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LOCAL LIPSCHITZ CONTINUITY OF THE METRIC PROJECTION OPERATOR

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1. Introduction

X is a normed linear space with unit sphere S, and Y is a closed linear subspace of X. If $x \in X$, then $P_Y x$ denotes the set of elements $y \in Y$ such that

$$||x-y|| = \inf_{y' \in Y} ||x-y'||.$$

The (set-valued) operator P_Y is called the *metric projection operator* (m.p.o.) *onto* Y, and here it will just be studied at points where it is single-valued; however, see Remark 1.

If $||x-y|| \geqslant ||x||$ for all $y \in Y$, then x is said to be *metrically orthogonal to* Y and after a translation (by $P_Y x$) we can (and shall) here assume that so is the case. For the basic properties of P_Y see Holmes and Kripke [5].

The purpose of this paper is to study the local continuity of P_Y at an element x_0 metrically orthogonal to a fixed subspace Y; more precisely, if $||x-x_0|| \leqslant \varepsilon$, how small is then $||P_Y x - P_Y x_0|| = ||P_Y x||$?

We say that P_Y is *locally* Lipα *at* x_0 if in a neighbourhood of x_0 we have

$$||P_Y x - P_Y x_0|| \leqslant C_{x_0} ||x-x_0||^{\alpha},$$

for some constant C_{x_0}. The magnitude of C_{x_0} depends on the size of the neighbourhood of x_0 considered and will not be investigated here.

There are two fairly general methods for the study of the continuity of the m.p.o.: the first method, described in Section 2, is frequently simple to apply but does not so often give sharp results. However, it can be used if one is interested just to establish that P_Y *is* continuous, cf. Kahane [6]. The second method (Section 3), which can be considered as a refinement of the first one, is applicable e.g. when geometric properties like convexity and smoothness of the unit sphere at $x_0/||x_0||$ are known.

This method gives sharp estimates in the classical L^p-spaces as is shown by counter-examples in Section 4. Section 4 also contains some applications of the results derived by the second method.

2. Continuity of the m.p.o. via strong unicity of the best approximation

Suppose that x_0 has a "strongly unique" best approximating element in Y in the sense that there exists a positive increasing function $\varkappa_Y(x_0, \cdot)$, such that

$$(1) \qquad \|x_0-y\|-\|x_0\| \geqslant \varkappa_Y(x_0, \|y\|),$$

for every $y \in Y$. By elementary inequalities,

$$\|x_0-P_Yx\| \leqslant \|x_0-x\|+\|x-P_Yx\| \leqslant \|x_0-x\|+\|x\| \leqslant 2\|x_0-x\|+\|x_0\|,$$

so $\varkappa_Y(x_0, \|P_Yx\|) \leqslant 2\|x_0-x\|$ and hence

$$(2) \qquad \|P_Yx\| \leqslant \varkappa_Y^{-1}(x_0, 2\|x_0-x\|),$$

where $\varkappa^{-1}$ denotes the inverse function to $\varkappa$.

Let us see what this rough method gives in some concrete cases. If Y is finite dimensional and x_0 has a unique best approximation in Y, then the existence of a $\varkappa$-function is obvious, but there remains to estimate it from below.

(i) Let Q be a compact Hausdorff space, let $X = C(Q)$ and Y be a Haar subspace of X. D. J. Newman and H. S. Shapiro proved in 1963 the following "strong unicity" theorem for f metrically orthogonal to Y (Shapiro [11], pp. 24–27):

$$\|f-p\|-\|f\| \geqslant c_{f,Y}\|p\| \qquad C(Q) \text{ real},$$

$$\|f-p\|-\|f\| \geqslant c_{f,Y}\|p\|^2 \qquad C(Q) \text{ complex},$$

if $p \in Y$ and in the complex case $\|p\| \leqslant K$. Thus, by (2), P_Y is locally Lip 1 in the real case, a fact proved directly by G. Freud already in 1958. It is also easily demonstrated that the Lipschitz constant C_f is unbounded as f ranges over S. In the complex case, (2) gives Lip(1/2) and it has been shown by the present author and, independently, by R. Wegmann ([13]) that this result is sharp.

(ii) Let I be a compact interval of the real line, let dx denote Lebesgue measure, and take X to be the continuous functions in $L^1(I, dx)$. Then we know (Jackson–Krein) that if Y is a Haar subspace, then P_Y is single-valued. When X is real, this unicity result can be strengthened as follows.

