APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

ON THE DUAL OF WEIGHTED $H^1(|z| < 1)$

RICHARD L. WHEEDEN

Rutgers University, Department of Mathematics, New Brunswick, N.J. 08903, U.S.A.

Introduction

We show that the dual of the weighted Hardy space H^1_w can be identified with the class BMO_w of functions of weighted bounded mean oscillation. If the weight function w is identically 1, this is C. L. Fefferman's result [3].

We restrict attention to the unit disc, where matters are as simple as possible. The proof can be extended to the half-plane, but there are several technical problems involved in extending it to the upper half-space in higher dimensions. The half-plane case has also been studied extensively by J. G. Cuerva in [2]. His method is different from ours; it is based on the "atomic" decomposition of H^1_w, while ours follows the basic outline of Fefferman's original proof.

Let $w(x)$ be periodic (with period 2π), non-negative, and integrable over $(-\pi, \pi)$, and let $m_w(E)$ denote the w-measure of a set E: $m_w(E) = \int_E w(x)\,dx$. We say that *w satisfies condition* A_∞ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that if E is a measurable subset of an interval I and $|E| < \delta|I|$, then $m_w(E) < \varepsilon m_w(I)$. (See, e.g., [1].)

A function $F(z)$, $z = re^{ix}$, analytic in $|z| < 1$ is said to *belong to* H^1_w if the expression

$$||F||_{H^1_w} = \sup_{0<r<1} \int_{-\pi}^{\pi} |F(re^{ix})|w(x)\,dx$$

is finite. It is known (see [4], p. 118) that an analytic F belongs to H^1_w, $w \in A_\infty$, if and only if its non-tangential maximal function $N(F)(x)$ belongs to L^1_w, where

$$L^1_w = \left\{f\colon\ ||f||_{L^1_w} = \int_{-\pi}^{\pi} |f(x)|w(x)\,dx < +\infty\right\}.$$

Moreover, $||F||_{H^1_w}$ and $||N(F)||_{L^1_w}$ are equivalent norms. See also [10].

To state our main result, we need a boundary value characterization of H^1_w. We assume that

$$(A_1) \qquad w^*(x) \leqslant cw(x),$$

where w^* is the Hardy–Littlewood maximal function of w. This is a strong condition compared to A_∞, but in some sense, it is natural for boundary value considerations. In fact, we need a condition which insures that every function in L^1_w has a Poisson integral. A_1 is such a condition, since if w satisfies it, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w(x)\,dx \leqslant cw(x);$$

thus, w is bounded below by a positive constant, and $L^1_w \subset L^1(-\pi,\pi)$. On the other hand, if we let $w(x) = |x|^\alpha$, $\alpha > 0$, and $f(x) = |x|^{-1}$, $|x| < \pi$, then w satisfies A_∞ (in fact, in the terminology of [6], it satisfies A_p for $\alpha+1 < p < \infty$) and $f \in L^1_w$, but $f \notin L^1(-\pi,\pi)$.

If w satisfies A_1, we also know that the conjugate function of f, defined by

$$\tilde{f}(x) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|t|<\pi} f(x+t)\frac{1}{2}\cot\frac{t}{2}\,dt,$$

exists a.e. (in the pointwise sense) for any $f \in L^1_w$. (See [5].)

The boundary value characterization of H^1_w is given in the following theorem. We use the notations $Pf = (Pf)(r,x)$ and $Qf = (Qf)(r,x)$ for the Poisson and conjugate Poisson integrals of f, resp.

THEOREM 1. *Suppose that w satisfies* A_1.

(i) *Let $F \in H^1_w$, let u and v be its real and imaginary parts, and let* $v(0,0) = 0$. *Then there is a function $f \in L^1_w$ such that $\tilde{f} \in L^1_w$, $u = Pf$ and $v = Qf = P\tilde{f}$. Moreover, there exists $c > 0$ such that*

$$c^{-1}\|F\|_{H^1_w} \leqslant \|f\|_{L^1_w} + \|\tilde{f}\|_{L^1_w} \leqslant c\|F\|_{H^1_w}. \tag{1}$$

(ii) *Let $f \in L^1_w$. If $\tilde{f} \in L^1_w$, then the function $F = Pf + iQf$ belongs to H^1_w. Moreover, $Qf = P\tilde{f}$, and* (1) *holds.*

It follows that we can identify H^1_w, $w \in A_1$, with

$$\{(f,\tilde{f})\colon f, \tilde{f} \in L^1_w\}$$

Moreover, the norm defined by

$$\|f\|_{H^1_w} = \|(f,\tilde{f})\|_{H^1_w} = \|f\|_{L^1_w} + \|\tilde{f}\|_{L^1_w}$$

is equivalent to the usual H^1_w norm. We will see later that $\{(f,\tilde{f})\colon f$ is a real trigonometric polynomial$\}$ is dense in H^1_w.

