{(m,n)}_f = \sup \| H^{(m,n)}_f \|.
\]

For \( 0 \leq k \leq m + 1 \) the lemma was established in [3]. In the case of \( k : -(m+1) \leq k \leq 0 \), the proof is quite similar and therefore it is omitted.

Now, the well-known Hardy-Littlewood maximal inequality, (cf. [13], p. 5), combined with Lemma 2.5, gives

**Corollary 2.1.** Let \( |k| \leq m + 1 \) and let \( 1 < p < \infty \). Then for some \( C_m \) we have

\[
\| H^{(m,n)}_f \|_p \leq C_m \frac{p}{p-1} \| f \|_p
\]

and

\[
\| H^{(m,n)}_f \| \leq C_m \| f \|^{1/p} \| f \|_p^{1-1/p} + C_m .
\]

**3. Unconditionality of the basis \( \{ h^{(m,n)} \} \) in \( L_p(I) \), \( 1 < p < \infty \)**

For given \( m \) and \( k, |k| \leq m + 1 \), a function \( f \) is called \( (m,k) \)-**polynomial** if it is of the form

\[
f = \sum_{j=-m}^{m} a_j h^{(m,n)} .
\]

Now, let the integers \( m, k, m' \), and \( k' \) be given such that \( |k| \leq m + 1, |k'| \leq m' + 1 \), and let \( e = \{ e_j = \pm 1, j \geq (|k| - m) \lor (|k'| - m') \} \) where \( a \lor b = \max(a, b) \). The operator \( T \) is defined on \( (m',-k') \)-polynomials as follows

\[
Tf = \sum_{j=-m'}^{m'} e_j f, h^{(m',k')}, h^{(m,n)} .
\]

Let us notice that, according to Lemma 2.1, \( T \) is well defined for \( f \) in \( L_p(I) \), as well.

**Theorem 3.1.** There exists a constant \( C_{m,m'} \) such that

\[
\left\| \{ f : |Tf| > y \} \right\|_p \leq C_{m,m'} \| f \|_p , \quad y > 0
\]

holds for all \( (m',-k') \)-polynomials \( f \), i.e. \( T \) is of weak type \((1,1)\).

This theorem is being proved in almost the same way as Bockariev's theorem in [5], and the proof below is given simply for the sake of completeness.
Proof. For fixed \( y > 0 \) let us define \( Q = \{ t \in I : M(t), y \} \) and \( P = \cap_{Q_i} \). Since the operator \( M \) is of weak type \((1,1)\), we have (cf. [13], p. 5) \( (3.2) \)

\[ |Q_i| \leq \frac{\delta}{y} ||f||_1. \]

Notice that (3.1) holds with \( C_{m,w} = 5 \) if only \( 0 < y \leq \frac{5}{2} ||f||_1 \). However, in the opposite case, \( P \) is non-empty, and in this case let \( (Q_i) \) be a Whitney decomposition of \( Q \) (see [13], pp. 167–168), i.e. each \( Q_i = \{ a_i, \beta_i \} \) is a dyadic interval of the form \((2^{-r} (i-1), 2^{-r})\) and \( (3.3) \)

\[ Q = \bigcup_{i=1}^{\infty} Q_i, \quad \text{int} Q_i \cap \text{int} Q_j = \emptyset, \quad i \neq j, \quad |Q_i| \leq \text{dist}(Q_i, P) \leq 4|Q_i|. \]

Now, (3.4) and the definition of \( P \) give (3.5)

\[ \frac{\delta}{y} ||f||_1 \leq 5|Q_i|. \]

The next step is to decompose \( f \) in suitable way into a sum of two functions \( f_1 \) and \( f_2 \). To do this let \( T_1, t \) denote the orthogonal projection of \( L_1(Q) \) onto the \((d+1)\)-dimensional subspace spanned by \( 1, \frac{t}{t}, \ldots, \frac{t^d}{t^d} \) restricted to \( Q_i \), where \( d = 2[(m+1) \sqrt{m(m+1)}] \). Let

\[ f_1(t) = \begin{cases} f(t) & \text{for } t \in P, \\ T_1 f(t) & \text{for } t \in \text{int} Q_i, \quad i = 1, 2, \ldots, \\ 0 & \text{for } t \in Q \setminus \bigcup_{i=1}^{\infty} \text{int} Q_i, \end{cases} \]

and let \( f_2 = f - f_1 \).