Let ω_f be the modulus of uniform continuity of f. Then, for δ sufficiently small,

$$(3) \qquad \|f-p\|-\|f\| \geqslant 2\int_0^{\omega_f^{-1}(c\delta)} c\delta-\omega_f(x)\,dx,$$

where $c = c_{f,Y}$ and $p \in Y$ satisfies $\|p\| \geqslant \delta$. In particular, if $\omega_f(\delta) = O(\delta^\alpha)$, then the integral in (3) *is of order $\delta^{1+\alpha^{-1}}$.*

Combination of (2) and (3) gives an estimate for the continuity of P_Y at f and, for any Haar subspace, there are functions such that this estimate is of the right order.

(iii) Let X be uniformly convex with modulus of uniform convexity δ_X and let $\|x_0\| = 1$. If $y \in Y$, one easily obtains

$$\|x_0-y\|-\|x_0\| \geqslant \|x_0-y\|\,\delta(\|y\|/\|x_0-y\|) \geqslant \delta(\|y\|/\|x_0-y\|),$$

so if $\|y\| \leqslant 3$, then $\|x_0-y\|-\|x_0\| \geqslant \delta(\|y\|/4)$. Hence, for $\|x-x_0\| \leqslant 1/2$,

$$(4) \qquad \|P_Yx\| \leqslant 4\delta^{-1}(2\|x-x_0\|),$$

i.e. essentially the inverse of δ_X gives a bound for P_Y. Note that the estimate is uniform in x_0, as x_0 varies over S. R. Wegmann ([13]) has shown that, if δ_X is convex (which is not always the case), one even has $\|P_Yx\| \leqslant 2\delta^{-1}(\|x-x_0\|)$. The most well-known uniformly convex spaces are the classical L^p-spaces, and for $1 < p < 2$, $\delta(\varepsilon) = (p-1)\varepsilon^2/8+O(\varepsilon^4)$ and in the range $2 < p < \infty$, $\delta(\varepsilon) = \varepsilon^p/p2^p+ O(\varepsilon^{2p})$. Hence, we can say that P_Y is locally Lip(1/2), respectively Lip$(1/p)$. However, these estimates are not sharp for any choise of x_0 and Y; see Corollary 3.

Finally, it should be mentioned that the strong unicity technique has been utilized by J. P. Kahane in his study of the m.p.o. onto closed translation-invariant subspaces of $L^1(T)$. He found e.g. that for every subspace Y of $L^1(T)$, such that P_Y is single-valued, P_Y is also continuous; Kahane [6], [7].

3. Continuity of the m.p.o. — geometric approach

For this method, two new geometric moduli must be introduced, but first let us recall the definitions of some well-known moduli which will be used in the corollaries. Besides the modulus of uniform convexity δ_X of X we have the modulus of uniform smoothness ϱ_X, defined as follows

$$(5) \qquad \varrho_X(\tau) = \sup_{\substack{\|x\|=1\\ \|y\|=\tau}} (\|x+y\|+\|x-y\|-2)/2.$$

X is *uniformly smooth* if $\varrho_X(\tau) = o(\tau)$ and then, in particular, X is smooth which means that through each point on S there is only one hyperplane supporting S; a point with this property is called a *smooth point*. M. Day showed that X is uniformly smooth if and only if its dual space X^* is uniformly convex, and this fact was given an exact quantitative formulation by the following duality relation of Lindenstrauss [8]:

$$(6) \qquad \varrho_X(\tau) = \sup_{0\leqslant\varepsilon\leqslant 2} \{\tau\varepsilon/2-\delta_{X^*}(\varepsilon)\}.$$

In this formula the positions of X and X^* can be changed and also ϱ and δ may change places if δ is replaced by its largest convex minorant.