We say (see [8]) that a real-valued periodic $b(x)$ *belongs to* BMO_w (*weighted* BMO) if $b \in L^1(-\pi,\pi)$ and there exists $c > 0$ such that

$$\int_I |b(x) - b_I|\,dx \leqslant cm_w(I), \qquad b_I = \frac{1}{|I|}\int_I b(x)\,dx,$$

for any interval I. (This condition for all $I \subset (-\pi,\pi)$ implies it for all I.) If $b \in \mathrm{BMO}_w$, we set

$$\|b\|_* = \sup_I \frac{1}{m_w(I)}\int_I |b(x) - b_I|\,dx.$$

Then $\|b\|_*$ is a semi-norm. One can obtain a norm by replacing $\|b\|_*$ by $\|b\|_* + \|b\|_{L^1}$, or by identifying functions which differ by a constant.

The next result gives the sense in which BMO_w is the dual of H^1_w.

THEOREM 2. *Let w satisfy* A_1.

(i) *If l is a real-valued continuous linear functional on H^1_w, there exists $b \in \mathrm{BMO}_w$ such that*

$$l(f,\tilde{f}) = \int_{-\pi}^{\pi} fb\,dx$$

for all real trigonometric polynomials f.

(ii) *There is a constant c such that*

$$\Big|\int_{-\pi}^{\pi} fb\,dx\Big| \leqslant c\|f\|_{H^1_w}\Big(\|b\|_* + \Big|\int_{-\pi}^{\pi} b\,dx\Big|\Big)$$

for all real trigonometric polynomials f and all $b \in \mathrm{BMO}_w$.

As a corollary of the proof, we obtain the following interesting characterizations of BMO_w.

THEOREM 3. *Let* $w \in A_1$.

(i) *The general element b of* $(\mathrm{BMO})_w$ *has the form* $b = \phi_1 w + (\phi_2 w)^\sim$, $\phi_1, \phi_2 \in L^\infty$.

(ii) *An integrable b belongs to* $(\mathrm{BMO})_w$ *if and only if there is a constant c such that for every interval I, $|I| \leqslant 1$,*

$$\iint_{B(I)} (1-r)|\nabla(Pb)|^2 e^{-P(\log w)} r\,dr\,dx \leqslant cm_w(I),$$

where $B(I) = \{re^{ix}\colon 1-r < |I|,\ x \in I\}$.

In the proofs, we will need several facts about weight functions, most of which will be listed as they arise. Here, we note that if w satisfies A_∞, then it also satisfies an inequality in which the roles of w-measure and Lebesgue measure are reversed: given $\varepsilon > 0$, there is a $\delta > 0$ such that if $E \subset I$ and $m_w(E) < \delta m_w(I)$, then $|E| < \varepsilon|I|$. Moreover, there exists p, $1 < p < \infty$, such that

$$\Big(\frac{1}{|I|}\int_I w\,dx\Big)\Big(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\,dx\Big)^{p-1} \leqslant c, \qquad 1 < p < \infty. \tag{A_p}$$

(See [7], [1].) The case $p = 2$ will be especially important. Note that w satisfies A_2 if and only if w^{-1} does. Also, if $b \in \mathrm{BMO}_w$ and w satisfies A_2, then

$$\int_I |b-b_I|^2 w^{-1}\,dx \leqslant c\|b\|_*^2 m_w(I),$$

with c independent of b and I (see [8], Theorem 4).

We will use the same letter c to denote different constants which may depend on w, but not on f, b, or I. We also use L^p, $1 \leqslant p \leqslant \infty$, to denote ordinary $(w \equiv 1)$ $L^p(-\pi, \pi)$.

