Added in proof

The conclusion of Lemma 3 remains valid for every Banach space E with the b.a.p. and every total subspace V of E^* with codim $_{E^*}V < \infty$ (see L. D. Menihes and A. L. Pličko, *Conditions of linear and finite-dimensional regularizability of linear inverse problems*, Dokl. Akad. Nauk SSSR 241 (1978), pp. 1027–1030 (Russian)). Consequently, the answer to the last question above is affirmative.

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A RELATION BETWEEN FOURIER TRANSFORMS IN ONE AND TWO VARIABLES

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Let $\psi \in C^\infty(I)$ be real-valued, where $I = [0, 1]$; define $\gamma\colon I \to \boldsymbol{R}^2$ by

$$\gamma(t) = (t, \psi(t)), \quad t \in I,$$

and let Γ denote the curve $\gamma(I)$. We let dS denote the arc length measure on Γ and set

$$Sf(x) = \int_\Gamma e^{ix\cdot t} f(t)\, dS(t), \quad f \in L^1(\Gamma; dS),\ x \in \boldsymbol{R}^2.$$

It follows from the restriction theorem of C. Fefferman and E. M. Stein that

$$(1) \qquad \|S(f\circ\gamma^{-1})\|_{L^q(\boldsymbol{R}^2)} \leqslant C_{q,p}\|f\|_{L^p(\boldsymbol{R})}, \quad f \in C_0^\infty(\boldsymbol{R}),$$

for $4 < q < \infty$ and $q/(q-3) \leqslant p \leqslant \infty$, if Γ has non-vanishing curvature at every point (see Fefferman [2], Hörmander [4] and Zygmund [8]).

We define an operator Q in the following way. Let ϕ be a fixed function in $C_0^\infty(\boldsymbol{R})$ with support contained in $(0, 1)$ and set $Qf = S((\phi\hat{f})\circ\gamma^{-1})$, $f \in C_0^\infty(\boldsymbol{R})$, where $\hat{f}(u) = \int e^{-iut} f(t)\, dt$. We shall here study the problem of determining for what values of α the estimate

$$(2) \qquad \Big(\int_{\boldsymbol{R}^2} |Qf(x)|^p (1+|x|)^{-\alpha}\, dx\Big)^{1/p} \leqslant C_p\|f\|_{L^p(\boldsymbol{R})}$$

is valid for all Γ and ϕ of the above type (here C_p may depend on Γ and ϕ, but not on f, and we do not assume that Γ has non-vanishing curvature). It turns out that the operator Q is closely related to the Fourier multipliers $[(y-\psi(x))_+]^\alpha \varrho(x, y)$, $\alpha > 0$, $\varrho \in C_0^\infty(\boldsymbol{R}^2)$, studied in Carleson and Sjölin [1], Fefferman [3], Hörmander [4] and Sjölin [6]. We shall prove the following theorem.

THEOREM 1. *Let Q be defined as above. Then* (2) *holds for*

$$(3) \qquad 1 \leqslant p \leqslant 2 \quad \textit{and} \quad \alpha > 2-p/2,$$

$$(4) \qquad 2 < p \leqslant 4 \quad \textit{and} \quad \alpha > 1$$

and for

$$(5) \qquad 4 < p < \infty \quad \textit{and} \quad \alpha > p/2-1.$$

We shall also prove that the conditions on α in (3) and (4) can not be relaxed and that it is not possible to have $\alpha < p/2-1$ in (5). In the case $p = 2$ the above estimate is a consequence of Plancherel's theorem. To treat the case $p > 2$ we shall use the following lemma which is proved in [6].