If in (5) x is kept fixed, one obtains a local modulus of smoothness $\varrho_X(x, \tau)$ and we can obtain a generalisation of Lindenstrauss duality relation, if we define a local modulus of convexity as follows. Assume, just for simplicity of notation, that x_0 is a smooth point on S and let f_0 be the unique element on the unit sphere of X^* (the "dual point") that peaks at x_0, i.e. satisfies $f_0(x_0) = \|f_0\| = 1$. Then, if

$$\delta_X(x_0, \varepsilon) = \inf_{\substack{x, y\in S\\ \|x-y\|\geqslant\varepsilon}} \{1-f_0(x+y)/2\},$$

we will have

(7) $$\varrho_X(x_0,\tau) = \sup_{0\le\varepsilon\le 2}\{\tau\varepsilon/2-\delta_{X^*}(f_0,\varepsilon)\}.$$

Using (7) and its variants, one can compare the shape of the unit spheres at dual points. However, the local moduli $\delta_X(x_0,\cdot)$ and $\varrho_X(x_0,\cdot)$, which are studied in the literature, are still not what we want to study the m.p.o. at x_0, so we introduce even subtler measures of the geometry of S at x_0.

If x_0 is a smooth point and f_0 its peaking functional, let $x_\alpha = (1-\alpha)x_0$ and $H_\alpha = \{x\colon f_0(x) = 1-\alpha\}$. Define, for $0\le\alpha\le 1$,

$$\omega(x_0,\alpha) = \inf_{x\in H_\alpha\cap S}\|x-x_\alpha\|,$$

$$\Omega(x_0,\alpha) = \sup_{x\in H_\alpha\cap S}\|x-x_\alpha\|.$$

(*Problem.* Find duality relations for ω and Ω.)

If x_0 is non-smooth, we have to write $H_{f_0,\alpha}$, $\omega(x_0,f_0,\alpha)$, etc., for each peaking functional f_0 and can then in Lemma 2 and Theorem 2 take the infimum over the set of functionals peaking at x_0. We assume from now on, that x_0 is a smooth point.

For the corollaries we need the following estimate.

LEMMA 1. (i) $\omega(x_0,\alpha)\ge(1-\alpha)\varrho^{-1}(x_0,\alpha/2(1-\alpha))$, *so also*

$$\omega(x_0,\alpha)\ge(1-\alpha)\varrho^{-1}(\alpha/2(1-\alpha)).$$

(ii) $\Omega(x_0,\alpha)\le\delta^{-1}(x_0,\alpha)$, *so also*

$$\Omega(x_0,\alpha)\le\delta^{-1}(\alpha).$$

Proof. (i) If $\theta>1$, there is an $x\in H_\alpha\cap S$ such that $u = x-x_\alpha$ satisfies $\|u\|\le\theta\omega(x_0,\alpha)$. Now,

$$\varrho\left(x_0,\frac{\|u\|}{1-\alpha}\right)\ge\frac{1}{2}\left(\left\|x_0+\frac{u}{1-\alpha}\right\|+\left\|x_0-\frac{u}{1-\alpha}\right\|\right)-1$$
$$\ge\frac{1}{2}\left(\frac{1}{1-\alpha}+1\right)-1 = \frac{\alpha}{2(1-\alpha)};$$

hence,

$$\|u\|\ge(1-\alpha)\varrho^{-1}(x_0,\alpha/2(1-\alpha))$$

and since $\theta\omega(x_0,\alpha)\ge(1-\alpha)\varrho^{-1}(x_0,\alpha/2(1-\alpha))$ holds for all $\theta>1$, x_0 and α being fixed, it must hold also for $\theta = 1$.

(ii) We have $f_0(x+y)/2 = 1-\alpha$, if $x,y\in H_\alpha\cap S$, so certainly

$$\sup_{\substack{x,y\in S\\ \|x-y\|\ge\Omega(x_0,\alpha)}} f_0(x+y)/2\ge 1-\alpha.$$

Then

$$\alpha\ge\inf_{\substack{x,y\in S\\ \|x-y\|\ge\Omega(x_0,\alpha)}}\{1-f_0(x+y)/2\} = \delta(x_0,\Omega(x_0,\alpha)).\ \blacksquare$$

Before proceeding, recall that x_0 is metrically orthogonal to Y if and only if there is an element f in the annihilator of Y peaking at x_0. The (solid) unit ball of X is denoted by B.