1. Proof of Theorem 1

The proof of Theorem 1 follows standard lines. The easiest way to show part (i) is to note that if w satisfies A_1, then (since $w(x) \geqslant c > 0$) H_w^1 is actually contained in the classical Hardy space H^1. Therefore, by well-known results (see [11]) if $F \in H_w^1$, then F has (radial) boundary values $f+ig$ a.e.:

$$\lim_{r\to 1} F(re^{ix}) = \lim_{r\to 1}[u(re^{ix})+iv(re^{ix})] = f(x)+ig(x) \text{ a.e.}$$

(See also [5].) Since the non-tangential maximal function of F, $N(F)(x)$, belongs to L_w^1, and since $|u(re^{ix})|, |v(re^{ix})| \leqslant N(F)(x)$, it follows by dominated convergence that $f, g \in L_w^1$. Moreover, if we convolve u and v with the Poisson kernel and use the maximum principle for harmonic functions, it is a simple matter to see that $u = Pf$, $v = Pg$.

Thus, $F = Pf+iQf$, since both sides are analytic in $|z| < 1$, have equal real parts there, and have equal imaginary parts at the origin. Thus, $Pg = Qf$. In particular, taking limits at the boundary, we have $g = \tilde{f}$ a.e., which shows that $\tilde{f} \in L_w^1$, $v = P\tilde{f} = Qf$.

To complete the proof of (i), we must verify that $\|f\|_{L_w^1} + \|\tilde{f}\|_{L_w^1}$ is equivalent to $\|F\|_{H_w^1}$. The inequality

$$\|f\|_{L_w^1} + \|\tilde{f}\|_{L_w^1} \leqslant 2\|F\|_{H_w^1}$$

is an immediate corollary of the convergence of F to $f+i\tilde{f}$ in L_w^1 norm. On the other hand,

$$\int_{-\pi}^{\pi} |u(re^{ix})|w(x)\,dx = \int_{-\pi}^{\pi} |(Pf)(re^{ix})|w(x)\,dx \leqslant \int_{-\pi}^{\pi} |f(x)|(Pw)(re^{ix})\,dx.$$

Since $(Pw)(re^{ix}) \leqslant cw^*(x)$ and w satisfies A_1, we obtain

$$\int_{-\pi}^{\pi} |u(re^{ix})|w(x)\,dx \leqslant c\|f\|_{L_w^1}.$$

A similar relation holds for v and $\tilde{f}$, so that

$$\|F\|_{H_w^1} = \sup_{0<r<1} \int_{-\pi}^{\pi} |u(re^{ix})+iv(re^{ix})|w(x)\,dx \leqslant c(\|f\|_{L_w^1} + \|\tilde{f}\|_{L_w^1}). \tag{2}$$

To prove (ii), suppose that $f, \tilde{f} \in L_w^1$, and consider the analytic function $F = Pf+iQf$. The easiest way to proceed is to note that f and $\tilde{f}$ belong to L^1, since

$w(x) \geqslant c > 0$. Hence, by the remarks at the bottom of p. 285 of [11], Vol. 1, we see that $F \in H^1$. This implies (by part (i), for example) that $Qf = P\tilde{f}$. It now follows exactly as in the proof of (2) that $F \in H_w^1$. The remaining part of (ii) follows from (i).

Before going on to Theorem 2, we prove a result mentioned in the introduction.

LEMMA 1. *If w satisfies A_1, then*

$$\{(f, \tilde{f}):\ f \text{ is a real trigonometric polynomial}\}$$

is dense in H_w^1.

Proof. Let $F \in H_w^1$, $F(z) = \sum_0^\infty c_n z^n$, $|z| < 1$, and let $p(z) = \sum_0^N c_n z^n$ for N to be chosen. By Theorem 1, the H_w^1 norm of $F(z)-p(rz)$, $0 < r < 1$, is equivalent to the L_w^1 norm of $F(e^{ix})-p(re^{ix})$, which is bounded by

$$\|F(e^{ix})-F(re^{ix})\|_{L_w^1} + \|F(re^{ix})-p(re^{ix})\|_{L_w^1}.$$

The first of these terms tends to zero as $r \to 1$. The second, for fixed r, is arbitrarily small when N is large, since p converges uniformly to F on compact subsets of $|z| < 1$. It follows that there exist N and r so that $F(z)-p(rz)$ has arbitrarily small H_w^1 norm, which completes the proof.

