However, this method and that in Remark 4 resort to ideas rather deeper than are needed for the problem, so perhaps the simple counterexample of the present note, the need for which was suggested by Ehrling, is not altogether superfluous.

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ON BANACH SPACES IN WHICH EVERY M-BASIS IS A GENERALIZED SUMMATION BASIS

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We recall that if E is a Banach space, a (countable) biorthogonal system (x_n, f_n) in (E, E^*) is called an M-*basis* (*Markushevich basis*) for E if $\{x_n\}$ is complete in E (i.e., the closed linear span$[x_n]$ of $\{x_n\}$ is E, so E must be separable) and $\{f_n\}$ is total over E (i.e., $\{x \in E |\ f_n(x) = 0\ (n = 1, 2, \ldots)\} = \{0\}$); it is well known that M-basis exist in every separable space E. Following M. I. Kadec [7], an M-basis (x_n, f_n) is called a *generalized summation basis* (g.s.b.) for E, if there exists a sequence $\{t_n\}$ of linear operators with $t_n\colon [x_i]_1^n \to [x_i]_1^n$ $(n = 1, 2, \ldots)$, such that $x = \lim_{n\to\infty} t_n s_n(x)$, for all $x \in E$, where $s_n(x) = \sum_{i=1}^{n} f_i(x) x_i$ $(x \in E, n = 1, 2, \ldots)$. A separable Banach space E is said to have the *bounded approximation property* (b.a.p.), if there exists on E a sequence of continuous linear operators $\{u_n\}$ of finite rank (i.e., $\dim u_n(E) < \infty$), such that $x = \lim_{n\to\infty} u_n(x)$, for all $x \in E$. Thus, if E has a g.s.b., then E is separable and has the b.a.p.; W. B. Johnson has proved that the converse is also true ([5], Theorem IV.1).

M. I. Kadec has shown ([7], Theorem 4) that if a reflexive space E has a g.s.b., then every M-basis for E is a g.s.b. By the above-mentioned result of Johnson, this is equivalent to the fact that, *in a separable reflexive space E with the* b.a.p., *every* M-*basis for E is a* g.s.b. ([5], Corollary IV.2). Therefore it is natural to raise the problem of characterizing the (separable) Banach spaces E with this property. Of course, a necessary condition is that such a space *E must have the* b.a.p. Furthermore, another necessary condition is that *E must be quasi-reflexive* (i.e., $\dim E^{**}/\pi(E) < \infty$, where $\pi\colon E \to E^{**}$ is the canonical isometrical embedding); indeed, for every separable non-quasi reflexive space E, the dual E^* contains a separable total subspace V of characteristic zero [2], and then E has an M-basis (x_n, f_n) with $[f_n] = V$ (see e.g. [5], Theorem III.1), but, as was observed by Kadec ([7], Theorem

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3 and [4], Theorem 2), such an M-basis is not a g.s.b. We shall prove that these two conditions are also sufficient.

LEMMA 1 (W. B. Johnson [6], Lemma 1). *Let V, G be two Banach spaces, where $\dim G < \infty$ and let Γ be a subspace of V^* with $\dim \Gamma < \infty$. Then, for every continuous linear operator $\tilde{u}$: $V^* \to G$ and every $\varepsilon > 0$, there exists a w^*-continuous linear operator $\tilde{v}$: $V^* \to G$ such that*

$$\tilde{v}|_{\Gamma} = \tilde{u}|_{\Gamma}, \tag{1}$$

$$\|\tilde{v}\| \leqslant \|u\| + \varepsilon. \tag{2}$$

Using this lemma, we shall prove

LEMMA 2. *Let E be a quasi-reflexive Banach space, G a finite-dimensional subspace of E and V a total closed linear subspace of E^*. Then, for every continuous linear operator u: $E \to G$, there exists a $\sigma(E, V)$-continuous linear operator v: $E \to G$ such that*

$$v|_G = u|_G, \tag{3}$$

$$\|v\| \leqslant C\|u\|, \tag{4}$$

where C is a constant which does not depend on $\dim G$.