LEMMA 2. *Let there be a positive* ε_0, *such that if* $\|x-x_0\|<\varepsilon_0$, *then* P_Y *is single-valued and* $\|P_Yx\|<\beta$ *for some* $\beta<1$. *Then, if* $\varepsilon\le\varepsilon_0/2$,

$$\sup_{\|x-x_0\|\le\varepsilon}\|P_Yx\|\le\sup_{x\in H_\alpha\cap B}\|P_Yx\|,$$

where

$$\alpha = \omega^{-1}\left(x_0,\frac{\beta+2}{1-\beta}\varepsilon\right).$$

Proof. Let $\|x-x_0\|<\varepsilon\le\varepsilon_0/2$, so $\|P_Yx\|<\beta<1$. The line through x and P_Yx intersects S at x', with $f(x')>0$. Define α by $f(x') = 1-\alpha$. Then,

$$x' = P_Yx+\frac{1-\alpha}{f(x)}(x-P_Yx),$$

so

$$\|x'-x_\alpha\| = \left\|P_Yx+\frac{1-\alpha}{f(x)}(x-P_Yx)-(1-\alpha)x_0\right\|$$
$$\le\left|\frac{f(x)-1+\alpha}{f(x)}\right|\|P_Yx\|+\frac{1-\alpha}{f(x)}\|x-f(x)x_0\|$$
$$\le\frac{\varepsilon+\alpha}{1-\varepsilon}\beta+\frac{(1-\alpha)2\varepsilon}{1-\varepsilon} = \frac{\alpha(\beta-2\varepsilon)+(\beta+2)\varepsilon}{1-\varepsilon}.$$

Since $\alpha\le\|x'-x_\alpha\|$, one obtains an estimate for α,

$$\alpha\le\frac{\beta+2}{1-\beta+\varepsilon}\varepsilon$$

which then gives (if $\varepsilon\le\beta/2$, which is the case since $\varepsilon_0\le\beta$)

$$\|x'-x_\alpha\|\le\frac{\beta+2}{1-\beta+\varepsilon}\varepsilon\le\frac{\beta+2}{1-\beta}\varepsilon,$$

so actually one has

$$\alpha\le\omega^{-1}(x_0,\varepsilon(\beta+2)/(1-\beta)).$$

Since $P_Yx'' = P_Yx$, if x'' is on the ray through x and P_Yx, the conclusion follows. $\blacksquare$

Remark 1. The condition on single-valuedness may be dropped if $\|P_Yx\|$ is replaced by $\sup\|y\|$ over the set of elements $y\in Y$, such that $x-y$ is metrically orthogonal to Y.

The following theorem is now a simple consequence of the geometry of $H_\alpha\cap S$; the more "rounded" $H_\alpha\cap S$ is (in the sense that ω and Ω are of similar order), the stronger continuity of P_Y at x_0. If $H_\alpha\cap S$ is "needle-shaped", P_Y may have poor continuity properties as is shown by counter-examples in Section 4. Hence, the relevant condition is how S curves away from the supporting hyperplane through x_0 in *different* directions.

THEOREM 1. *Let there be a positive ε_0 such that if $\|x-x_0\| < \varepsilon_0$, then P_Y is single-valued and $\|P_Y x\| < \beta$ for some $\beta < 1$. Then, if $\varepsilon \leqslant \varepsilon_0/2$,*

$$\sup_{\|x-x_0\|<\varepsilon} \|P_Y x\| \leqslant 2\Omega(x_0, \alpha), \tag{8}$$

where

$$\alpha = \omega^{-1}\left(x_0, \frac{\beta+2}{1-\beta}\varepsilon\right).$$

Proof. Instead of considering elements within distance ε ($\leqslant \varepsilon_0/2$) of x_0 we can, by Lemma 2 (and its proof), consider elements in $H_\alpha \cap B$ where $\alpha = \omega^{-1}\left(x_0, \frac{\beta+2}{1-\beta}\varepsilon\right)$ and assume they have unique best approximations in Y. So, let x be such and consider the affine subspace $x+Y = \{x+y\colon y \in Y\}$ which lies in H_α. Now contract B until it has just one point x' in common with $x+Y$ (which is possible, because P_Y is single-valued at x). Evidently, $P_Y x' = 0$ so if $x' = x-y$, $y \in Y$, $P_Y x = y+ +P_Y x' = y$. Since both x and x' are in $H_\alpha \cap B$, $\|y\| = \|x-x'\| \leqslant 2\Omega(x_0, \alpha)$. ∎

Remark 1 applies again. Note that the estimate (8) is uniform over all subspaces Y such that x_0 is metrically orthogonal to Y (but if x_0 is non-smooth, remember the remark preceding Lemma 1), so for certain "directions" of Y, P_Y may be much better than what follows from (8). Another level of uniformity has been studied by V. I. Berdyshev, see [2], namely uniform continuity of P_Y as x_0 ranges over S and Y over all closed subspaces of X.