2. Lemmas for Theorem 2

Aside from several facts which are already in print, we will need three lemmas to prove Theorem 2. These are given below.

A weight w is said to *satisfy condition* B_2 if there is a constant c such that for every interval I,

$$\int_{-\infty}^{\infty} w(x)\frac{|I|}{|I|^2+(x-x_I)^2}\,dx \leqslant c\frac{m_w(I)}{|I|}, \tag{B_2}$$

where x_I is the center of I. This condition was introduced in [5], (2.3), and holds if w satisfies A_1 (in fact, A_2 is enough). If w satisfies B_2 and I is the interval with center x and length $1-r$, $0 < r < 1$, it follows that

$$(Pw)(re^{ix}) \leqslant \frac{c}{1-r}\int_{|x-t|<1-r} w(t)\,dt. \tag{3}$$

Note also that if αI, $\alpha > 0$, denotes the interval concentric with I whose length is $\alpha|I|$, then, by restricting integration in (B_2) to $2I$, we see that any w satisfying B_2 also satisfies the doubling condition

$$m_w(2I) \leqslant cm_w(I). \tag{A}$$

LEMMA 2. *If w satisfies B_2, there is a constant c such that for any $b \in \mathrm{BMO}_w$ and any I,*

$$(4)\qquad \int_{-\infty}^{+\infty} |b(x)-b_I| \frac{|I|}{|I|^2+(x-x_I)^2}\,dx \leqslant c\|b\|_* \frac{m_w(I)}{|I|},$$

where $b_I = \int_I b(x)\,dx/|I|$, *and* x_I *is the center of* I.

Proof. Fix I, and let β denote the expression on the left in (4). Then β is the sum of similar integrals extended over I and $R_k = 2^kI-2^{k-1}I$, $k = 1, 2, \ldots$. Therefore,

$$\beta \leqslant \frac{1}{|I|}\int_I |b-b_I|\,dx + c\sum_{k=1}^{\infty} \frac{1}{2^{2k}|I|}\int_{2^kI} |b-b_I|\,dx.$$

For $k \geqslant 1$, we have

$$\int_{2^kI} |b-b_I|\,dx \leqslant \int_{2^kI} |b-b_{2^kI}|\,dx + \sum_{j=1}^{k} 2^k|I|\,|b_{2^jI}-b_{2^{j-1}I}|.$$

Since

$$|b_{2^jI}-b_{2^{j-1}I}| = \left|\frac{1}{2^{j-1}|I|}\int_{2^{j-1}I} (b-b_{2^jI})\,dx\right|$$
$$\leqslant \frac{2}{2^j|I|}\int_{2^jI} |b-b_{2^jI}|\,dx \leqslant \frac{2\|b\|_*}{2^j|I|} m_w(2^jI),$$

we obtain

$$\int_{2^kI} |b-b_I|\,dx \leqslant 2\|b\|_*[m_w(2^kI)+2^k\sum_{j=1}^{k}\frac{1}{2^j}m_w(2^jI)], \qquad k \geqslant 1.$$

Also, $\int_I |b-b_I|\,dx \leqslant \|b\|_* m_w(I)$, so that

$$\beta \leqslant \frac{c\|b\|_*}{|I|}\left[m_w(I)+\sum_{k=1}^{\infty}\frac{1}{2^{2k}}m_w(2^kI)+\sum_{k=1}^{\infty}\frac{1}{2^k}\sum_{j=1}^{k}\frac{1}{2^j}m_w(2^jI)\right].$$

Changing the order of summation in the last term on the right, we get

$$\beta \leqslant \frac{c\|b\|_*}{|I|}\sum_{k=0}^{\infty}\frac{1}{2^{2k}}m_w(2^kI).$$

Since w satisfies B_2,

$$cm_w(I) \geqslant \int_{-\infty}^{+\infty} w(x)\frac{|I|^2}{|I|^2+(x-x_I)^2}\,dx = \int_I + \sum_{k=1}^{\infty}\int_{R_k}$$
$$\geqslant c_1\left[m_w(I)+\sum_{k=1}^{\infty}\frac{1}{2^{2k}}m_w(R_k)\right].$$

But $m_w(R_k)$ is comparable to $m_w(2^kI)$ by the doubling condition, so that $m_w(I) \geqslant c\sum_{k=0}^{\infty} m_w(2^kI)/2^{2k}$. Therefore, $\beta \leqslant c\|b\|_* m_w(I)/|I|$, as claimed.

