ON UNCONDITIONAL CONVERGENCE OF HAAR SERIES

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The following result of A. Pelczynski [10] is well known: there are no unconditional bases in the space $L(0, 1)$. In particular, the Haar system is not an unconditional basis in the space $L(0, 1)$. Moreover, as was shown by V. F. Gaposhkin [4], [5], the Haar system is an unconditional basis in an Orlicz space if and only if it is reflexive (for the sufficiency of the theorem see also [3]). According to the results of A. M. Olevski [8], reflexivity is a necessary and sufficient condition for the existence of an unconditional basis in an Orlicz space. P. L. Uljanov [13] and M. B. Petrovskaja [9] considered the question under what conditions the function $f$ has its Fourier-Haar series unconditionally convergent in the metric of the space $L(0, 1)$. Analogical questions about the unconditional convergence in classes of spaces, containing in particular the spaces $L\ln^\alpha(0, 1)$ ($\alpha > 0$), have been considered in [8], [11], and [12]. In this paper we consider necessary conditions for the unconditional convergence of multidimensional Fourier-Haar series in certain nonreflexive spaces of functions.

Let $\mathbb{R}^n$ be the $n$-dimensional real Euclidean space of points $\vec{x} = (x_1, ..., x_n)$, $I^n = [0, 1] \times \cdots \times [0, 1]$ the $n$-dimensional unit cube, and $N^n \subset \mathbb{R}^n$ the subset of positive integer points $\vec{m} = (m_1, ..., m_n)$ and $|\vec{m}| = \sum_{i=1}^{n} m_i$.

Moreover, we use below the following notation; $r\vec{m} = (rm_1, ..., rm_n)$, $\vec{1} = (1, ..., 1)$, $(a_{\vec{m}})_{\vec{n}, \vec{m}} = (a_{m_1, ..., m_n})_{n_1, ..., m_n}$, $\sum_{\vec{m}} (\cdot) = \sum_{m_1=1}^{\infty} ... \sum_{m_n=1}^{\infty} (\cdot)$.

The convergence $M \to \infty$ is understood in the sense of Pringsheim, i.e. it is equivalent to $M_i \to \infty$ for all $1 \leq i \leq n$, and $\Phi(L)$ denotes the class of measurable functions $f(x)$ on the cube $I^n$ for which $\int_{I^n} \Phi(f(x)) \, dx < \infty$. In the sequel, $L^\alpha_n = L^\alpha_n(I^n)$ denotes a Banach space of functions defined on the cube $I^n$ which is generated by $N$-functions with $\Phi$ as their principal part (for details cf. [7]). We assume that the function $\Phi$ considered below satisfies condition $A_2$ (i.e. $\Phi(2u) < C\Phi(u)$ for $u \geq u_0$). This condition is necessary and sufficient for the Orlicz space $L^\alpha_n$ to be separable.
In the sequel the following notation will be used: \( C \) — absolute positive constants, \( \lambda_{2} = \log_{2} u \),

\[
\Delta^{l}(p(m)) = \sum_{j=0}^{k} (-1)^{j} \frac{\binom{k}{j}}{p(m+j)}, \quad D_{l}(p(m)) = \Delta^{l}(p(m)),
\]

\[
D_{l}(p(m)) = p(m) - p(m_{l}), \quad m_{l} = (m_{1}, \ldots, m_{l-1}, m_{l+1}, \ldots, m_{n}),
\]

\[
\Delta_{m_{l}} \Phi \left( q(m) \right) = \Delta_{m_{l}} \left( \Delta_{m_{l-1}} \ldots \Delta_{m_{1}} \Phi \left( q(m) \right) \right),
\]

\[
\Phi(q(m)) = 2^{-m} \Phi(2^{m} \rho) \quad \forall \mho \in \mathbb{N}^{*} \quad \text{(the function } \Phi \text{ is defined below)},
\]

\[
b_{m} = \left| \rho \right|^{-m} \Phi^{-1}(\rho).
\]

**Definition 1.** Let \( B \) be a Banach space and let \( A \) be a subset of \( B \). The sequence \( \{ x_{k} \}_{k=1}^{n} \) in the space \( B \) is called an (unconditional) basis of \( A \) in the norm of the space \( B \) if for every \( x \in A \) there exists a unique sequence of scalars \( \{ \alpha_{k} \}_{k=1}^{n} \) such that the series \( \sum_{k=1}^{n} \alpha_{k} x_{k} \) (unconditionally) converges to \( x \).