If $\delta(x_0, \alpha) > 0$ for $\alpha > 0$, then P_Y is continuous at x_0, so for x sufficiently close to x_0 we will have $\|P_Y x\| < \beta = 1/4$ and for this β, $(\beta+2)/(1-\beta) = 3$. If $\alpha \leqslant 1/2$, then by Lemma 1, $\omega(x_0, \alpha) \geqslant (1/2)\varrho^{-1}(x_0, \alpha/2)$, i.e. $\omega^{-1}(x_0, \alpha) \leqslant 2\varrho(x_0, 2\alpha)$. Hence, $\varepsilon \leqslant 1/6$ implies $\omega^{-1}(x_0, 3\varepsilon) \leqslant 2\varrho(x_0, 6\varepsilon)$.

COROLLARY 1. *Let x_0 ($\|x_0\| = 1$) be metrically orthogonal to the closed subspace Y and suppose $\delta(x_0, \varepsilon) > 0$ for $\varepsilon > 0$. Then, for $\varepsilon \leqslant \varepsilon_{x_0}$,*

$$\sup_{\|x-x_0\|\leqslant\varepsilon} \|P_Y x\| \leqslant 2\delta^{-1}\big(x_0, 2\varrho(x_0, 6\varepsilon)\big).$$

We see that convexity is the decisive condition, but that smoothness of S at x_0 improves the continuity of the m.p.o. at that point; then we also get a better result than what is in general obtainable by the strong unicity method, cf. (4). Note that we will have similar estimates for P_K if K is a closed convex set in X; the constants may be bigger if $\mathrm{dist}(x_0, K)$ is large. In a uniformly convex space the estimate is uniform over the unit sphere:

COROLLARY 2. *In the uniformly convex space X, let x_0 ($\|x_0\| = 1$) be metrically orthogonal to the closed subspace Y. Then, for $\varepsilon \leqslant \varepsilon_0$,*

$$\sup_{\|x-x_0\|\leqslant\varepsilon} \|P_Y x\| \leqslant 2\delta^{-1}\big(2\varrho(6\varepsilon)\big).$$

It is known, that many Orlicz spaces have uniformly convex and uniformly smooth norms, or at least have an equivalent norm with these properties, and T. Figiel has shown how under certain conditions (essentially the Δ_2 condition), the moduli of uniform convexity and smoothness can be estimated by means of the Orlicz function, see [3]. Hence, Corollary 2 is immediately applicable to such spaces, but here we content ourselves to the following simple (but important) application.

COROLLARY 3. *Let x_0 have norm one and be metrically orthogonal to the closed subspace Y of $L^p\{\mu\}$. Then there are constants ε_p and c_p, just depending on p, much that for $\varepsilon \leqslant \varepsilon_p$,*

$$\sup_{\|x-x_0\|\leqslant\varepsilon} \|P_Y x\| \leqslant c_p \varepsilon^{p/2}, \quad 1 < p \leqslant 2,$$

$$\sup_{\|x-x_0\|\leqslant\varepsilon} \|P_Y x\| \leqslant c_p \varepsilon^{2/p}, \quad 2 \leqslant p < \infty.$$

c_p remains bounded as $p \to \infty$ but, as derived here, behaves like $(p-1)^{-1/2}$ as $p \to 1$.

4. Applications and counterexamples

4.1. Normed linear spaces isomorphic to inner-product spaces

It is known that inner-product spaces are the most convex spaces in the sense that, for every space X, $\delta_X(\varepsilon) \leqslant \delta_H(\varepsilon)$, where $\delta_H(\varepsilon) = 1-\big(1-(\varepsilon/2)^2\big)^{1/2} = \varepsilon^2/8+O(\varepsilon^4)$ is the modulus of uniform convexity for an inner-product space. They are also the most globally smooth spaces, since always $\varrho_X(\tau) \geqslant \varrho_H(\tau) = (1+\tau^2)^{1/2}-1 = \tau^2/2+ +O(\tau^4)$. D. A. Senechalle has shown in [10] that, if for some null-sequence (ε_i), $\delta_X(\varepsilon_i)/\delta_H(\varepsilon_i) \to 1$, then X is (linearly isometric to) an inner-product space. Now, if we relax somewhat on the convexity condition but instead require high smoothness, then we have the following result.