LEMMA 1. *Let ϕ and $\psi \in C^\infty(I)$ and assume that ψ is real-valued. Set*

$$K_N(x) = N\int_I e^{iN(x_1u+x_2\psi(u))}\phi(u)\,du, \quad x = (x_1, x_2) \in \mathbf{R}^2,\ N \geqslant 2,$$

and

$$T_Nf(x) = \int_0^1 K_N(x_1-t, x_2)f(t)\,dt, \quad f \in L^1(0, 1).$$

Then, if $4 < p \leqslant \infty$ and $\varepsilon > 0$, there exists a constant C_p, depending only on ψ, ϕ, p and ε, such that

$$\|T_Nf\|_{L^p(D)} \leqslant C_pN^{1/2-2/p+\varepsilon}\|f\|_{L^p(0,1)},$$

where $D = \{x \in \mathbf{R}^2;\ |x_i| \leqslant 10,\ i = 1, 2\}$.

Theorem 1 is a consequence of the following lemma.

LEMMA 2. *Let ϕ and $\psi \in C^\infty(I)$ and assume that ψ is real-valued and ϕ has support in the interior of I. Set*

$$Pf(x) = \int_I e^{i(x_1u+x_2\psi(u))}\phi(u)\hat{f}(u)\,du, \quad f \in C_0^\infty(\mathbf{R}),\ x \in \mathbf{R}^2.$$

Then

$$\Big(\int_{\mathbf{R}^2} |Pf(x)|^p(1+|x|)^{-\alpha}dx\Big)^{1/p} \leqslant C_p\|f\|_{L^p(\mathbf{R})}, \tag{6}$$

if p and α satisfy the conditions in Theorem 1.

Proof. Case 1. $p = 2$. From Plancherel's theorem it follows that for every x_2,

$$\int |Pf(x)|^2dx_1 = \frac{1}{2\pi}\int |\phi(u)|^2|\hat{f}(u)|^2du \leqslant C\int |\hat{f}(u)|^2du = C\int |f(t)|^2dt$$

and we obtain (6) with $p = 2$ if $\alpha > 1$.

Case 2. $p = 1$. We assume $\alpha > 3/2$. The Schwartz inequality and the first equality in Case 1 yield

$$\int |Pf(x)|(1+|x|)^{-\alpha}dx \leqslant \Big(\int |Pf(x)|^2(1+|x|)^{-2\alpha/3}dx\Big)^{1/2}\Big(\int (1+|x|)^{-4\alpha/3}dx\Big)^{1/2}$$

$$\leqslant C\Big(\int |\phi(u)|^2|\hat{f}(u)|^2du\Big)^{1/2} \leqslant C\|\hat{f}\|_{L^\infty(\mathbf{R})} \leqslant C\|f\|_{L^1(\mathbf{R})},$$

since $2\alpha/3 > 1$ and $4\alpha/3 > 2$.

Case 3. $1 < p < 2$. The desired estimate follows from interpolation between the cases $p = 1$ and $p = 2$ (see Stein [7]).

Case 4. $4 < p < \infty$. For $l = 0, 1, 2, \ldots$ we let Ω_l denote the set of all intervals $\omega_{lk} = (k2^l, (k+1)2^l)$, $k \in \mathbf{Z}$, and Ω_l^* the set of all intervals $\omega_{lk}^* = (k2^l, (k+2)2^l)$, $k \in \mathbf{Z}$. For $x \in \mathbf{R}_+^2 = \{x \in \mathbf{R}^2;\ x_2 > 0\}$ let $\omega_l^*(x)$ denote the interval in Ω_l^* which contains x_1 in its middle half, and let $n(x)$ be an integer defined by $2^{n(x)-1} \leqslant x_2 < 2^{n(x)}$, if $x_2 \geqslant 1$, and $n(x) = 0$, if $0 < x_2 < 1$. We set $\omega_l(x) = \omega_{l+1}^*(x)\setminus\omega_l^*(x)$ for $l > n(x)$ and $\omega_l(x) = \omega_{l+1}^*(x)$ if $l = n(x)$. It follows from the construction that, for $l > n(x)$, $\omega_l(x)$ is the union of two intervals $\omega_l^1(x)$ and $\omega_l^2(x)$ belonging to Ω_l, and that there exist two positive constants c_1 and c_2 such that $c_12^l < \operatorname{dist}(x, \omega_l^i(x)) < c_22^l$, for $i = 1, 2$. For $l = n(x) > 0$, $\omega_l(x)$ is a union of four intervals $\omega_l^i(x)$ which also satisfy the above inequality. We have