The projections \( T_1 \) can be represented in terms of the Legendre orthonormal polynomials \( l_{a_1}, \ldots, l_{a_d} \), given on \( I \). Indeed, let

\[ l_{a_i}(t) = \left[ \frac{t - a_i}{|Q_i|} \right]^{1/2}, \quad t \in Q_i. \]

Then \( T_1 f(t) = \int_{Q_i} f(t) \left( \sum_{j=0}^{d} l_{a_j}(t) l_{a_j}(s) \right) ds, \quad t \in Q_i, \)

whence we infer

\[ |T_1 f(t)| \leq \frac{C_d}{|Q_i|} \int_{Q_i} |f(s)|, \quad t \in Q_i, \]

where

\[ C_d = \max_{s \in I} \left( \sum_{j=0}^{d} |l_{a_j}(s)| \right). \]

Thus, the definitions of \( P \) and \( f_1 \), and (3.5) give (3.6)

\[ ||f_1(t)||_1 \leq 5C_d y \text{ a.e. in } t \in P. \]

The following properties of \( f_2 \) are going to be needed. According to definition, \( f_2(t) = 0 \) for \( t \in P \), and \( f_2 \) restricted to \( Q_i \) is orthogonal to all polynomials of degree at most \( d \), i.e.

\[ \int_{Q_i} f_2 \cdot P = 0, \quad i = 1, 2, \ldots, \]

where \( w \) is a polynomial of degree less than or equal to \( d \). Moreover, since \( f_2 = f - f_1 \), we have from (3.5) and (3.6)

\[ \int_{Q_i} |f_2| \leq 5(C_d + 1) ||f_1||_1, \quad i = 1, 2, \ldots \]

The estimate for \( T_1 f_2 \). According to Lemma 2.1 we have

\[ ||T_1 f_2||_1 \sim \sum_{j=1}^{\infty} \sum_{|Q_j| < \delta} (f_j, h_m \cdot k_j^r)^2 \leq \sum_{j=1}^{\infty} (f_j, h_m \cdot k_j^r)^2 \sim ||f||_2, \]

and therefore for some constant \( C_{m,w} \) we have

\[ ||T_1 f_2||_1 \leq C_{m,w} ||f||_2. \]

Consequently, (3.9), (3.6), and (3.2) give

\[ ||f(t)| > y|| \leq \frac{||T_1 ||_1}{y^2} \lessapprox \frac{C_{m,w}}{y^2} ||f||_1 \leq \frac{C_{m,w}}{y^2} \left( \int_{Q_i} |f_1|^2 + \int_{Q_i} |f_2|^2 \right) \]

\[ \leq \frac{C_{m,w}}{y^2} \left( \frac{1}{y} \int_{Q_i} |f|^2 + \frac{1}{y^2} \int_{Q_i} |f_2|^2 \right) \leq \frac{C_{m,w}}{y^2} ||f||_1. \]

We denote, here and later on in each step, by the same letter different constants.

