where $G = \omega(1)$; in particular, the calculation of the best constant G in inequality (3.10) may always be reduced to the problem of the best approximation in $L(0,\infty)$ of the function θ by functions $J^*_{\sigma-1}\zeta$.

References

[1] [H. Bateman and A. Erdelyi] Г. Бейтмен, А. Эрдейи, *Таблицы интегральных преобразований*, Наука, Москва 1970.

[2] [W. N. Gabushin] В. Н. Габушин, *Неравенства для норм функции и ее производных в метриках L_p*, Матем. заметки 1 (3) (1967), pp. 291–298.

[3] —, *Наилучшее приближение функционалов на некоторых множествах*, ibid. 8 (5) (1970), pp. 551–562.

[4] [S. P. Geisberg] С. П. Гейсберг, *Обобщение неравенства Адамара. Исследования по некоторым проблемам конструктивной теории функций*, Сб. науч. тр. ЛМИ 50, pp. 42–54, Ленинград 1965.

[5] —, *Дробные производные ограниченных на оси функций*, Изв. ВУЗов, Математика 11 (1968), pp. 51–69.

[6] J. Hadamard, *Sur le module maximum d'un fonction et de ses dérivées*, Soc. Math. France, Comptes rendus des Séances 41 (1914), pp. 68–72.

[7] G. H. Hardy and J. E. Littlewood, *Contribution to the arithmetic theory of series*, Proc. London Math. Soc. (2) 11 (1912), pp. 411–478.

[8] [A. N. Kolmogorov] А. Н. Колмогоров, *О неравенствах между верхними гранями последовательных производных произвольной функции на бесконечном интервале*, Уч. зап. Моск. Ун-та 30, Математика кн. 3 (1939), pp. 3–16.

[9] E. Landau, *Einige Ungleichungen fur zweimal differentierbare Funktionen*, Proc. London Math. Soc. (2) 13 (1913), pp. 43–49.

[10] [A. P. Matorin] А. П. Маторин, *О неравенствах между наибольшими значениями абсолютных величин функции и ее производных на полупрямой*, Укр. мат. журн. (1955), pp. 262–266.

[11] L. Neder, *Abschätzungen fur die Ableitungen einer reellen Funktion eines reellen Argumentes*, Math. Z. 31 (1930), pp. 356–365.

[12] I. J. Schoenberg and A. Cavaretta, *Solution of Landau's problem concerning higher derivatives on the halfline*, Proc. of Conference on Approximation theory, Varna 1970, pp. 297–308, Sofia 1972.

[13] [S. B. Stechkin] С. Б. Стечкин, *Наилучшее приближение линейных операторов*, Матем. заметки 1 (2) (1967), pp. 137–148.

[14] —, *О неравенствах между верхними гранями производных произвольной функции на полупрямой*, ibid. 1 (6) (1967), pp. 665–674.

[15] [E. Titchmarsh] Е. Титмарш, *Введение в теорию интегралов Фурье*, Гостехиздат, Москва–Ленинград 1948.

[16] [A. Zygmund] А. Зигмунд, *Тригонометрические ряды*, ГОНТИ, Москва–Ленинград 1939.

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ON THE UNIFORM CONTINUITY OF METRIC PROJECTION

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Let X denote a normed space, M a convex set in X, for $x \in X$, $xM = \inf\{\|x-y\|: y \in M\}$ the distance from x to M, $x_M = \{y \in M: \|x-y\| = xM\}$ the set of the elements from M of the best approximation for x. In what follows, M is assumed to be an existence set [9], i.e. x_M is nonempty for any $x \in X$. A map $x \to x_M$ (set-valued, in general) is called a *metric projection*. It is well known that the metric projection is single-valued for any existence set in the strictly convex space.

In Section 1 there is considered a problem of estimating $\|x_M - y_M\|$ from above via $\|x-y\|$ and geometric characteristics of X under the condition of strict convexity. Section 2 is devoted to the metric projection from $L(S,\mu)$ onto its finite-dimensional subspace. The main results are contained in Theorems 3 and 4.