$$Pf(x) = \int\Big(\int e^{i((x_1-t)u+x_2\psi(u))}\phi(u)\,du\Big)f(t)\,dt$$

and we denote the inner integral by $K(x_1-t, x_2)$. For $x_2 > 0$ we have

$$Pf(x) = \sum_{l=n(x)}^{\infty}\int_{\omega_l(x)} K(x_1-t, x_2)f(t)\,dt \tag{7}$$

$$= \sum_{l=0}^{\infty}\sum_{k=-\infty}^{\infty}\chi_{lk}(x)\int_{\omega_{lk}} K(x_1-t, x_2)f(t)\,dt,$$

where χ_{lk} is the characteristic function of the set $E_{lk} = \{x \in \mathbf{R}_+^2;\ \omega_{lk} \subset \omega_l(x)\}$. From the above inequalities it follows that there exist positive constants c_3 and c_4 such that

$$E_{lk} \subset \{x;\ c_32^l < \operatorname{dist}(x, \omega_{lk}) < c_42^l\} \quad \text{if} \quad l > 0.$$

We denote the last sum in (7) by $P_lf(x)$ so that $Pf(x) = \sum_{l=0}^{\infty}P_lf(x)$. The Hölder inequality yields

$$|P_lf(x)|^p \leqslant C\sum_{k=-\infty}^{\infty}\chi_{lk}(x)\Big|\int_{\omega_{lk}} K(x_1-t, x_2)f(t)\,dt\Big|^p, \tag{8}$$

since there exists a constant A such that, for every l and every $x \in \mathbf{R}_+^2$ x is contained in at most A of the sets E_{lk}, $k \in \mathbf{Z}$. Let

$$E'_{lk} = \{x \in E_{lk};\ x_2 > b(k2^l-x_1) \text{ and } x_2 > b(x_1-(k+1)2^l)\}$$

and $E''_{lk} = E_{lk}\setminus E'_{lk}$, where we choose the positive constant b so small that $|x_1-t+x_2\psi'(u)| > b_02^l$ for $t \in \omega_{lk}$, $x \in E''_{lk}$, $u \in I$ and some constant $b_0 > 0$. Then

$$x_2 > c_52^l, \quad x \in E'_{lk},\ l > 0, \tag{9}$$

where c_5 is a positive constant. Denoting the integral in (8) by $P_{lk}f(x)$, we shall prove that

$$\Big(\int_{E_{lk}} |P_{lk}f(x)|^p(1+x_2)^{-\alpha}dx\Big)^{1/p} \leqslant C_p2^{l(1/2-1/p-\alpha/p+\varepsilon)}\Big(\int_{\omega_{lk}} |f(t)|^pdt\Big)^{1/p} \tag{10}$$

and from translation invariance we may assume that $k = 0$. Using the notation of

Lemma 1 we have

$$P_{l0}f(x) = \int_0^1 K_N(x_1'-t', x_2')f(Nt')\,dt',$$

where $N = 2^l$ and $x = Nx'$. Lemma 1 yields

$$\Big(\int_{E_{lk}'} |P_{l0}f(x)|^p dx\Big)^{1/p} \leqslant \Big(\int_{C_1 D} |P_{l0}f(Nx')|^p dx'\Big)^{1/p} N^{2/p}$$

$$\leqslant C_p N^{1/2+\varepsilon}\Big(\int_0^1 |f(Nt')|^p dt'\Big)^{1/p} \leqslant C_p N^{1/2-1/p+\varepsilon}\Big(\int_{\omega_{l0}} |f(t)|^p dt\Big)^{1/p},$$

where we have used the obvious fact that Lemma 1 holds also if the square D is replaced by $C_1 D = \{C_1 x;\ x \in D\}$, for some constant C_1. Using (9), we now obtain (10) with E_{lk} replaced by E_{lk}'. For $t \in \omega_{lk}$, $x \in E_{lk}''$ it follows from repeated partial integrations in the integral defining K that $|K(x_1-t, x_2)| \leqslant CN^{-10}$, since $|x_1-t+ +x_2\psi'(u)| > b_0 N$ for $u \in I$. Hence (10) holds with E_{lk} replaced by E_{lk}'' and (10) is completely proved. From (8) and (10) it follows that