**Definition 2.** The function \( \Phi \) satisfies condition (**) iff \( \Phi \) is an \( N \)-function and the following conditions hold:

(1) \( \Phi(\alpha^{*}) \leq C|\alpha|\Phi(\alpha) \quad \text{for } |\alpha| \leq 1 \).

**Lemma 1.** Let the function \( \Phi \) satisfy condition (**) Set

\[
\Psi(\alpha) = \begin{cases} 
\Phi(\alpha) \frac{1}{|\alpha|^{s+1}} \Phi(\alpha) & \text{for } |\alpha| > 1, \\
0 & \text{for } |\alpha| \leq 1.
\end{cases}
\]

Then the following inequalities hold:

(3) \( \Phi(\alpha^{*}) \leq C|\alpha|\Phi(\alpha) \).

(4) \( \Phi(\alpha) \leq C\Phi(\|\alpha\|^{2}) \).

(5) \( C^{-1}\Phi(\alpha)\Phi^{1/2}(\lambda) \leq \Psi(\lambda) \leq C\Phi(\alpha)\Phi^{1/2}(\lambda) \).

(6) \( \Phi(\alpha^{*}) \leq C(\Phi(\alpha) + \Phi(\alpha)) \).

Proof. It follows from Definition 2 for \( \Psi(\alpha) \) that

\[
\Phi(\alpha^{*}) \leq 2^{l(2^{l-1})} \Phi(2^{l-1}),
\]

whence for large \( |\alpha| \), \( \Phi(\alpha^{*}) \leq 2^{l(2^{l-1})} \Phi(2^{l-1}) \), and this implies (3).

According to the results of P. L. Ul'yanov ([14], p. 664) (3) implies (4). Since, for \( u > 1 \),

\[
\Psi(\alpha) \geq \Psi \left( \frac{1}{\lambda} \right) \Phi(\alpha^{r-1}) \Phi(\alpha) \geq (2C)^{-r} \Phi(\alpha) \Phi(\alpha),
\]

it follows that the left-hand inequality in (5) holds. Since the function \( u^{-r} \Phi(\alpha) \) is monotone for \( u > 1 \), the right-hand inequality in (5) holds, too. Finally, let \( m_{l} \geq m_{l} \); then

\[
\Phi(2^{l-1} \rho) \leq 2^{l-1} \left( 1 - \frac{C}{m_{l} + m_{l} - 1} \right) \cdots \left( 1 + \frac{C}{m_{l}} \right) \Phi(2^{l-1}) \leq 2^{m_{l}} \left( 1 - \frac{C}{m_{l}} \right) \Phi(2^{l-1}) \leq C^{2} \Phi(2^{l-1}).
\]

Applying this estimation, we obtain (6).

**Lemma 2.** Let \( \{ x_{k} \}_{k=1}^{n} \) denote the Haar system on the interval \( I^{2} = [0, 1] \) defined as follows (cf. [13], pp. 54-55):

\[
\chi_{k}(x) = 1 \quad \text{for } 0 < x < 1,
\]

\[
\chi_{k}(x) = \begin{cases} 
\sqrt{2^{2}} & \text{for } \frac{2k - 2}{2^{k+1}} < x < \frac{2k - 1}{2^{k+1}} , \\
\sqrt{2} & \text{for } \frac{2k - 1}{2^{k+1}} < x < \frac{2k}{2^{k+1}} , \\
0 & \text{elsewhere in } I^{2},
\end{cases}
\]

whence \( \Phi(\alpha^{*}) \leq C(\Phi(\alpha) + \Phi(\alpha)) \).

for \( m = 2^{l-1}k, \quad p = 0, 1, \ldots, k = 1, \ldots, 2^{l} \). Let \( \{ x_{k} \}_{k=1}^{n} \) denote the Haar system defined on \( I^{2} \), where \( \chi_{k}(x) = x_{k}(x) \), \( x_{k}(x) \).