1

Further we will denote by θ the zero of the space X and $V(x,r) = \{y \in X: \|x-y\| \leqslant r\}$ $(r > 0)$, $V = V(\theta, 1)$. Let $d(M) = \sup\{\|x-y\|: x, y \in M\}$ be the diameter of the set $M \subset X$. For a given set M contained in the plane $P \subset X$, we denote by $s(M)$ the width of M with respect to P, namely

$$s(M) = s(M)_P = \inf\{\sup_{x \in M-p} f(x) - \inf_{x \in M-p} f(x): f \in (P-p)^*, \|f\| = 1\},$$

where p is an element from P and $(P-p)^*$ is a conjugate space to $(P-p)$. Assume [1]

$$(1) \qquad \Omega(t) = \Omega(t)_X = \sup d(M \cap V) \quad (t \geqslant 0),$$

where the supremum is taken over all the hyperplanes $P \subset X$ such that $s(P \cap V)_P \leqslant t$. The function $\Omega(t)$ is nondecreasing and continuous in $(0, 2]$.

Let us estimate $\|x_M - y_M\|$ by means of Ω in the case of a strictly convex X.

LEMMA 1. *Let $M \subset X$ be the convex existence set, $x^1, x^2 \notin M$, $\|x^1_M - x^2_M\| > 0$; then the inequalities*

$$(2) \qquad \|x^1_M - x^2_M\| \leqslant \|x^1_L - x^2_L\|, \quad x^iM - 2\|x^1 - x^2\| \leqslant x^iL \leqslant x^iM \quad (i = 1, 2)$$

hold for the line $L = \{(1-\lambda)x^1_M + \lambda x^2_M: |\lambda| < \infty\}$.

Proof. The first inequality and the inequalities $x^iL \leqslant x^iM$ $(i = 1, 2)$ were established in [1] (see also [10]). It is easily seen that the element $z \in [x^1, x^2]$ with the property $z_L \cap M \neq \emptyset$ exists. Then

$$x^2L \geqslant zL - ||z-x^2|| \geqslant zM - ||z-x^2|| \geqslant x^2M - 2||z-x^2|| \geqslant x^2M - 2||x^1-x^2||. \ \blacksquare$$

THEOREM 1. *If X is a strictly convex space, and M is a convex existence set in X, $x, y \in X$, $||x-y|| = t$, $xM = r$, $r > 2t$, then*

$$(3) \qquad ||x_M - y_M|| \leqslant \min\left\{r\Omega\left(\frac{2t}{r-2t}\right) + 2t,\ 2(r+t)\right\}.$$

Proof. The equality $||x_M - y_M|| \leqslant 2(xM + ||x-y||)$ is obvious. In [1], Corollary, p. 802, it is shown that the inequality

$$(4) \qquad ||x_L - y_L|| \leqslant xL \cdot \Omega\left(2\frac{||x-y||}{xL}\right) + 2||x-y||$$

holds for line L and $x, y \in L$ in a strictly convex space. Assuming $L = \{(1-\lambda)x_M + \lambda y_M : |\lambda| < \infty\}$ and using (4), (2), we obtain (3). ■

If M is an existence plane in X, then under the conditions of Lemma 1, we have $x_L^i = x_M^i$ $(i = 1, 2)$ and, consequently, the inequality

$$(5) \qquad ||x_M^1 - x_M^2|| \leqslant x^1M \cdot \Omega\left(2\frac{||x^1-x^2||}{x^1M}\right) + 2||x^1-x^2||,$$

for $x^1, x^2 \notin M$.

Let us recall the definitions of

$$(6) \qquad \varepsilon(\alpha) = \sup\{||x^1-x^2|| : ||x^1|| = ||x^2|| = 1,\ 1 - ||x^1-x^2||/2 \leqslant \alpha\}, \quad 0 \leqslant \alpha < 1,$$

an inverse function of the convexity modulus [7] of the space X,

$$(7) \qquad \varrho(\tau) = \tfrac{1}{2}\sup\{||x+y|| + ||x-y|| - 2 : ||x|| = 1,\ ||y|| = \tau\}, \qquad \tau \geqslant 0,$$

the smoothness modulus of X [8]. In [2] another smoothness modulus was introduced, namely

$$(8) \qquad \varrho_1(\tau) = \sup\left\{1 - \frac{||x^1+x^2||}{2} : ||x^1|| = ||x^2|| = 1,\ ||x^1-x^2|| \leqslant \tau\right\}; \quad 0 \leqslant \tau \leqslant 2,$$

and it was proven, in particular, that

$$(9) \qquad \varrho_1(\tau) \leqslant \varrho(\tau/(2-\tau)), \qquad 0 \leqslant \tau < 2.$$