If w satisfies A_1 (even A_∞), then $\log w$ is of bounded mean oscillation (in the usual unweighted sense) — see [5], Lemma 5. Let $P(\log w)(r, x)$ denote its Poisson integral, and let ∇ and Δ denote the gradient and Laplace operators, respectively.

LEMMA 3. *If* w *satisfies* A_2, *there is a constant* c *such that for* $0 < r < 1$,

(i) $(1-r)|\nabla P(\log w)(r, x)| \leqslant c$,

(ii) $c^{-1}e^{P(\log w)(r,x)} \leqslant \dfrac{1}{1-r}\displaystyle\int_{|x-t|<1-r} w(t)\,dt \leqslant ce^{P(\log w)(r,x)}$.

Proof. Part (i) is a corollary of the fact that $\log w$ is of bounded mean oscillation; it follows from Theorem 3 (iii) of [3] by using the mean-value property of harmonic functions.

To prove (ii), observe by Jensen's inequality for convex functions and (3) that

$$\exp(P\log w)(r, x) \leqslant (Pw)(r, x) \leqslant \frac{c}{1-r}\int_{|x-t|<1-r} w(t)\,dt.$$

Applying this to w^{-1} (which also satisfies A_2), we obtain

$$\exp(-P\log w)(r, x) \leqslant \frac{c}{1-r}\int_{|x-t|<1-r} w^{-1}(t)\,dt \leqslant c_1\left(\frac{1}{1-r}\int_{|x-t|<1-r} w(t)\,dt\right)^{-1}.$$

This proves (ii).

Although we shall not need the result, it is interesting to note that Lemma 3 is true if w satisfies A_∞. In fact, w then satisfies A_p for some p, $1 \leqslant p < \infty$. If $p \leqslant 2$, the fact that any A_p function satisfies A_2 gives (ii). If $p > 2$, then $w^{-\frac{1}{p-1}}$ satisfies $A_{p'}$, $p' = \dfrac{p}{p-1} < 2$, which implies that $\exp\left\{-\dfrac{1}{p-1}(P\log w)(r, x)\right\}$ $(= \exp P(\log w^{-\frac{1}{p-1}})(r, x))$ and $\dfrac{1}{1-r}\displaystyle\int_{|x-t|<1-r} w^{-\frac{1}{p-1}}(t)\,dt$ are equivalent. Part (ii) follows by applying condition A_p.

LEMMA 4. *Let* f *be a real trigonometric polynomial, and let* $b \in L^1$. *If either* $\int_{-\pi}^{\pi} f\,dx = 0$ *or* $\int_{-\pi}^{\pi} b\,dx = 0$, *then*

$$(5)\qquad \left|\int_{-\pi}^{\pi} fb\,dx\right| \leqslant c\int_0^1\int_{-\pi}^{\pi} (1-r^2)|\nabla(Pf)(r, x)||\nabla(Pb)(r, x)|\,dr\,dx,$$

with c *independent of* f *and* b.

Proof. If $b \in L^2$, then (5) is derived in the course of proving (3.21) in Chapter 14 of [11]: see specifically p. 215, line 11. If $b \in L^1$, choose $b_k \in L^2$ such that $|b_k| \leqslant |b|$ and $b_k \to b$ in L^1. Then (5) holds for each b_k, and since $fb_k \to fb$ in L^1, it is enough to show that the expression on the right side of (5) is the limit of the same expression with b replaced by b_k. We shall use the dominated convergence theorem. Note that $|\nabla(Pb_k)|$ converges pointwise to $|\nabla(Pb)|$, and that $(1-r^2)|\nabla(Pb_k)| \leqslant cP(|b_k|) \leqslant cP(|b|)$. Since $|\nabla(Pf)|$ is bounded and

$$\int_0^1\int_0^{2\pi} P(|b|)\,dr\,dx \leqslant \int_0^1\Big[\int_0^{2\pi}|b|\,dx\Big]dr = \|b\|_1 < +\infty,$$

the result follows.