THEOREM 2. *Let the normed linear space X have a uniformly convex norm which satisfies*

$$\delta_X(\varepsilon) \geqslant \lambda\varepsilon^2, \tag{9}$$

and an equivalent uniformly smooth norm which satisfies

$$\varrho_X(\tau) \leqslant \eta\tau^2. \tag{10}$$

Then X is isomorphic to an inner-product space.

Proof. First we use Asplunds' renorming technique (see [1]) to find a third equivalent norm on X which simultaneously satisfies inequalities like (9) and (10), and henceforth we work with that norm. We shall show that every closed subspace is the range of a uniformly continuous linear projection and thus is complemented. Let Y be a closed subspace of X; since X is reflexive, there are elements x_0 on S which are metrically orthogonal to Y and, by Corollary 2 and (9), (10), we have for $\varepsilon \leqslant \varepsilon_0$

$$\sup_{\|x-x_0\|\leqslant\varepsilon} \|P_Y x\| \leqslant K\varepsilon$$

with ε_0 and K independent of x_0. But then (cf. Holmes and Kripke [5], pp. 227–228),

$$||P_Y x_1 - P_Y x_2|| \leqslant \max(K, 2+2\varepsilon_0^{-1})\, ||x_1 - x_2||,$$

for all x_1, x_2 in X, i.e. P_Y is uniformly continuous. Then, by a result of Lindenstrauss, [9], p. 270, there exists a bounded *linear* projection onto Y (with no bigger norm that of P_Y). Hence, Y is complemented and by the Complemented Subspace Theorem the conclusion follows. ∎

This result was also derived in Lindenstrauss [8] under an additional basis condition. That basis condition is also removed in Figiel and Pisier [4], where the above theorem is proved by probabilistic methods. Possibly the conditions could be weakened somewhat, since (9) and (10) give a bound on the P_Y uniform over Y which is not necessary for our purposes.

4.2. Inheritance of smoothness

H. S. Shapiro has in [12] studied the regularity properties of the element of best approximation for the case when Y is a closed translation-invariant subspace of $L^p(T)$, T denoting the unit circle. He showed that $P_Y f$ might possess less regularity than f: the regularity being measured by the integral modulus of continuity: $f \in \Lambda_\alpha^p$, $0 < \alpha \leqslant 1$, if $||f - f_\tau||_p = O(\tau^\alpha)$. f may belong to Λ_1^p but $P_Y f$ belongs to no smoother class than Λ_β^p where $\beta = p^{-1} + (p-1)^{-1}$ ($\leqslant 1$). He also showed, just using the uniform convexity of L^p, that if $f \in \Lambda_\alpha^p$, then $P_Y f \in \Lambda_{\alpha/p}^p$, $2 < p < \infty$.

Since translation on T commutes with the operation of taking best approximation, it follows from Corollary 3 that

If $f \in \Lambda_\alpha^p$, then $P_Y f \in \Lambda_{p\alpha/2}^p$ for $1 < p < 2$ and $P_Y f \in \Lambda_{2\alpha/p}^p$ for $2 < p < \infty$.

Problem. Can this result be improved? A negative answer to this question would also solve an important problem on saturation of Fourier multipliers, see [12], p. 138.

4.3. Counterexamples in L^p-spaces

Now we shall see that the m.p.o. may possess no better continuity properties than what follows from the general geometric theory of Section 3.

THEOREM 3. *There exists a closed subspace Y of $X = L^p(I, dx)$ and an element f_0 in X such that, for $\varepsilon \leqslant \varepsilon_0$,*

(i) $$\sup_{||f-f_0|| \leqslant \varepsilon} ||P_Y f - P_Y f_0|| \geqslant c_p \varepsilon^{p/2}, \quad 1 < p < 2,$$

(ii) $$\sup_{||f-f_0|| \leqslant \varepsilon} ||P_Y f - P_Y f_0|| \geqslant c_p \varepsilon^{2/p}, \quad 2 < p < \infty,$$

for some positive constant $c_p = c_{p,Y,f_0}$.