A space X is called *uniformly convex* if $\varepsilon(\alpha) \xrightarrow[\alpha \to 0]{} 0$ [7]. The following theorem is valid:

THEOREM (B. Björnestål [5]). *Let X be a uniformly convex space, and P a closed subspace in X, $x^0 \in X$, $x^0P = 1$; then there exists a number t_0 dependent on X only and such that*

$$(10) \qquad \sup_{||x^0-x|| \leqslant t} ||x_P^0 - x_P|| \leqslant 2\varepsilon(2\varrho(6t)).$$

The estimate (10) is of interest because in the case of L_p $(1 < p < \infty)$ spaces it cannot be improved with respect to the order of t, $t \to 0$, an the class of closed subspaces $P \in L_p$ [6].

Using (4), we will come to a similar estimate for a convex existence set.

THEOREM 2. *The following inequality holds:*

$$(11) \qquad \Omega(t) \leqslant \varepsilon(2\varrho_1(t)) \leqslant \varepsilon\big(2\varrho(t/(2-t))\big), \qquad 0 < t < 1.$$

Proof. For any $\gamma > 0$ there exists a plane $P \subset X$ such that $s(V \cap P) \leqslant t$, $d(V \cap P) \geqslant \Omega(t) - \gamma$. Let us have $x_1, x_2 \in V \cap P$ such that $||x_1 - x_2|| \geqslant d(V \cap P) - \gamma$ and write $x = (x_1+x_2)/2$, $x' = x/||x||$; then

$$(12) \qquad \Omega(t) \leqslant ||x_1 - x_2|| + 2\gamma \leqslant \varepsilon(||x-x'||) + 2\gamma.$$

Owing to $s(V \cap P) \leqslant t$ there exist $v_1, v_2 \in P$ satisfying the conditions $||v_1|| = ||v_2|| = 1$, $||v_1 - v_2|| \leqslant t$, $x \in [v_1, v_2]$. If $v = (v_1+v_2)/2$, $v' = v/||v||$, then

$$(13) \qquad ||v - v'|| \leqslant \varrho_1(t).$$

Now we estimate $||x-x'||$ by $||v-v'||$. Let w, $w \in [v_1, v_2]$, be such that the segments $[x', w]$, $[v', v]$ are parallel. The element w exists because $||v_1 - v_2|| \leqslant t < 1$. As can easily be seen,

$$||x' - w|| \leqslant 2||v - v'||, \qquad \frac{||x'-x||}{||x'-w||} = \frac{||x||}{||v||} = \frac{1 - ||x'-x||}{1 - ||v-v'||};$$

thus $||x-x'|| \leqslant 2||v-v'||/(1+||v-v'||)$. The last inequality and (12), (13) lead to $\Omega(t) \leqslant \varepsilon(2\varrho_1(t))$. Using (9), we complete the proof of the theorem. From (3) and (11) follows

THEOREM 3. *Let X be a strictly convex space, M a convex existence set in X and $r > 0$, $0 < t < \min\{1, r/3\}$; then*

$$(14) \qquad \sup\{||x_M - y_M|| : xM = r, ||x-y|| \leqslant t\} \leqslant r\Omega(2t/(r-2t)) + 2t \leqslant r\varepsilon\big(2\varrho_1(2t(r-2t))\big) + 2t \leqslant r\varepsilon\big(2\varrho(t/(r-3t))\big) + 2t.$$

2

In what follows we will deal with a space $L(S, \mu)$ of real functions integrable on the space (S, μ) with nonnegative measure μ. For $f \in L(S, \mu)$ denote as usual $||f|| = \int_S |f(t)|\, d\mu$. We will formulate several known results. Let P be a subspace in $L(S, \mu)$ (possibly infinite-dimensional). For $f \in L(S, \mu)$, $x \in P$ write

$$\Delta = f - x, \qquad Z = \{t \in S : f(t) = x(t)\}.$$

THEOREM [11]. *An element $x \in P$ is the best approximation element for $f \in L(S, \mu)$ iff*

$$(15) \qquad \left|\int_S y(t)\operatorname{sing}\Delta(t)\, d\mu\right| \leqslant \int_Z |y(t)|\, d\mu \qquad \forall y \in P.$$

THEOREM [3]. *The best approximation element $x \in P$ for $f \in L(S, \mu)$ is unique iff for any $y \in P$, $y \neq \theta$, at least one of the following conditions is true:*