The estimate for \( T_1 f_2 \). Let us expand \( f_2 \) with respect to \( h_m \cdot k_j^r \), \( j \geq |k| - m^2 \); \( f_2 \) is bounded, so we have

\[ f_2 = \sum_{j=1}^{\infty} b_j h_m \cdot k_j^r, \]

where \( b_n = (f_2, h_m \cdot k_j^r) \). Since \( f_2 \) vanishes on \( P \), it follows by (3.7) that \( b_{|k| - m^2} = \ldots = b_1 = 0 \), and consequently for \( n \geq 2 \) we have

\[ b_n = \sum_{j=0}^{d} f_2 h_m \cdot k_j^r. \]

Now, if \( 2^n < n \leq 2^{n+1} \), then \( h_m \cdot k_j^r \) is a polynomial of degree at most \( 2(m + 1) \leq d \) on each dyadic interval of length \( 2^{-n} \). Thus, according to (3.7), for an \( i \) such that \( |Q_i| \leq 2^{-n} \) we have

\[ \int_{Q_i} f_2 h_m \cdot k_j^r = 0, \]
and therefore

\[ b_n = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} f_k \mathbb{H}^{(m,x)} dt. \]

Consequently,

\[ |\mathcal{T}_f(t)| = \sum_{n=(k-n)^{x-1}}^{\infty} \left| \sum_{\mu} \sum_{\nu < \mu} \int f_k \mathbb{H}^{(m,x)}(t) dt \right| \leq \sum_{\mu} \sum_{\nu < \mu} \int f_k \mathbb{H}^{(m,x)}(t) dt \]

\[ = \sum_{\mu} \int f_k(t) \left( \sum_{\nu < \mu} \int f_k \mathbb{H}^{(m,x)}(t) dt \right) dt \]

\[ \leq \sum_{\mu} \int f_k(t) \left( \sum_{\nu < \mu} \int f_k \mathbb{H}^{(m,x)}(t) dt \right) dt \]

Let us now assume that \( t \in P \) and \( s \in Q \). Then (3.4) implies that \( |r-s| > |Q| \), and therefore, for \( t \in P \), by Lemma 2.2 and by (3.8)

\[ |\mathcal{T}_f(t)| \leq \sum_{\mu} |Q| \left( \int f_k \mathbb{H}^{(m,x)}(t) dt \right) \left( \sum_{\nu < \mu} \int f_k \mathbb{H}^{(m,x)}(t) dt \right) \]

\[ \leq C_{m,n} \sum_{\mu} \left( \int f_k \mathbb{H}^{(m,x)}(t) dt \right)^2 \left( \sum_{\nu < \mu} \int f_k \mathbb{H}^{(m,x)}(t) dt \right)^2 \]

\[ \leq C_{m,n} \sum_{\mu} \left( \int f_k \mathbb{H}^{(m,x)}(t) dt \right)^2 \left( \sum_{\nu < \mu} \int f_k \mathbb{H}^{(m,x)}(t) dt \right)^2 \]

Notice that (3.4) implies \( |Q| \leq \text{dist}(s, P) \) for \( s \in Q \) and that \( \text{dist}(r, Q) \geq |r-s|/2 \) for \( s \in Q \), \( t \in P \). Thus, for each \( i \) we have

\[ |Q| \left( \int f_k \mathbb{H}^{(m,x)}(t) dt \right)^2 \leq 4 \left( \int f_k \mathbb{H}^{(m,x)}(t) dt \right)^2 dt, \]

and therefore

\[ |\mathcal{T}_f(t)| \leq C_{m,n} \left( \int f_k \mathbb{H}^{(m,x)}(t) dt \right)^2 dt, \quad t \in P. \]

It now follows from a property of Marcinkiewicz integral (see [13], pp. 14–15) that for some constant \( C_{m,n} \),

\[ \left\| \mathcal{T}_f \right\|_p \leq C_{m,n} y \left[ Q \right]. \]

This and (3.2) give finally

\[ |\{ t \in I : |\mathcal{T}_f(t)| > y \}| \leq \left[ Q \right] + \left\{ t \in P : |\mathcal{T}_f(t)| > y \right\} \]

\[ \leq \left[ Q \right] + \frac{1}{y} \left\| \mathcal{T}_f \right\|_p \leq C_{m,n} \left[ Q \right] \]

\[ \leq C_{m,n} \frac{p^2}{p-1} \left\| f \right\|_p, \]

and therefore the proof is complete.