We shall use below the following facts:

(7) \( \sum_{n=1}^{\infty} |m|^{1-\alpha} = \infty \).

(8) \( \sum_{n=1}^{\infty} |m|^{1-\alpha} < \infty \quad \forall \alpha > 0. \)

It follows directly from (7) that

(9) \( \sum_{n=1}^{\infty} b_{n} \rho(\rho) = \infty \)

and (1), (3), (6) imply that

(10) \( \sum_{n=1}^{\infty} \Delta^{l}(b_{n}) \rho(\rho) = \sum_{n=1}^{\infty} \Delta^{l}(\rho^{-1} \rho^{-1}(\rho)) \rho(\rho) \leq C \sum_{n=1}^{\infty} |m|^{1-\alpha} < \infty. \)
Let $e(t)$ be such a function that $e(t) \downarrow 0$ while $t \to \infty$, $0 < e(t) < 1$ and let $(l_k)_{k=0}^{\infty}$ ($l_0 = 0$) be an increasing sequence of integers satisfying some conditions specified below.

Let us define the following sets:

1. $V_k = \{ \bar{m} : l_k + 1 \leq \max_{1 \leq i \leq k} m_i \leq l_{k+1} \}$, $k = 0, 1, 2, \ldots$,
2. $V_k^* = \{ \bar{m} : l_k + 1 < \max_{1 \leq i \leq k} m_i \leq l_{k+1} \}$, $k = 0, 1, 2, \ldots$,
3. $S_k = V_k \setminus V_k^*$, $k = 1, 2, \ldots$,
4. $W_k = \{ \bar{m} : l_k + 1 \leq m_i \leq l_{k+1}, \ 1 \leq i \leq n \}$, $k = 0, 1, 2, \ldots$.

In addition, we set

1. $D_k = \sum_{\bar{m} \in V_k} |\bar{m}|^{-\alpha}, \ k = 1, 2, \ldots$,
2. $A_k = \sum_{\bar{m} \in V_k^*} |\bar{m}|^{-\alpha}, \ k = 1, 2, \ldots$,
3. $a_k = \left\{ \begin{array}{ll} c A_k^{k} |\bar{m}|^{-\alpha} & \text{for } \bar{m} \in V_k, \ k \geq 1, \\ |\bar{m}|^{-\alpha} & \text{for } \bar{m} \in V_0. \end{array} \right.$

Let us note that

1. $\Delta_{j_1, \ldots, j_k}(a_k) \geq 0, \ 1 \leq j_1 < \ldots < j_k \leq n$.

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In view of $e(t) \downarrow 0$ for $t \to \infty$, using (17) we can choose a sequence $\{l_k\}_{k=1}^\infty$ in such a way that the conditions introduced below are fulfilled for $k = 1, 2, \ldots$.

Consequently, the following inequalities hold:

1. $e(|\bar{m}|) < k^{-3}, \ \text{for } |\bar{m}| > l_k$,
2. $\sum_{1 \leq k \leq \infty} |\bar{m}|^{-\alpha} e(|\bar{m}|) A_k < A_k^{-1} k^{-1}, \ k = 1, 2, \ldots$.

**Lemma 2.** Let $\Theta$ be a function satisfying condition $(*)$ and let $e(t) \downarrow 0$ for $t \to \infty$, $0 < e(t) < 1$. Then the sequence $\{a_k\}_{k=1}^\infty$, constructed according to (18), (21)–(25), is decreasing for every $m_i$ ($1 \leq i \leq n$), tends to 0, and satisfies the inequalities

1. $\sum_{k=1}^\infty \sum_{\bar{m} \in W_k} a_k e(|\bar{m}|) = \infty$,
2. $\sum_{n=1}^\infty \sum_{|\bar{m}| \leq n} a_k e(|\bar{m}|) < \infty, \ 1 \leq j_1 < \ldots < j_k \leq n$,
3. $\sum_{n=1}^\infty 2^{-|\bar{m}|} (2|\bar{m}| d(\bar{m})) \Theta(2|\bar{m}| d(\bar{m})) \Theta(2|\bar{m}| d(\bar{m})) < \infty$.