(16) $$\mu\{t \in S \setminus Z:\ \operatorname{sign}\Delta(t) = \operatorname{sign} y(t),\ |y(t)| > |\Delta(t)|\} > 0,$$

(17) $$\int_S y(t)\operatorname{sign}\Delta(t)\,d\mu < \int_Z |y(t)|\,d\mu.$$

THEOREM [12]. *If a space (S, μ) has no atoms and P is a finite-dimensional subspace of $L(S, \mu)$, then there exists a $\psi \in L(S, \mu)$ such that*

(18) $$|\psi(t)| = 1 \quad \forall t \in S, \quad \int_S y(t)\psi(t)\,d\mu = 0 \quad \forall y \in P.$$

In what follows all the subsets from S in question are supposed to be measureable, and the symbol $\forall'$ means "for almost all".

LEMMA 2. *Let P be a finite-dimensional subspace in $L(S, \mu)$. For any $\gamma > 0$ there exist $\varepsilon > 0$ and a set $e \subset S$, $\mu e > 0$, such that*

(19) $$(f \in P,\ \|f\| < \varepsilon) \Rightarrow (|f(t)| < \gamma \ \ \forall t \in e).$$

Proof. The proof is evident.

LEMMA 3. *If the measure μ has no atoms and P is a finite-dimensional subspace of $L(S, \mu)$, then for any $\eta > 0$ there exists a set $E = E_\eta \subset S$, $0 < \mu E \leqslant \eta$ and a function $\psi \in L(E, \mu)$, $0 < \psi(t) < 1$ $\forall' t \in E$, such that*

(20) $$(\varphi \in P,\ \varphi(t) \leqslant \psi(t)\ \ \forall' t \in E) \Rightarrow (\varphi(t) \leqslant 0\ \ \forall' t \in S).$$

Proof. It is sufficient to verify (20) for $\varphi \in P$, $\|\varphi\| \leqslant 1$. The set

$$Q_m = \Big\{f \in P:\ \|f\| \leqslant 1,\ \int_{\{t \in S:\ (t) > 0\}} f(t)\,d\mu \geqslant 1/m\Big\} \quad (m = 1, 2, \ldots)$$

is compact. For every $f \in Q_m$ there exists a number $0 < \gamma_f < 1$ with the property $\mu\{t \in S:\ f(t) \geqslant \gamma_f\} > 0$. Using Lemma 2, we find a number $\varepsilon_f > 0$ and a set $e_f \subset \{t \in S:\ f(t) \geqslant \gamma_f\}$ such that $\mu e_f > 0$ and

(21) $$(\varphi \in P,\ \|f - \varphi\| < \varepsilon_f) \Rightarrow (\varphi(t) > f(t) - \gamma_f/2 \geqslant \gamma_f/2 \quad \forall' t \in e_f).$$

Thus Q_m is covered with the sets $V(f, \varepsilon_f) \subset P$, $(f \in Q_m)$. Owing to the compactness of Q_m, we can select a finite number of neighbourhoods $V_i^m = V(f_i^m, \varepsilon_{f_i^m})$, such that $Q_m \subset \bigcup_{i=1}^{N_m} V_i^m$. Write $\gamma_m = \min\{\gamma_{f_i^j}:\ i = 1, \ldots, N^j;\ j = 1, \ldots, m\}$. Since the measure μ is non-atomic, there exists a set $E_i^m \subset e_{f_i^m}$, $0 < \mu E_i^m \leqslant \eta/N_m 2^{m+1}$ $(i = 1, \ldots, N_m)$. Let $E^m = \bigcup_{i=1}^{N_m} E_i^m$; then $0 < \mu E^m \leqslant \eta/2^{m+1}$ and (see (21))

(22) $$\forall \varphi \in Q_m \exists e \subset E^m:\ \mu e > 0, \quad \varphi(t) > \gamma_m \quad \forall' t \in e.$$

Now we write

$$E = \bigcup_{m=1}^{\infty} E^m, \quad \psi(t) = \inf_{\{m:\ t \in E^m\}} \gamma_m \quad (t \in E);$$

then $\mu E \leqslant \eta$, $\mu\{t \in E:\ \psi(t) = 0\} = 0$. Thus for $\eta > 0$ we have built the set $E \subset S$, $0 < \mu E \leqslant \eta$ and the function $\psi \in L(E, \mu)$, $\psi(t) > 0$ $\forall' t \in E$, such that

(23) $$\forall \varphi \in \bigcup_{m=1}^{\infty} Q_m \exists e \in E:\ \mu e > 0, \quad \varphi(t) > \psi(t) \quad \forall' t \subset e.$$

Conditions (20) and (23) are equivalent.