$$\int_{R_+^2} |P_l f(x)|^p (1+x_2)^{-\alpha} dx \leqslant C_p 2^{l(p/2-1-\alpha+p\varepsilon)} \int |f(t)|^p dt$$

and, since $\alpha > p/2-1$, we can choose ε so small that $p/2-1-\alpha+p\varepsilon = -\delta$, where $\delta > 0$. Hence

$$\Big(\int_{R_+^2} |P_l f(x)|^p (1+|x_2|)^{-\alpha} dx\Big)^{1/p} \leqslant C_p 2^{-l\delta/p} \|f\|_{L^p_{(R)}}$$

(and the same inequality holds, if R_+^2 is replaced by R_-^2) and summing over l we obtain Lemma 2 in the case $4 < p < \infty$.

Case 5. $2 < p \leqslant 4$. Interpolation between the cases $p = 2$ and $p > 4$ gives the desired estimate.

The proof of Theorem 1 is complete.

We shall finally give a description of the counterexamples mentioned after the statement of Theorem 1. It is clear that, for every value of p, $\alpha > 1$ is a necessary condition for (2) to hold. This follows from the case when Γ is a straight line segment. A counterexample for $1 \leqslant p < 2$ is obtained by assuming that Γ has positive curvature and choosing f and ϕ so that $\hat{f}\phi = 1$ in an interval. Then $|Qf(x)| \geqslant c|x|^{-1/2}$, where c is a positive constant, when $|x|$ is large and x belongs to a certain cone with vertex at the origin (see e.g. Littman [5]). If (2) holds, the integral $\int_{|x|>1} |x|^{-p/2-\alpha} dx$ has to be convergent, and hence $\alpha > 2-p/2$.

It remains to treat the case $4 < p < \infty$. A counterexample can be obtained from the connection between the operator Q and the multipliers mentioned above, but we shall give a direct argument (cf. [4], pp. 9–10).

We set

$$Pf(x) = \int_{-1}^{1} e^{i(x_1 u + x_2 u^2)}\phi(u)\hat{f}(u)\,du,$$

where $\phi \in C_0^\infty(R)$ has support in the interior of $[-1, 1]$ and $\phi(u) = 1$ for $|u| \leqslant 1/2$, and assume that inequality (6) in Lemma 2 holds. We shall prove that then necessarily $\alpha \geqslant p/2-1$. We choose $f(t) = g(t/N)e^{it^2/N}$, where $g \in C_0^\infty(R)$ equals 1 in a neighbourhood of the origin. Then $\|f\|_{L^p(R)} = C_p N^{1/p}$ and

$$Pf(Nx) = N\iint e^{iN(x_1 u - tu + x_2 u^2 + t^2)} g(t)\phi(u)\,dt\,du.$$

Setting $v = t-u/2$ we obtain $x_1 u - tu + x_2 u^2 + t^2 = x_1 u + (x_2 - 1/4)u^2 + v^2$. It therefore follows from the stationary phase method that

$$|Pf(Nx)| \geqslant c_1 |x_2 - 1/4|^{-1/2} \quad \text{for} \quad |x_1| < c_2|x_2-1/4|,\ c_3 < N|x_2-1/4|,$$

where c_1, c_2, c_3 are positive constants. Hence $|Pf(x)| > c_1 N^{1/2}|x_2 - N/4|^{-1/2}$ for $|x_1| < c_2|x_2-N/4|$, $c_3 < |x_2-N/4|$. It follows that

$$\int |Pf(x)|^p (1+|x|)^{-\alpha} dx \geqslant c_4 N^{p/2-\alpha},$$

where c_4 is a positive constant. Inequality (6) yields $N^{p/2-\alpha} \leqslant C_p N$, but this can hold for large values of N only if $\alpha \geqslant p/2-1$.

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