**Proof.** (27) follows immediately from (14) and (24).

Let us note that

1. $\Delta_{j_1, \ldots, j_k}(a_k) \geq 0, \ 1 \leq j_1 < \ldots < j_k \leq n$.

In view of $e(t) \downarrow 0$ for $t \to \infty$, using (17) we can choose a sequence $\{l_k\}_{k=1}^\infty$ in such a way that the conditions introduced below are fulfilled for $k = 1, 2, \ldots$.

Consequently, the following inequalities hold:

1. $\sum_{|\bar{m}| > l_k} |\bar{m}|^{-\alpha} e(|\bar{m}|) A_k < A_k^{-1} k^{-1}, \ k = 1, 2, \ldots$.

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Consequently, the following inequalities hold:

1. $\sum_{|\bar{m}| > l_k} |\bar{m}|^{-\alpha} e(|\bar{m}|) A_k < A_k^{-1} k^{-1}, \ k = 1, 2, \ldots$.
Now we have the following decomposition:
\[
\sum_{n=1}^{m} 2^{-|\vec{m}|} \Phi(E(2^{m}|d(a_n))) \Phi(2^{|\vec{m}|} \Phi(d(a_n))) \leq \sum_{n=1}^{m} A_{m+1} \sum_{\vec{m}_n \in \mathbb{N}} |\vec{m}|(\Phi(\vec{m}))^2 = \sum_{n=1}^{m} \mathcal{I}_n.
\]
According to (4) and the properties of the function \(e(t)\) we have
\[
\mathcal{I}_n \leq \sum_{n=1}^{m} 2^{-|\vec{m}|} |\vec{m}|(\Phi(\vec{m})) < \infty.
\]
Further, by (1) and (26) we get
\[
\mathcal{I}_n \leq \sum_{n=1}^{m} A_{m+1} \sum_{\vec{m}_n \in \mathbb{N}} e(\vec{m}) |\vec{m}|(\Phi(\vec{m})) < \sum_{n=1}^{m} A_{m+1} \sum_{\vec{m}_n \in \mathbb{N}} |\vec{m}|^{-1} e(\vec{m}) < \infty.
\]
Finally,
\[
\mathcal{I}_n \leq \sum_{n=1}^{m} A_{m+1} \sum_{\vec{m}_n \in \mathbb{N}} e(\vec{m}) |\vec{m}|(\Phi(\vec{m})) < \sum_{n=1}^{m} A_{m+1} < \infty,
\]
and this completes the proof of Lemma 2.

Let \(a_n\) denote either \(a_n\) or \(b_n\), and let us set, for a positive integer \(r\),
\[
T(a_n) = \sum_{i=1}^{r} \sum_{a_n \in M_i} \sum_{x_1 \in \mathbb{N}_{+}} \sum_{x_2 \in \mathbb{N}_{+}} \cdots \sum_{x_r \in \mathbb{N}_{+}} 2^{-m_1 \cdots - m_r} x_1^{m_1} \cdots x_r^{m_r}.
\]

The following equality will be useful: for arbitrary positive integer \(r\) and such \(x\) that \(x_i \in \{2^{-r+1}, 2^{-r}, \ldots, 2^{-r+1}\}, 1 \leq i \leq r, 1 \leq i \leq n,\)
\[
\sum_{n=1}^{N} 2^{-m_n} e(x_n) = T(a_n).
\]
It is easy to verify this fact by induction with respect to \(n\).

Lemma 3. For a sufficiently large \(r\) the following inequalities hold:
\[
T(b_n) \geq 2^{|\vec{m}|} r^{-|\vec{m}|},
\]
\[
T(a_n) \geq 2^{|\vec{m}|} r^{-|\vec{m}|}.
\]