3. Proof of Theorem 2

The proof of part (i) of Theorem 2, which is relatively short, is as follows. Let l be a real-valued continuous linear functional on H^1_w, and let w satisfy A_1. By Theorem 1, H^1_w can be identified with a subset of the direct sum

$$L^1_w \oplus L^1_w = \{(f, g)\colon \|f\|_{L^1_w}+\|g\|_{L^1_w} < +\infty\}.$$

We first claim that H^1_w is a closed subset of $L^1_w \oplus L^1_w$. To check this, suppose that $(f_k, \tilde{f}_k) \in H^1_w$, and that f_k and $\tilde{f}_k$ converge in L^1_w norm to f and g, resp. Then both f and g belong to L^1_w, and we only need to show that $g = \tilde{f}$ a.e. However, since w satisfies A_1, $\tilde{f}_k$ converges in w-measure to $\tilde{f}$: in fact, by Theorem 1(d) of [5],

$$m_w\{|\tilde{f}_k-\tilde{f}| > \alpha\} \leqslant \frac{c}{\alpha}\int_{-\pi}^{\pi}|f_k-f|\,w\,dx, \qquad \alpha > 0.$$

Hence, since $\tilde{f}_k$ also converges in w-measure to g, the claim follows.

By the Hahn–Banach theorem, l has an extension $\bar{l}$ to a continuous linear functional on $L^1_w \oplus L^1_w$. Hence, there exist $\phi_1, \phi_2 \in L^\infty$ such that

$$\bar{l}(f, g) = \int_{-\pi}^{\pi}(f\phi_1+g\phi_2)\,w\,dx, \qquad f, g \in L^1_w.$$

Thus, if $(f,\tilde{f}) \in H^1_w$,

$$l(f,\tilde{f}) = \int_{-\pi}^{\pi}(f\phi_1+\tilde{f}\phi_2)\,w\,dx. \tag{6}$$

We next observe that if f is any trigonometric polynomial, then

$$\int_{-\pi}^{\pi}\tilde{f}\phi_2 w\,dx = -\int_{-\pi}^{\pi}f(\phi_2 w)^{\sim}\,dx.$$

In fact, since w (and so $\phi_2 w$) belongs to L^p for some $p > 1$ (see [6]), this is a well-known corollary of the boundedness of the conjugate function on L^p, $p > 1$. Hence, by (6), if f is a polynomial,

$$l(f,\tilde{f}) = \int_{-\pi}^{\pi}f(\phi_1 w-(\phi_2 w)^{\sim})\,dx.$$

Let $b = \phi_1 w-(\phi_2 w)^{\sim}$. Clearly, $\phi_1 w \in \mathrm{BMO}_w$ since it is a bounded function times w. Moreover, $(\phi_2 w)^{\sim} \in \mathrm{BMO}_w$ by Theorem 1 of [8]. Hence, $b \in \mathrm{BMO}_w$, which completes the proof of part (i).

To prove part (ii), let $b \in \mathrm{BMO}_w$, and write $b = b_0+b_1$ where $b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} b\,dx$. Then

$$\Big|\int_{-\pi}^{\pi} fb_0\,dx\Big| \leqslant \|f\|_{L^1}|b_0| \leqslant c\|f\|_{L^1_w}|b_0| \leqslant c\|f\|_{H^1_w}|b_0|,$$

if w satisfies A_1 (since then $w(x) \geqslant c > 0$). It follows that we may assume b has integral zero. We may also assume that $\|b\|_* = 1$. By (5),

$$\Big|\int_{-\pi}^{\pi} fb\,dx\Big| \leqslant c\iint_{|z|<1}(1-r^2)|\nabla(Pf)|\,|\nabla(Pb)|\,dr\,dx. \tag{7}$$

The part of the integral on the right with $r \leqslant 1/2$ is clearly bounded by a multiple of $\|f\|_{H^1_w}$, since in this range $|\nabla(Pf)| \leqslant c\|f\|_{L^1}$ and $|\nabla(Pb)| \leqslant c\|b\|_{L^1}$, while $\|f\|_{L^1} \leqslant c\|f\|_{H^1_w}$ and $\|b\|_1 = \|b-b_0\|_{L^1} \leqslant cm_w(-\pi, \pi)$ (recall that $\|b\|_* = 1$).