Proof. If $\dim E^{**}/\pi(E) = n < \infty$, then, by [8], Lemma 4, we have $0 \leqslant \operatorname{codim}_{E^*} V = k \leqslant n$ and $\operatorname{codim}_{E^{**}}(\pi(E) \oplus V^{\perp}) = n-k$. The latter equality implies, by [8], proof of Theorem 1 (implication 2° ⇒ 3°), that $\operatorname{codim}_{V^*} \varphi(E) = n-k$, where φ denotes the canonical mapping of E into V^*, defined by $\varphi(x)(f) = f(x)$ $(x \in E, f \in V)$; note that since $\operatorname{codim}_{E^*} V < \infty$, the characteristic $r(V)$ of V is > 0 and hence φ is an isomorphism with $\|\varphi^{-1}\| \leqslant 1/r(V)$ (see [3]). Let p be any continuous linear projection of V^* onto $\varphi(E)$. If u: $E \to G$ is a continuous linear operator, define $\tilde{u}$: $V^* \to G$ by $\tilde{u} = u\varphi^{-1}p$. By Lemma 1 above, with $\Gamma = \varphi(G) \subset V^*$, for $0 < \varepsilon \leqslant \|\tilde{u}\|$, there exists a w^*-continuous linear operator $\tilde{v}$: $V^* \to G$ such that

$$\begin{aligned}\tilde{v}(\varphi(y)) &= \tilde{u}(\varphi(y)) = u\varphi^{-1}p(\varphi(y)) \\ &= u\varphi^{-1}\varphi(y) = u(y) \quad (y \in G),\end{aligned} \tag{5}$$

$$\|\tilde{v}\| \leqslant \|\tilde{u}\| + \varepsilon \leqslant 2\|\tilde{u}\| \leqslant 2\|\varphi^{-1}\|\,\|p\|\,\|u\| \leqslant \frac{2\|p\|}{r(V)}\|u\|. \tag{6}$$

Now define v: $E \to G$ by $v = \tilde{v}\varphi$. Then, since $\tilde{v}$ is $\sigma(V^*, V)$-continuous, v is $\sigma(E, V)$-continuous. Also, by (5) we have (3) and by (6) and $\|\varphi\| \leqslant 1$, we have (4) with $C = 2\|p\|/r(V)$, which completes the proof.

LEMMA 3. *Let E be a separable quasi-reflexive Banach space with the* b.a.p. *and let V be a total closed linear subspace of E^*. Then there exists on E a sequence of continuous linear operators $\{v_n\}$ of finite rank, such that*

$$x = \lim_{n\to\infty} v_n(x) \quad (x \in E), \tag{7}$$

$$\bigcup_{n=1}^{\infty} v_n^*(E^*) \subset V. \tag{8}$$

Proof. By our assumption, there exists a sequence of continuous linear operators $\{u_n\}$ of finite rank, such that $x = \lim_{n\to\infty} u_n(x)$ $(x \in E)$, whence $\sup_n \|u_n\| = \lambda < \infty$; since this convergence is uniform on compact subsets of E, we may assume (by passing to a subsequence of $\{u_n\}$) that

$$\|x - u_n(x)\| < \frac{1}{n}\|x\| \quad \Big(x \in \Big[\bigcup_{i=1}^{n-1} u_i(E)\Big],\ n = 2, 3, \ldots\Big). \tag{9}$$

By Lemma 2 (with $G = G_n = [\bigcup_{i=1}^{n} u_i(E)]$), for each $n = 1, 2, \ldots$ there exists a $\sigma(E, V)$-continuous linear operator v_n: $E \to G_n$ such that

$$v_n|_{G_n} = u_n|_{G_n} \quad (n = 1, 2, \ldots), \tag{10}$$

$$\|v_n\| \leqslant C\|u_n\| \leqslant C\lambda < \infty \quad (n = 1, 2, \ldots), \tag{11}$$

where C and λ do not depend on n. Then, from (9) and (10) it follows that $x = \lim_{n\to\infty} v_n(x)$ $(x \in \bigcup_{m=1}^{\infty} u_m(E))$, whence, by (11) and since $\bigcup_{m=1}^{\infty} u_m(E)$ is dense in E, we obtain (7). Finally, since v_n is $\sigma(E, V)$-continuous, we can write $v_n(x) = \sum_{i=1}^{m_n} h_i^{(n)}(x) y_i^{(n)}$, where $\{y_i^{(n)}\}_1^{m_n}$ is a basis of G_n and $\{h_i^{(n)}\} \subset V$. But then $v_n^*(f) = \sum_{i=1}^{m_n} f(y_i^{(n)}) h_i^{(n)} \in V$ $(f \in E^*)$, which proves (8).

THEOREM. *For a separable Banach space E the following statements are equivalent:*

(1) *Every* M-*basis for E is a* g.s.b.

(2) *E is quasi-reflexive and has the* b.a.p.

Proof. The implication (1) ⇒ (2) was observed before Lemma 1 above.