Proof. First we shall prove that for an arbitrary \(r > 0\) one can find such an \(R(r)\) that for \(r > R(r)\),
\[
G(b_n) = T(b_n) - 2^{-|\vec{m}|} r^{-|\vec{m}|} \leq 2^{-|\vec{m}|} r^{-|\vec{m}|}.
\]
Indeed, \(G(b_n)\) can be represented as a finite combination of sums of the form
\[
H = \sum_{\vec{m}_n} \sum_{n=1}^{m_n} b_{n}^{2^{m_1} \cdots m_n} \sum_{\vec{m}_n} \sum_{n=1}^{m_n} b_{n}^{2^{m_1} \cdots m_n} \times |\vec{m}_n|^{-1} e(\vec{m}_n) (0 \leq s \leq k \leq n, k > 1),
\]
where the first group of sums is degenerated for \(s = 0\) and the second one for \(s = k\). However, from (3), (4), (6) and the estimate
\[
|\vec{m}_n|^{-1} e(\vec{m}_n) (0 \leq s \leq k \leq n, k > 1),
\]
we obtain
\[
H \leq C \left( 2 + \sum_{s=1}^{k} |p_{n}|^{-1} \right) \left( \frac{2}{2} \sum_{s=1}^{k} |p_{n}| \right) \left( \frac{2}{2} \sum_{s=1}^{k} |p_{n}| \right) 2^{-|\vec{m}|} r^{-|\vec{m}|} \times (2^{-1})^{-2^{-|\vec{m}|} / 2},
\]
which is integral to the last term tends to 0 while \(r \to \infty\) (because of the first factor for \(s = 0\) and of the second one for \(s > 0\)). Now it is easy to find such an \(R(r)\) that for \(r > R(r)\) we get (37). Setting \(r = 1/2\), we obtain (35). Since for a sufficiently large \(r\)
\[
b_{n}^{2^{-|\vec{m}|} r^{-|\vec{m}|}} \geq 2^{-1/2} r^{-|\vec{m}|},
\]
(36) immediately follows from (37).

Lemma 4. There are a positive integer \(R\) and a set \(P \subset \mathbb{N}^n\) such that for \(r > R\) we have
\[
\sum_{n \in P} e(x_n) = \infty,
\]
\[
T(a_n) \geq 2^{-|\vec{m}|} r^{-|\vec{m}|} V \in P,
\]
\[
T(a_n) \geq 2^{-|\vec{m}|} r^{-|\vec{m}|} V \in P.
\]

Proof. Set
\[
P = \bigcup_{k=1}^{M} Y_k,
\]
where
\[
Y_k = \{ \vec{m} \in W_k ; m_k \leq k \leq k+1 ; 1 \leq i \leq n, A_k \leq \min(2^{m_1}, \ldots, 2^{m_n}) \},
\]
(44) \[ Z_k = \{ \tilde{m}: \tilde{m} \in \mathbb{N}^n, 4L \leq m_i \leq L_{i+1}, 1 \leq i \leq n, \, A_2 \geq \min (2^m, k) \}. \]

Now by (44) and (27) we get
\[
\begin{align*}
\sum_{\tilde{m} \in Z_k} a_{\tilde{m}} & \sum_{|\tilde{m}|} \sum_{p \in \mathbb{P}_k} a_{\tilde{m}} e(|\tilde{m}|) - \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} a_{\tilde{m}} \sum_{p \in \mathbb{P}_k} a_{\tilde{m}} e(|\tilde{m}|) - \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} \sum_{p \in \mathbb{P}_k} a_{\tilde{m}} e(|\tilde{m}|) \\
& - C \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} \sum_{P \in \mathbb{P}_k} a_{\tilde{m}} e(|\tilde{m}|) - C = C = 0.
\end{align*}
\]

Hence, taking into account the monotonicity of the function \( a_{\tilde{m}} e(|\tilde{m}|) \) (for separated coordinates \( p_i, 1 \leq i \leq n \)), we obtain (39).

Now we shall prove (40). Since for an arbitrary positive integer \( r \) there exist infinitely many \( \tilde{p} \in P \) (as follows from (39)), there is a \( k_0 \) such that \( \tilde{p} \in \mathbb{P}_{k_0} \). Since the set \( P \) is symmetric with respect to the diagonal of an \( n \)-dimensional matrix, then it is enough to show that for any \( \epsilon > 0, 0 < L \leq \epsilon, \) and for a sufficiently large \( r \), the following inequality is satisfied:
\[
U = \sum_{|\tilde{m}|=2^\ell} \sum_{p \in \mathbb{P}_k} a_{\tilde{m}} e(|\tilde{m}|) - \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} a_{\tilde{m}} e(|\tilde{m}|) - \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} \sum_{p \in \mathbb{P}_k} a_{\tilde{m}} e(|\tilde{m}|) - C < 0.
\]