THEOREM 4. *Let (S, μ) be a space with a non-negative σ-finite measure without atoms, P a finite dimensional subspace of $L(S, \mu)$, and y_0 an element of P. For any $\varepsilon > 0$, $\gamma > 0$ there exist functions $f_1, f_2 \in L(S, \mu)$ such that $\|f_1 - f_2\| < \varepsilon$, $\|f_i\| \leqslant \|y_0\| + \gamma$ $(i = 1, 2)$; moreover, y_0 (coresp. θ) is the only best approximation element from P for f_1 (coresp. f_2).*

Proof. Let the function $\psi \in L(S, \mu)$ satisfy conditions (18), $S^{\pm} = \{t \in S:\ \psi(t) = \pm 1\}$. The proof will be given for the case $\mu S^+ > 0$, $\mu S^- > 0$. If $\mu S^+ = 0$ or $\mu S^- = 0$, then the proof is nearly the same. As follows from Lemma 2, for $\eta > 0$ there exist sets $e^+ \subset S^+$, $e^- \subset S^-$ $(0 < \mu e^{\pm} < \eta/2)$ and functions $\varphi^+ \in L(e^+, \mu)$, $\varphi^- \in L(e^-, \mu)$ such that $\varphi^+(t) > 0$ $\forall' t \in e^+$, $\varphi^-(t) < 0$ $\forall' t \in e^-$, and

(24) $$\begin{aligned} &(\varphi \in P,\ \varphi(t) \leqslant \varphi^+(t)\ \ \forall' t \in e^+) \Rightarrow (\varphi(t) \leqslant 0\ \ \forall' t \in S^+),\\ &(\varphi \in P,\ \varphi(t) \geqslant \varphi^-(t)\ \ \forall' t \in e^-) \Rightarrow (\varphi(t) \geqslant 0\ \ \forall' t \in S^-). \end{aligned}$$

Let us have $q \in L(S, \mu)$, $q(t) > 0$ $\forall t \in S$, and denote $e = e^+ \cup e^-$,

(25) $$f_1(t) = \begin{cases} \psi(t)\big(q(t) + |y_0(t)|\big), & t \in S \setminus e,\\ y_0(t) + \varphi^+(t), & t \in e^+,\\ y_0(t) + \varphi^-(t), & t \in e^-, \end{cases}$$
$$f_2(t) = \begin{cases} f_1(t), & t \in S \setminus e,\\ \varphi^+(t), & t \in e^+,\\ \varphi^-(t), & t \in e^-. \end{cases}$$

Then

(26) $$\operatorname{sign}\big(f_1(t) - y_0(t)\big) = \operatorname{sign} f_2(t) = \psi(t).$$

As follows from (26) (see (15)), y_0 and θ are the elements of the best approximation according to f_1 and f_2. To prove uniqueness we will verify the inequalities (for any $y \in P$, $y \neq \theta$)

(27) $$\mu\{t \in S:\ \operatorname{sign}\big(f_1(t) - y_0(t)\big) = \operatorname{sign} y(t),\ |y(t)| > |f_1(t) - y_0(t)|\} > 0,$$

(28) $$\mu\{t \in S:\ \operatorname{sign} f_2(t) = \operatorname{sign} y(t),\ |y(t)| > |f_2(t)|\} > 0.$$

If

(29) $$y(t) \leqslant 0\ \ \forall' t \in S^+ \quad \text{and} \quad y(t) \geqslant 0 \quad \forall' t \in S^-,$$

then

$$\int_S y(t)\psi(t)\,d\mu = \int_{S^+} y(t)\,d\mu - \int_{S^-} y(t)\,d\mu < 0,$$

which contradicts (18). Hence at least one of the inequalities (29) is wrong. Let the first one be wrong. Then $\mu\{t \in S^+: y(t) > 0\} > 0$ and (see (24)) $\mu\{t \in e^+: y(t) > \varphi^+(t)\} > 0$. But it is exactly (27). Inequality (28) is verified as (27). Using (24) and the inequalities $\mu e < \eta$, $|\varphi^\pm(t)| < 1$ $(\forall t \in e^\pm)$, we obtain $\|f_i\| \leqslant \|y_0\| + \|q\| + \eta$. Then $\|f_1 - f_2\| \leqslant \int_e |y_0(t)|\, d\mu$. Choosing the $\|q\|$ and η small enough, one obtains inequalities $\|f_i\| \leqslant \|y_0\| + \gamma$ $(i = 1, 2)$, $\|f_1 - f_2\| \leqslant \varepsilon$.