Remark 2. A simpler version of the theorem, just giving the Lipschitz-exponents can be given a more intuitive proof. This proof is based on the fact that through a point x_n within distance ε_n from $x = 2^{(-1/p)}(1,1,0)$, one may draw a tangent, perpendicular to $(1,1,0)$, with touches the unit sphere of $l^p(3)$ at a distance of order $\varepsilon_n^{p/2}$ ($\varepsilon_n^{2/p}$) from x if $1 < p < 2$ $(2 < p < \infty)$; Fig. 1.a, 1.b. Let Y_n be

the line through the origin (subspace) parallel to this tangent. Now, in the consecutive 3-dimensional subspaces of $X = l^p(N)$ construct the corresponding Y_n (for some null-sequence (ε_n)) and let Y be the closed linear span of the Y_n. If x_0

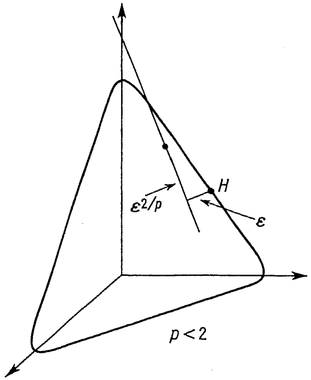

Fig. 1.a

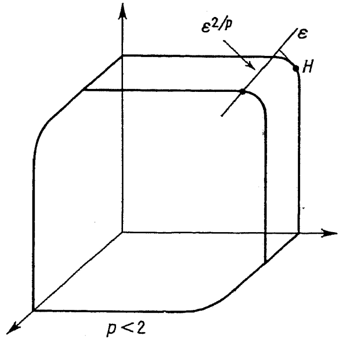

Fig. 1.b

$= (1, 1, 0, 2^{-1}, \ldots, n^{-1}, n^{-1}, 0, \ldots)$, we can then have an element x_n such that $||x_0 - x_n|| = n^{-1}\varepsilon_n$, $P_Y x_n = P_{Y_n} x_n$ and

$$||P_{Y_n} x_n|| \geqslant c_p n^{-1} \varepsilon_n^{p/2} = c_p n^{(p/2)-1} (n^{-1}\varepsilon_n)^{p/2}$$

if $1 < p < 2$. Hence, by choosing the ε_n rapidly decreasing, we see that an inequality $||P_Y x_n|| < K\varepsilon_n^{(p/2)+\delta}$ is impossible for any $\delta > 0$.

Similarly an inequality $||P_Y x_n|| < K\varepsilon_n^{(2/p)+\delta}$ is impossible if $2 < p < \infty$. One cannot work around the point $(1, 1, 0, 0, \ldots)$, since then x_n does not pick up its best approximation just from Y_n and we do not obtain the effect wanted here. Note, that x is a point where the unit sphere of $l^p(3)$ curves maximally in one direction and minimally in another; then the "global" Corollary 2 may give a sharp result.

Proof of Theorem 3. We shall work in $L^p([0,2], dx)$ and I supply the details for $2 < p < \infty$. Let $\varphi_n(x) = \mathrm{sgn}(\sin 2^n x)$ on $[0,1]$ (Rademacher functions on $[0,1]$) and $\varphi_n(x) = 0$ on $(1,2]$. $\psi_n(x) = 0$ on $[0,1]$ and $\psi_n(x) = \varphi_n(x-1)$ for $1 < x \leqslant 2$. Let $(\varepsilon_n)_{n=1}^\infty$ be a strictly decreasing null-sequence and let Y_n denote the one-dimensional subspace spanned by $\tilde{\varphi}_n = \varepsilon_n \varphi_n + \varepsilon_n^{2/p} \psi_n$ and $Y = \overline{\mathrm{span}}\{Y_1, Y_2, \ldots\}$. We shall study P_Y at $f_0(x)$, the characteristic function of $[0,1]$. To prove that $f_0(x)$ is metrically orthogonal to Y it suffices to show that f_0 is orthogonal to each Y_n (this is a property of L^p-spaces; see e.g. Shapiro [11], p. 56), i.e.