Let
\[
X = \{ \tilde{q}: \tilde{q} \in \mathbb{P}_k, 1 \leq i \leq k, \, q_i = p_i, \, k+1 \leq i \leq n \};
\]

One may check that
\[
U = \sum_{|\tilde{m}|=2^\ell} \sum_{\tilde{q} \in X} a_{\tilde{m}} e(|\tilde{m}|) - \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} a_{\tilde{m}} e(|\tilde{m}|) - \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} \sum_{\tilde{q} \in X} a_{\tilde{m}} e(|\tilde{m}|) - C < 0.
\]

Hence, by the properties of \( \Theta \), we obtain
\[
\sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} \sum_{\tilde{q} \in X} a_{\tilde{m}} e(|\tilde{m}|) - C = 0.
\]

Let
\[
\Lambda = \{ \tilde{q}: \tilde{q} \in \mathbb{Q} \}, \quad 1 \leq i \leq k, \quad \tilde{q} = q_i, \quad k+1 \leq i \leq n \};
\]

We obtain
\[
\sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} \sum_{\tilde{q} \in \Lambda} a_{\tilde{m}} e(|\tilde{m}|) - C = 0.
\]

Whence by (43) we get (44).

Now we shall prove (45). Since for an arbitrary positive integer \( r \) there exist infinitely many \( \tilde{p} \in P \) (as follows from (39)), there is a \( k_0 \) such that \( \tilde{p} \in \mathbb{P}_{k_0} \). Since the set \( P \) is symmetric with respect to the diagonal of an \( n \)-dimensional matrix, then it is enough to show that for any \( \epsilon > 0, 0 < L \leq \epsilon, \) and for a sufficiently large \( r \), the following inequality is satisfied:
\[
U = \sum_{|\tilde{m}|=2^\ell} \sum_{\tilde{q} \in X} a_{\tilde{m}} e(|\tilde{m}|) - \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} a_{\tilde{m}} e(|\tilde{m}|) - \sum_{\ell=2}^m \sum_{|\tilde{m}|=2^\ell} \sum_{\tilde{q} \in X} a_{\tilde{m}} e(|\tilde{m}|) - C < 0.
\]

Theorem 1. Under condition (*) the function
\[
f(x) = \left\langle \phi(x), \{c_n\} \right\rangle = \sum_{n=-\infty}^{\infty} c_n \phi_n(x)
\]

has the following property: there exists a subseries of Haar–Fourier series which converges to \( f \) in the norm of the Orlicz space \( L_{\Phi}(\mathbb{R}) \), though \( f \in L_{\Phi}(\mathbb{R}) \).

Proof. According to the definition of the function \( f \) and according to conditions (1), (8) and (28), by using properties of function \( \Theta \), we obtain
\[
\sum_{n=-\infty}^{\infty} c_n \phi_n(x) dx < \infty.
\]
Thus \( f \in L^p_s(I) \).

Now, choosing by Lemma 3 and 4 a sufficiently large \( r \), let us consider the series

\[
\sum_{n=1}^{\infty} 2^{\gamma_{\tau}} c_n \varphi(\varphi^{-1}(\xi))
\]

and let

\[
E_M = \{ \eta \in M : \eta = 0 \} \quad \text{for} \quad c_n = a_n, \\
E_M = \{ \eta \in M : \eta = 0 \} \quad \text{for} \quad c_n = b_n.
\]

Taking into account (34), (35) and (40) and the fact that \( \Phi \) is even and using the definition of the set \( E_M \), we get

\[
B = \left\{ \varphi \left( \sum_{n=1}^{\infty} 2^{\gamma_{\tau}} c_n \varphi(\varphi^{-1}(\xi)) \right) dx \geq \sum_{n=1}^{\infty} 2^{\gamma_{\tau} - \eta} \right\}
\]

\[
\Phi \left( \sum_{n=1}^{\infty} 2^{\gamma_{\tau}} c_n \varphi(\varphi^{-1}(\xi)) \right) dx \geq 2^{\gamma_{\tau} - \eta} \Phi \left( \varphi(\varphi^{-1}(\xi)) \right) \geq C \sum_{n=1}^{\infty} 2^{\gamma_{\tau} - \eta} \Phi \left( \varphi(\varphi^{-1}(\xi)) \right),
\]