**Theorem 3.2.** Let \( p, m, k, m', k' \) and \( \varepsilon \) be given such that \( 1 < p < \infty \), \( |k| \leq m+1 \) and \( |k'| \leq m'+1 \). Then there exists a constant \( C_{m,n} \) such that

\[ (3.10) \quad \left\| \mathcal{T}_f \right\|_p \leq C_{m,n} \frac{p^2}{p-1} \left\| f \right\|_p. \]

**Proof.** Inequality (3.10) for \( 1 < p < \infty \) is a consequence of Lemma 2.1, Theorem 3.1, and the interpolation theorem of Marcinkiewicz (cf. [14], vol. II, (4.6)). Let now \( T^* : L_p(\mathbb{R}) \to L_p(\mathbb{R}), p^{-1} + q^{-1} = 1 \), denote the conjugate operation to \( T \). It is easily seen that

\[ T^* \mathbb{H} = \sum_{j=m}^{\infty} \sum_{k} \alpha_j \mathbb{H}^{(m,x)}(t) \mathbb{H}^{(m,x)} \mathbb{H}^{(m,x)}, \quad g \in L_p(\mathbb{R}). \]

According to Theorem 3.1, \( T^* \) is of weak type \((1,1)\) and, by Lemma 2.1, it is of strong type \((2,2)\) and therefore we have

\[ \left\| T^* \mathbb{H} \right\|_p \leq C_{m,n} \frac{p^2}{p-1} \left\| \mathbb{H} \right\|_p, \quad 1 < p < \infty. \]

Now, simple conjugacy argument gives (3.10) for all \( p, 1 < p < \infty \).

**Corollary 3.1.** Let \( m, k, p \) and \( \varepsilon \) be given such that \( |k| \leq m+1 \), \( 1 < p < \infty \). Then \( \mathbb{H}^{(m,x)}(t) \) is an unconditional basis in \( L_p(\mathbb{R}) \), and for some constant \( C_{m,n} \) we have

\[ \left\| \sum_{j=m}^{\infty} \sum_{k} \alpha_j \mathbb{H}^{(m,x)}(t) \mathbb{H}^{(m,x)} \mathbb{H}^{(m,x)} \right\|_p \leq C_{m,n} \frac{p^2}{p-1} \left\| \sum_{j=m}^{\infty} \sum_{k} \alpha_j \mathbb{H}^{(m,x)}(t) \mathbb{H}^{(m,x)} \mathbb{H}^{(m,x)} \right\|_p. \]

To obtain Corollary 3.1 we use Theorem 3.2 with \( m = m' \) and \( k = -k' \).

**Remarks.** Corollary 3.1 gives in particular the unconditionality of Haar (J. Marcinkiewicz [8]) and Franklin (S. V. Bolikari [1]) bases in \( L_p(\mathbb{R}), 1 < p < \infty. \) The first case corresponds to \( m = -1, k = 0 \), and the second case to \( m = k = 0. \) If
m is arbitrary and k = 0, then Corollary 3.1 can be obtained by direct extension of Bolchovtsev's result (cf. S. Rolka [11]).

COROLLARY 3.2. Let |k| \leq m+1, 1 \leq p \leq \infty, and let for f \in L_p(I)

\[ f = \sum_{n=-m}^{m} a_n h_n(k). \]

Then there is a constant C_{m,p} such that

\[ \|f\|_p \leq \left( \sum_{n=-m}^{m} |a_n h_n(k)|^p \right)^{1/p} \leq C_{m,p} \|f\|_p. \]

This follows from Corollary 3.1 by the known argument with the Khinchine inequality.

4. Equivalence of the spline bases in the L_p(I) spaces

The main object of this section is to prove for given p, 1 < p < \infty, the equivalence of all bases in L_p(I) belonging to the family \( \{h_n^{(m)}, m \geq |k| - m \} \) indexed by the pair of integers (m, k): |k| \leq m+1. It is sufficient of course to prove that the Haar system \( \{s_{n,m}\}, n \geq |k| - m \) for each pair (m, k): |k| \leq m+1. To do this we need two lemmas.