$$\int_0^2 \tilde{\varphi}_n(x)\, \mathrm{sgn} f_0(x)\, |f_0(x)|^{p-1} dx = 0$$

and obviously the integral vanishes here for each n. Put $f_n = f_0 + \varepsilon_n \varphi_n$; $f_n - P_{Y_n} f_n = f_n - t_n \tilde{\varphi}_n$ is orthogonal to every Y_m, since

$$\int_0^2 \tilde{\varphi}_m \operatorname{sgn}(f_n - P_{Y_n} f_n) \, |f_n - P_{Y_n} f_n|^{p-1} dx$$

$$= \varepsilon_m \int_0^1 \varphi_m \operatorname{sgn}\big(1 + (1-t_n)\varepsilon_n \varphi_n\big) \, |1 + (1-t_n)\varepsilon_n \varphi_n|^{p-1} dx +$$

$$+ \varepsilon_m^{2/p} \int_1^2 \psi_m \operatorname{sgn}(-t_n \psi_n) \, |t_n \varepsilon_n^{2/p} \psi_n|^{p-1} dx = 0.$$

If $m \neq n$, the integral over $[1, 2]$ trivially vanishes, and to see that also that over $[0, 1]$ is zero, just note that $|1 + (1-t_n)\varepsilon_n \varphi_n|^{p-1} = \alpha + \beta \varphi_n$. Hence $f_n - P_{Y_n} f_n$ is orthogonal to Y, which may be stated as $P_Y f_n = P_{Y_n} f_n$. Now, the norm of $P_{Y_n} f_n$ is easily estimated; we just compute the value $t = t_n$ for which $||f_n - t\tilde{\varphi}_n||$ is minimal and $||P_{Y_n} f_n|| = ||t_n \tilde{\varphi}_n||$:

$$||f_n - t\tilde{\varphi}_n||^p = \int_0^1 |1 + (1-t)\varepsilon_n \varphi_n|^p dx + \int_1^2 |t\varepsilon_\varepsilon^{2/p} \psi_n|^p dx$$

$$= (1/2) \left(|1 + (1-t)\varepsilon_n|^p + |1 - (1-t)\varepsilon_n|^p \right) + |t|^p \varepsilon_n^2.$$

It is easily verified that the minimum occurs for some t in $(0, 2)$ and hence for

$$(p\varepsilon_n/2) \Big(-\big(1 + (1-t)\varepsilon_n\big)^{p-1} + \big(1 - (1-t)\varepsilon_n\big)^{p-1} + 2t^{p-1}\varepsilon_n \Big) = 0.$$

Accordingly,

$$-(p-1)(1-t_n) + O(\varepsilon_n^2) + t_n^{p-1} = 0,$$

so certainly t_n cannot tend to zero as ε_n tends to zero, say $t_n \geqslant c_p$. In conclusion, we have

$$||f_0 - f_n|| = \varepsilon_n \quad \text{but} \quad ||P_Y f_0 - P_Y f_n|| = ||P_{Y_n} f_n|| \geqslant c_p \varepsilon_n^{2/p}.$$

For $1 < p < 2$ we use the same functions f_0, φ_n and ψ_n as above, but change the "direction" of Y_n (remember Fig. 1.a, 1.b); here let Y_n be spanned by $\tilde{\varphi}_n = \varepsilon_n^{p/2} \varphi_n + \varepsilon_n \psi_n$ and take $f_n(x) = f_0(x) + \varepsilon_n \psi_n(x)$. ■

Finally, a few comments about metric projections onto finite-dimensional subspaces of real L^p-spaces. Holmes and Kripke proved in [5] that, if $2 < p < \infty$ and Y is finite-dimensional, then P_Y is locally Lip 1. However, this result is no longer true for $1 < p < 2$; in $L^p([-1, 1], dx)$ take $f_0(x) = \operatorname{sgn} x \cdot |x|^{(2-p)^{-1}}$ and Y to be the constant functions on $[-1, 1]$. Let $f_\delta(x) = \delta$ on $[0, \delta^{2-p}]$ and $f_\delta(x) = f_0(x)$ on the rest of $[-1, 1]$. Then $||f_0 - f_\delta|| < \delta^{2/p}$ but

$$||P_Y f_0 - P_Y f_\delta|| = ||P_Y f_\delta|| \geqslant c_p |\delta/\ln\delta|,$$

i.e. P_Y *satisfies no higher Lipschitz-condition than* Lip$(p/2)$ *at* f_0.

On the other hand, *if $X = L^p$ is finite dimensional, then P_Y is always locally* Lip 1.

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