whence by (47) it follows that \( B \to \infty \) for \( M \to \infty \) (49). Now let us compute the Fourier coefficients of the function \( f \):

\[
\hat{f}(k) = \sum_{n=1}^{\infty} 2^{\gamma_{\tau} - \eta} \sum_{l \in \mathbb{Z}} (-1)^{k+l} \int_0^{2\pi} \cdots \int_0^{2\pi} \cdots \int_0^{2\pi} f(x) dx = 2^{\gamma_{\tau} - \eta} \hat{g}(k).
\]

Thus by the properties of the function \( \Phi \) and by (46), (50) and (49) we obtain

\[
\lim_{M \to \infty} \int \Phi \left( \sum_{n=1}^{\infty} 2^{\gamma_{\tau}} c_n \varphi(\varphi^{-1}(\xi)) \right) dx \geq \lim_{M \to \infty} \int \Phi \left( \sum_{n=1}^{\infty} 2^{\gamma_{\tau}} c_n \varphi(\varphi^{-1}(\xi)) \right) dx - \infty
\]

\[
= -C \lim_{M \to \infty} \int \phi \left( \sum_{n=1}^{\infty} 2^{\gamma_{\tau}} \theta_n \varphi(\varphi^{-1}(\xi)) \right) dx = \infty.
\]

However, it follows from (51) (see [7], p. 92) that for the Haar–Fourier series of the function \( f \) there exists a subseries which diverges in the norm of the Orlicz space \( L^p_s(I) \):

\[
\sum_{n=1}^{\infty} \varphi(\varphi^{-1}(\xi)) \Phi(2^\alpha \varphi(\varphi^{-1}(\xi))) \Phi(2^\alpha \varphi(\varphi^{-1}(\xi))) < \infty,
\]

and this completes the proof.

References

ТЕОРИЯ ЭКСТРЕМАЛЬНЫХ ЗАДАЧ И ТЕОРИЯ ПРИБЛИЖЕНИЙ

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1. Постановки некоторых экстремальных задач теории приближений

Постановки экстремальных задач сопровождают всю историю теории приближений. Еще в 18 веке Леманн написал, выражаясь современным языком, полиномы наименьшего уклонения от нуля в метриче пространства \( \mathcal{V}_d([-1, 1]) \). Простой пример, по которому он разрешил следующую проблему минимизации (1):

\[
 f_{10}(x) = \int \left( t^2 + \sum_{k=1}^{10} x_k t^{k-1} \right) dt = \| t^2 + p_1(t) \|_{\mathcal{V}_d([-1, 1])} \to \inf.
\]

Получившиеся в результате решения задач (1) полиномы имеют вид:

\[
 T_{12}(t) = \frac{r^1}{(2\pi)^1} \frac{d^r}{dt^r} (t^2 - 1)^2;
\]

они пропорциональны полиномам \( P_r(t) \), называемым полиномами Леманна. Чебышев решил аналогичную задачу в двух других метриках: \( C([-1, 1]) \) и \( L_1([-1, 1]) \). Решение задачи

\[
 f_{max}(x) = \max \{ 1 \leq k \leq 10 \} \left| t^2 + \sum_{k=1}^{10} x_k t^{k-1} \right| = \| t^2 + p_1(t) \|_{C([-1, 1])} \to \inf
\]

является полиномом Чебышева: \( T_{12}(t) = 2^{-0.5} \cos(\pi \cos(x)) \), а решением задачи

\[
 f_{10}(x) = \int \left( t^2 + \sum_{k=1}^{10} x_k t^{k-1} \right) dt = \| t^2 + p_1(t) \|_{L_1([-1, 1])} \to \inf
\]

(*) Если \( X \) — некоторое множество, \( f : X \to \mathbb{R} \) — функционал на нем, \( C \subseteq X \) — подмножество \( X \), наименьшее ограничение, то задача минимизации на \( C \) обозначается как \( f \to \inf \), \( x \in C \).

18 Banach Center t. IV