Let \( S_n(I) \) denote the linear span of \( \{h_n^{(m)}, -m \leq j \leq n\} \), and let \( I_{n,j} = (s_{n+1,j}, s_{n,j}) \) for \( j = 1, \ldots, n-1 \) and \( I_{n,0} = (s_{n,0}, s_{n+1,1}) \). The partition 0 = \( s_{n,0} < \ldots < s_{n,n+1} = 1 \) is defined as in the proof of Lemma 2.2.

**LEMMA 4.1.** Let \( m \geq 0 \) be given. Then for some constant C_m,

\[ \frac{\phi(t) - \phi(s)}{t-s} \leq C_m n \frac{1}{|I_{n,j}|} \int_{|I_{n,j}|} |\phi|, \quad t, s \in I_{n,j} \]

holds for \( j = 1, \ldots, n; \phi \in S_n(I) \).

**Proof.** The case of \( m = 0 \) is trivial. Let us assume, therefore, that \( m \geq 0 \). The argument is now carried out in two steps.

**First step.** For given n we introduce the family of partitions \( \pi_{n,j} = \{d_1^{(m)}, \ldots, d_{n+1}^{(m)}\} \), \( j = -m, \ldots, m+2 \), which is described by the following properties:

\[ d_1^{(m)} \leq \ldots \leq d_{n+1}^{(m)} = 0, \quad s_0^{(m)} = 0, \quad s_0^{(m)} = 1, \quad \text{and} \quad s_{n+1}^{(m)} = \cdots = \frac{1}{2} s_{n+1}^{(m)}, \quad j = -m, \ldots, m+2, \]

where the square brackets stand for the desired difference. For each \( j = -m \leq i \leq m+1 \), \( s_i^{(m)} \) is the subinterval of \( S_n(I) \), i.e., in the space of polynomials on I of degree not exceeding n+1. Thus, each \( \phi \in S_n(I) \) has unique representation

\[ \phi = \sum_{i=-m}^{m} a_i s_i^{(m)}. \]

Writing \( a_i^{(m)} = (a_0, \ldots, a_{|k|}) \) and \( \|a_i^{(m)}\|_1 = \|a_0^{(m)} + \cdots + a_{|k|}^{(m)}\|_1 \), we find easily that there is a constant C_m such that

\[ C_m \|a_i^{(m)}\|_1 \leq \|\phi\|_1 \leq C_m \|a_i^{(m)}\|_1, \quad j = -m, \ldots, m+2. \]

**Second step.** Let us assume that \( \phi \in S_n(I) \). The function \( \phi \) has therefore the unique representation

\[ \phi = \sum_{j=-m}^{m} \xi_j N_{n,j}^{(m)}(c), \]

where \( N_{n,j}^{(m)} \), \( i = -m, \ldots, n \), are the B-splines corresponding to the dyadic partition \( \{s_{n,i}, i = 0, \pm 1, \pm 2, \ldots\} \).

Let now \( t, s \in I_{n,j} \). Then

\[ \phi(t) - \phi(s) = \sum_{|l| = m+1} \xi_l N_{n,j}^{(m)}(t) - N_{n,j}^{(m)}(s) = \sum_{|l| = m+1} \xi_l \int_{s}^{t} DN_{n,j}^{(m)}(s) \, ds, \]

whence we infer

\[ |\phi(t) - \phi(s)| \leq |t-s| \sum_{|l| = m+1} \xi_l \|DN_{n,j}^{(m)}\|_\infty \leq |t-s| 2n \sum_{|l| = m+1} |\xi_l|. \]

On the other hand,

\[ \frac{1}{|I_{n,j}|} \int_{I_{n,j}} |\phi| = \frac{1}{|I_{n,j}|} \int_{|I_{n,j}|} \sum_{|l| = m+1} \xi_l N_{n,j}^{(m)}(c) \, dc = \frac{1}{2} \sum_{|l| = m+1} \sum_{|l| = m+1} \xi_l N_{n,j}^{(m)}(c_{n,j} + s_{n,j+1}) |\phi|. \]

Now, to complete the proof it is sufficient to apply the left-hand side of (4.1) to (4.3) and to combine the obtained result with (4.2).