Remark. In the proof just given the uniqueness of the best approximation element is established by means of (16). The functions f_1, f_2 can be constructed in such a way that the uniqueness can be proved by means of (17).

The Hausdorff distance between the sets A, B in the normed space is further denoted by $h(A, B)$. The following assertion is straightforward consequence of Theorem 4.

THEOREM 5. *Under the conditions of Theorem* 4

$$\sup\{h(x_p, y_p): \|x-y\| \leqslant t,\ \|x\| \leqslant 1\} \geqslant 1 \quad \forall t > 0; \tag{30}$$

consequently the metric projection from the unit ball of the space $L(S, \mu)$ onto each finite-dimensional subspace $P \in L(S, \mu)$ is not uniformly continuous.

Remark. Theorems 4, 5 were proved in [3] for $S = [a, b]$ in another way.

THEOREM 6. *Let (S, μ) be a space with non-negative σ-finite measure μ, $n \in \{1, 2, \ldots\}$. Then the following conditions are equivalent:*

(a) *there exists an n-dimensional subspace in $L(S, \mu)$ with a uniformly continuous metric projection,*

(b) *the space (S, μ) contains at least n atoms.*

Proof. Suppose that (S, μ) contains only k atoms $\{t_i\}_{i=1}^k$, $k < n$. If P is an n-dimensional subspace in $L(S, \mu)$, then there exists a $y_0 \in P$; $\|y_0\| = 1$, $y_0(t_i) = 0$, $i = 1, \ldots, k$. Applying Theorem 4 to the space $(S \setminus \{t_i\}_1^k, \mu)$, we obtain the functions $f^1, f^2 \in L(S \setminus \{t_i\}_1^k, \mu)$ for any $\varepsilon > 0$. The functions $\bar{f}^i$: $\bar{f}^i(t) = f^i(t)$, $t \notin \{t_j\}_1^k$; $\bar{f}^i(t_j) = 0$, $j = i, \ldots, k$, $(i = 1, 2)$ have the properties $\|\bar{f}^1 - \bar{f}^2\| \leqslant \varepsilon$, $\bar{f}^1_P = y_0$, $\bar{f}^2_P = \theta$. So (a) implies (b). The converse implication is evident.

References

[1] V. I. Berdyshev, *On the modulus of continuity of the operator of best approximation* Mat. Zametki 15 (1974), pp. 797–808 (in Russian).

[2] —, *An estimate of continuity modulus of operator* GRAD*F*, ibid. 16, 2 (1974), pp. 349–360 (in Russian).

[3] —, *A metric projection onto finite dimensional subspace from C and L*, ibid. 18, 4 (1975), pp. 473–488 (in Russian).

[4] —, *On the uniform continuity of the operator of best approximation*, Proc. Conf. at Poznań 1972, Warszawa 1975, pp. 21–32.

[5] B. O. Björnestål, *Continuity of the metric projection operator, 1*, Royal Inst. of Technology, Stockholm 1975, 12 pp. (preprint).

[6] —, *Continuity of the metric projection operator, 3*, Royal Inst. of Technology, Stockholm 1975, 27 pp. (preprint).

[7] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. 40, 3 (1936), pp. 396–414.

[8] M. M. Day, *Normed linear spaces*, Moscow 1961.

[9] N. V. Efimov, S. B. Stechkin, *Approximative compactness and Chebyshev sets*, Dokl. Akad. Nauk SSSR 140 (1961), pp. 522–524 (in Russian).

[10] R. Holmes and B. Kripke, *Smoothness of approximation*, Michigan Math. J. 15 (1968), pp. 225–248.

[11] B. R. Kripke and T. J. Rivlin, *Approximation in the metric of $L^1(X, \mu)$*, Trans. Amer. Math. Soc. 119, 1 (1965), pp. 101–122.

[12] R. R. Phelps, *Uniqueness of Hahn–Banach extention and unique best approximation*, ibid. 95, 2 (1960), pp. 238–255.

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