Conversely, assume that we have (2). From [5], proof of Theorem IV. 1, it follows that *if $\{v_n\}$ is a sequence of continuous linear operators of finite rank on E, satisfying* (7), *then every* M-*basis (x_n, f_n) for E with $\bigcup_{n=1}^{\infty} v_n^*(E^*) \subset [f_n]$ is a* g.s.b. Now let (x_n, f_n) be an arbitrary M-basis for E. Then, by Lemma 3 (with $V = [f_n]$), we can find $\{v_n\}$ of finite rank satisfying (7) and $\bigcup_{n=1}^{\infty} v_n^*(E^*) \subset [f_n]$. Hence, by the above observation, (x_n, f_n) is a g.s.b. Thus, (2) ⇒ (1), which completes the proof.

Remark. We do not know any characterization (a) of all Banach spaces E for which Lemma 2 or Lemma 3 remains valid, with "V total" replaced by "V norming" (i.e., $r(V) > 0$); (b) of the Banach spaces E in which every norming M-basis (x_n, f_n) (i.e., such that $r([f_n]) > 0$) is a g.s.b. By the above, every quasi-reflexive space (respectively, having also the b.a.p.) has these properties, but there might exist non-quasi-reflexive ones, too. For example, we do not know whether the non-quasi-reflexive spaces E such that every norming subspace V of E^* is of finite codimension in E^* have these properties (it is known that such spaces exist [1]).

Added in proof

The conclusion of Lemma 3 remains valid for every Banach space E with the b.a.p. and every total subspace V of E^* with codim $_{E^*}V < \infty$ (see L. D. Menihes and A. L. Pličko, *Conditions of linear and finite-dimensional regularizability of linear inverse problems*, Dokl. Akad. Nauk SSSR 241 (1978), pp. 1027–1030 (Russian)). Consequently, the answer to the last question above is affirmative.

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A RELATION BETWEEN FOURIER TRANSFORMS IN ONE AND TWO VARIABLES

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Let $\psi \in C^\infty(I)$ be real-valued, where $I = [0, 1]$; define $\gamma\colon I \to \boldsymbol{R}^2$ by

$$\gamma(t) = (t, \psi(t)), \quad t \in I,$$

and let Γ denote the curve $\gamma(I)$. We let dS denote the arc length measure on Γ and set

$$Sf(x) = \int_\Gamma e^{ix\cdot t} f(t)\, dS(t), \quad f \in L^1(\Gamma; dS),\ x \in \boldsymbol{R}^2.$$

It follows from the restriction theorem of C. Fefferman and E. M. Stein that

$$\|S(f\circ\gamma^{-1})\|_{L^q(\boldsymbol{R}^2)} \leqslant C_{q,p}\|f\|_{L^p(\boldsymbol{R})}, \quad f \in C_0^\infty(\boldsymbol{R}), \tag{1}$$

for $4 < q < \infty$ and $q/(q-3) \leqslant p \leqslant \infty$, if Γ has non-vanishing curvature at every point (see Fefferman [2], Hörmander [4] and Zygmund [8]).

We define an operator Q in the following way. Let ϕ be a fixed function in $C_0^\infty(\boldsymbol{R})$ with support contained in $(0, 1)$ and set $Qf = S((\phi\hat{f})\circ\gamma^{-1})$, $f \in C_0^\infty(\boldsymbol{R})$, where $\hat{f}(u) = \int e^{-iut} f(t)\, dt$. We shall here study the problem of determining for what values of α the estimate

$$\Big(\int_{\boldsymbol{R}^2} |Qf(x)|^p (1+|x|)^{-\alpha} dx\Big)^{1/p} \leqslant C_p \|f\|_{L^p(\boldsymbol{R})} \tag{2}$$

is valid for all Γ and ϕ of the above type (here C_p may depend on Γ and ϕ, but not on f, and we do not assume that Γ has non-vanishing curvature). It turns out that the operator Q is closely related to the Fourier multipliers $[(y-\psi(x))_+]^\alpha \varrho(x, y)$, $\alpha > 0$, $\varrho \in C_0^\infty(\boldsymbol{R}^2)$, studied in Carleson and Sjölin [1], Fefferman [3], Hörmander [4] and Sjölin [6]. We shall prove the following theorem.

THEOREM 1. *Let Q be defined as above. Then* (2) *holds for*

$$1 \leqslant p \leqslant 2 \quad \text{and} \quad \alpha > 2 - p/2, \tag{3}$$

$$2 < p \leqslant 4 \quad \text{and} \quad \alpha > 1 \tag{4}$$

and for

$$4 < p < \infty \quad \text{and} \quad \alpha > p/2 - 1. \tag{5}$$