**COROLLARY 4.1.** If \( m \geq 0 \) and \( \phi \in S_n(I) \), then

\[ |D\phi(t)| \leq C_m \cdot n^{-1} \frac{1}{|I_{n,j}|} \int_{|I_{n,j}|} |\phi|, \quad t \in I_{n,j}, \quad j = 1, \ldots, n. \]

For further use let for given m, k, \( |k| \leq m+1 \), and \( n > 1 \), the integers \( r, \mu \) and \( \lambda \) be given as follows: \( r = m+1-|k| \), \( 2^{-r+1} < 2^r \), \( 2^r < n \leq 2^{r+1} \). Notice that \( r > 0 \). Moreover, let \( f(t) = f(1-t), \ t \in I \).

**LEMMA 4.2.** Let m and k, \( |k| \leq m+1 \) be given. Moreover, let \( n > 2^{r+1} \), i.e. \( \lambda > \mu \). Then for some constant C_m we have

(i) \( \|x(k)\|_{C_m} \leq C_m M_{2^{r+1}}^{(m, 2^{r+1})} \) for \( 2^{r+1} < n \),

(ii) \( \|x(k)\|_{C_m} \leq C_m M_{2^{r+1}}^{(m, 2^{r+1})} \) for \( n < 2^{r+1} \),

(iii) \( \|x(k)\|_{C_m} \leq C_m M_{2^{r+1}}^{(m, 2^{r+1})} \) for \( 2^{r+1} < n \),

(iv) \( \|x(k)\|_{C_m} \leq C_m M_{2^{r+1}}^{(m, 2^{r+1})} \) for \( n < 2^{r+1} \).
Proof. Notice that $S_{n-r}^{m-k} I(f)$
Let now assume at first $2^l+r < n$. Then $n-r = 2^l + l$, $1 \leq l \leq 2^l$ and $n = 2^l + l'$, $1 \leq l' = 1 + r < 2^l$.

The proof of (i). In the case of $k = m+1$ we have $S_{n-r}^{m-k} I(f)$ and therefore (i) is a consequence of the properties (4) and (i) ($p = \infty$) of Section 2. For $k < m+1$, inequality (i) follows by Corollary 4.1 and properties (i) ($p = \infty$) and (4) of Section 2.

The proof of (ii). This inequality follows by property (5) of Section 2.

The second case corresponds to $2^l + r \geq n$, and therefore $2^l - 1 - r < 2^l$. Thus, $n-r = 2^l - 1 + l$ with $1 \leq 2^l - 1 - l < 2^l$, and $n = 2^l + l'$ with $1 \leq l' \leq r < 2^l$. In particular, we infer that $r > 0$.

The proof of (ii). Corollary 4.1, properties (i) and (4) give (ii).

The proof of (iv). It is sufficient to apply property (5).

THEOREM 4.1. Let $m$, $k$, and $p$ be given such that $|k| \leq m+1$, $1 < p < \infty$. Then the Haar system \( \{ \varphi_n \}_{n=1}^{\infty} \) and \( \{ k_{n-k}^{m-k} \}_{n=|k|-m}^{\infty} \) are equivalent bases in the space $L_q(I)$, i.e. the following two series

\[
\sum_{n=1}^{\infty} a_n \varphi_n \tag{4.4}
\]

and

\[
\sum_{n=|k|-m+1}^{\infty} a_{n-k} k_{n-k}^{m-k} \tag{4.5}
\]

are equiconvergent in the space $L_q(I)$.