Let D denote the part of the integral on the right of (7) with $1/2 < r < 1$. To estimate D, let $\Gamma_h(x)$, $0 < h \leqslant 1/2$, denote the 45° triangle with vertex e^{ix}, altitude h and axis along the radius from $z = 0$ to $z = e^{ix}$. Let

$$h(x) = \sup\Big\{h\colon 0 < h \leqslant 1/2,\ \Big(\iint_{\Gamma_h(x)}|\nabla(Pb)(r, \theta)|^2 r\,dr\,d\theta\Big)^{1/2} \leqslant c_1 w(x)\Big\},$$

where c_1 is a constant to be chosen. The theorem will follow by showing that D is less than a constant times

$$\int_{-\pi}^{\pi}\Big[\iint_{\Gamma_{h(x)}(x)}|\nabla(Pf)|\,|\nabla(Pb)|\,dr\,d\theta\Big]dx. \tag{8}$$

In fact, (8) is at most

$$2\int_{-\pi}^{\pi}\Big[\iint_{\Gamma_{h(x)}(x)}|\nabla(Pf)|\,|\nabla(Pb)|\,r\,dr\,d\theta\Big]dx$$

$$\leqslant 2\int_{-\pi}^{\pi}\Big[\iint_{\Gamma_{h(x)}(x)}|\nabla(Pf)|^2 r\,dr\,d\theta\Big]^{1/2}\Big[\iint_{\Gamma_{h(x)}(x)}|\nabla(Pb)|^2 r\,dr\,d\theta\Big]^{1/2}dx$$

$$\leqslant 2c_1\int_{-\pi}^{\pi}(Af)(x)\,w(x)\,dx,$$

where $(Af)(x)$ is the Lusin area integral of f. By [4] or [9], D is therefore bounded by a constant times $\|f\|_{H^1_w}$, which is the desired result.

To show that D is at most a constant times (8), note that (8) equals

$$\iint_{1/2<|z|<1}|\nabla(Pf)|\,|\nabla(Pb)|\,|E_{r,\theta}|\,dr\,d\theta,$$

where $E_{r,\theta} = \{x\colon re^{i\theta} \in \Gamma_{h(x)}(x)\}$. We will be done if we show that $|E_{r,\theta}| \geqslant c(1-r)$, $1/2 < r < 1$. This follows from showing that for every $re^{i\theta}$, $1/2 < r < 1$, $h(x) \geqslant 1-r$ for x is a subset of $I = \{x\colon |x-\theta| < 1-r\}$ whose measure exceeds a multiple (independent of $re^{i\theta}$) of $1-r$. We must thus show that if c_1 is large enough and I is an interval of length at most 1, then

$$\Big|\Big\{x \in I\colon \Big(\iint_{\Gamma_{\frac{1}{2}|I|}(x)} |\nabla(Pb)|^2 r\,dr\,d\theta\Big)^{1/2} \leqslant c_1 w(x)\Big\}\Big| \geqslant \frac{1}{2}|I|. \tag{9}$$

Fix I, let $\Gamma(x) = \Gamma_{\frac{1}{2}|I|}(x)$, and let E be the complementary subset of I:

$$E = \Big\{x \in I\colon \iint_{\Gamma(x)} |\nabla(Pb)|^2 r\,dr\,d\theta\, w^{-1}(x) \geqslant c_1^2 w(x)\Big\}$$

Integrating over E and applying Fubini's theorem, we obtain

$$c_1^2 m_w(E) \leqslant \int_E \Big(\iint_{\Gamma(x)} |\nabla(Pb)|^2 r\,dr\,d\theta\Big) w^{-1}(x)\,dx \tag{10}$$
$$\leqslant \iint_B |\nabla(Pb)|^2 \Big(\int_{|x-\theta|<1-r} w^{-1}(x)\,dx\Big) r\,dr\,d\theta,$$

where B is the "box" $\{re^{i\theta}\colon 1-r < 2|I|,\ \theta \in 2I\}$. The strategy is to find a constant c independent of I such that the last integral is majorized by $cm_w(I)$. This will give $c_1^2 m_w(E) \leqslant cm_w(I)$. By then choosing c_1 large and using A_∞ (in the form mentioned near the end of the Introduction), we get $|E| < \frac{1}{2}|I|$, thereby proving (9).