Proof. Let $f$ denote the sum of (4.4), and let $g$ be the sum of (4.5). It is enough to show the existence of such $C_{n,q}$ that

\[ C_{n,q}^{-1} ||f|| \leq ||g|| \leq C_{n,q} ||f||. \]

However, this follows by Corollary 3.2, Lemma 4.2, and Lemma 2.4.

5. Almost everywhere convergence for bounded spline systems

In order to state the maximal inequality we introduce the following notation:

\[
W_{k,n}^{m-k} f = \sum_{j=|k|-m}^{n} (f, w_{j}^{m-k}) w_{j}^{m-k},
\]

\[
W_{k,n}^{m-k} f = \sup_{n} |W_{k,n}^{m-k} f|.
\]

THEOREM 5.1. Let $m$, $k$, and $p$ be given such that $|k| \leq m+1$, $1 < p < \infty$. Then there exists constant $C_n$ such that

\[ ||W_{k,n}^{m-k} f|| \leq C_{n-q} \frac{p^2}{(p-1)^2} ||f||_p, \quad f \in L_q(I). \]

Proof. The idea of the proof is related to the argument used in [5] and therefore the present reasoning is being restricted to the main steps only. Let us define

\[ T_f = \sum_{j=|k|-m}^{n} (f, k_{j}^{m-k}) k_{j}^{m-k}. \]

Then the following inequalities can be derived

\[
W_{k,n}^{m-k} f \leq W_{k,n}^{m-k} f + \sup_{n=1}^{m} \left| (W_{k,n}^{m-k} f - W_{k,n}^{m-k} f) \right| \leq H_{k,n}^{m-k} f + C_n M \left( \sup_{n=m}^{m+1} \left| (W_{k,n}^{m-k} f - W_{k,n}^{m-k} f) \right| \right) \leq H_{k,n}^{m-k} f + C_n M W_{k,n}^{m-k} f + C_n M H_{k,n}^{m-k} f.
\]

It now follows by Theorem 3.2 that

\[ ||T_f||_p \leq C_{n-q} \frac{p^2}{(p-1)^2} ||f||_p. \]

Corollary 2.1 gives

\[ ||H_{k,n}^{m-k} f||_p \leq C_{n-q} \frac{p^2}{(p-1)^2} ||f||_p, \quad |k| \leq m+1. \]

The Hardy–Littlewood theorem implies

\[ ||M||_p \leq C_{n-q} \frac{p^2}{(p-1)^2} ||f||_p. \]

Finally, P. Sjölin [12] proved that

\[ ||W_{k,n}^{m-k} f||_p \leq C_{n-q} \frac{p^2}{(p-1)^2} ||f||_p. \]

Combining the five inequalities (5.1)–(5.5), we obtain the required inequality in the statement of Theorem 5.2.

COROLLARY 5.1. Let $k$, $m$, and $p$ be given such that $|k| \leq m+1$, $1 < p < \infty$. Then, for every $f \in L_q(I)$, we have

\[ f(t) = \sum_{n=|k|-m}^{n} (f, w_{j}^{m-k}) w_{j}^{m-k}(t) a.e. \text{ in } I. \]

Moreover, the series converges in the $L_q$ norm.

6. Final remarks

The limiting cases of $p = 1$ and $p = \infty$ could be discussed in similar way as in [5]. However, the results suggested by work [5] need not to be final, and the delicate discussion how sharp the estimates are requires new ideas.
The question whether \((5.6)\) does not hold in the case of \(k \neq 0\) for some \(f \in L_1(I)\) remains open. For \(k = 0\), \(\|f\|_{\infty} = 0\), \(n \gg -m\) is a uniformly bounded orthonormal system and therefore, by a recent result of S. V. Bočkariev, there is a function \(f \in L_1(I)\) such that the series in \((5.6)\) diverges on a set of positive Lebesgue measure. (Cf. S. V. Bočkariev, Divergent on a set of positive measure Fourier series for arbitrary bounded orthonormal set, Mat. Sb. 98 (1975), pp. 436-449.)

References


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