To show that (10) is majorized by $cm_w(I)$, let $J = 4I$, and write

$$\nabla(Pb)(r,\theta) = \int_{-\pi}^{\pi} [b(t)-b_J]\nabla P(r,\theta-t)\,dt = \int_J + \int_{(-\pi,\pi)-J} = \beta_1+\beta_2,$$

where $P(r,t)$ is the Poisson kernel. We will consider the parts of (10) arising from β_1 and β_2 separately. The part from β_2 is relatively simple. If $\theta \in 2I$,

$$\beta_2(r,\theta) \leqslant c \int_{t\notin J} |b(t)-b_J| \frac{dt}{(\theta-t)^2} \leqslant c\frac{m_w(I)}{|I|^2},$$

by Lemma 2 ($\|b\|_* = 1$) and the doubling condition. The corresponding part of (10) is thus majorized by a multiple of

$$\frac{m_w^2(I)}{|I|^4}\iint_B \Big(\int_{|x-\theta|<1-r} w^{-1}(x)\,dx\Big) r\,dr\,d\theta \leqslant c\frac{m_w^2(I)}{|I|^4} m_{w^{-1}}(J) \int_{1-2|I|}^{1} (1-r)\,dr \leqslant cm_w(I),$$

since w satisfies A_2.

Note that $\beta_1(r,\theta) = \nabla(Pb_1)(r,\theta)$, where $b_1 = (b-b_J)\chi_J$. Hence, by changing the order of integration, we see that the part of (10) corresponding to β_1 is at most

$$\int_{-\pi}^{\pi} (Ab_1)^2(x) w(x)^{-1}\,dx,$$

where Ab_1 is the Lusin area integral of b_1. Since $w(x)^{-1}$ satisfies A_2, it follows from [4] that the last integral is bounded by

$$c\int_{-\pi}^{\pi} b_1^2(x) w(x)^{-1}\,dx.$$

Observe that $b_1 \in L^2_{w^{-1}}$; in fact,

$$\int_{-\pi}^{\pi} b_1^2(x) w(x)^{-1}\,dx = \int_J |b(x)-b_J|^2 w(x)^{-1}\,dx \leqslant cm_w(J) \qquad (\|b\|_* = 1)$$

(see the Introduction). Combining estimates and using the doubling condition, we obtain the desired bound. This completes the proof of Theorem 2.

The proof of Theorem 3 follows by examination of that of Theorem 2 (cf. [3]). In particular, for the sufficiency of part (ii), note that the proof of Theorem 2 (ii) gives $\left|\int fb\,dx\right| \leqslant c\|f\|_{H^1_w}$ for polynomials f if b is any function in L^1 for which (10) is majorized by $cm_w(I)$. Then, by Lemma 3, note that (10) is essentially the double integral in part (ii) of Theorem 3.

References

[1] R. R. Coifman and C. L. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), pp. 241–250.

[2] J. G. Cuerva, *Weighted H^p spaces*, Ph. D. dissertation, Washington University.

[3] C. L. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), pp. 137–193.

[4] R. F. Gundy and R. L. Wheeden, *Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh–Paley series*, Studia Math. 49 (1974), pp. 107–124.

[5] R. A. Hunt, B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), pp. 227–251.

[6] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, ibid. 165 (1972), pp. 207–226.

[7] —, *The equivalence of two conditions for weight functions*, Studia Math. 49 (1974), pp. 101–106.

[8] B. Muckenhoupt and R. L. Wheeden, *Weighted bounded mean oscillation and the Hilbert transform*, ibid. 54 (1976), pp. 221–237.

[9] C. Segovia and R. L. Wheeden, *On weighted norm inequalities for the Lusin area integral*, Trans. Amer. Math. Soc. 176 (1973), pp. 103–123.

[10] R. L. Wheeden, *On the radial and non-tangential maximal functions for the disc*, Proc. Amer. Math. Soc. 42 (1974), pp. 418–422.

[11] A. Zygmund, *Trigonometric series*, 2nd ed., Cambridge Univ. Press, New York 1959.

Presented to the Semester
Approximation Theory
September 17–December 17, 1975