linear manifold of functions in $C(T)$. The above argument establishes that the closure of E in $C(T)$ equals the intersection with $C(T)$ of the closure of E in $L^1(T)$.

Thus the essential feature of E is the invariance of its elements under rotations of T. In like manner, the example in Section 4 can be generalized, replacing E by any linear manifold in H^p invariant with respect to the backward shift operator, that is the operator

$$V\colon (a_0, a_1, a_2, \ldots) \to (a_1, a_2, \ldots)$$

acting on the Taylor coefficients.

Thus, we might hope for a theorem of some generality in which X and Y are topological spaces of functions defined on some semigroup S, and $E \subset X$ is assumed invariant with respect to some transformation(s) of S. However, I could not so far even extend the example in Section 5 in this spirit. The problem here is: *Suppose E is a linear manifold in (say) $H^2(U)$, $VE \subset E$, and E is not dense in $L_a^2(U)$. Is E an $(H^2(U), L_a^2(U))$ manifold*? (That is: does the L_a^2 closure of E, intersected with H^2, equal the H^2 closure of E?) I would guess the answer is affirmative. This is a crucial test problem for ascertaining whether or not a theorem of some generality ultimately can be hoped for.

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Presented to the Semester
Approximation Theory
September 17–December 17, 1975

APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

A COUNTEREXAMPLE IN HARMONIC ANALYSIS

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Let B be a commutative Banach algebra with identity, and suppose $x \in B$ has norm 1 and satisfies

$$(1) \qquad |\hat{x}(m)| \geqslant \delta > 0, \quad \text{all } m \in M$$

where $\hat{x}$ denotes the Gelfand transform of x, and M the maximal ideal space of B. Then x is invertible; Gunnar Ehrling has raised the question, in connection with a problem arising in theoretical physics, whether there exists a constant $N(B;\delta)$ depending only on B and δ such that $\|x^{-1}\| \leqslant N(B;\delta)$ for all $x \in B$ satisfying (1).

In this note we show that the answer is *no* in the case where B is the algebra of absolutely convergent Taylor series. More precisely: let A denote the Banach algebra of functions f analytic on the open unit disc U whose series of Taylor coefficients is absolutely convergent:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \|f\| = \sum_{n=0}^{\infty} |a_n| < \infty$$

with multiplication in A defined as the usual pointwise multiplication of functions of U. The maximal ideal space of A is the closure U^- of U. We shall denote $\max_{z \in U^-} |f(z)|$ by $\|f\|_\infty$. Our result is, then:

THEOREM. *There exists a sequence $\{f_n\}_1^\infty \subset A$ and a positive absolute constant δ such that*

(i) $\|f_n\| = 1$, $n = 1, 2, \ldots$,

(ii) $|f_n(z)| \geqslant \delta$, $n = 1, 2, \ldots$; $z \in U^-$,

(iii) $\lim_{n\to\infty} \|f_n^{-1}\| = \infty$.

It will be convenient to precede the proof by some lemmas.

LEMMA 1. *For $f \in A$ we have*

$$(2) \qquad \|f\| \leqslant |f(0)| + \frac{1}{2}\int_0^{2\pi} |f'(e^{i\theta})|\,d\theta.$$

Proof. This is a classical inequality of Hardy and Littlewood. (See [2], vol. I, p. 286, Theorem (8.7).)

LEMMA 2. *Suppose $f \in A$ does not vanish in U^-, $f(0) = 1$, and n is a positive integer. Then there is an element $g \in A$ such that $g(0) = 1$, $g^n = f$, and $\|g\|^n \leqslant C$ where C is a constant depending only on f (we emphasize that it is independent of n).*

Proof. The existence of $g = f^{1/n}$ in A follows from the Wiener–Lévy theorem. Applying (2) now yields

$$\|g\| \leqslant 1+\frac{1}{2n}\int_0^{2\pi} |f(e^{i\theta})|^{1/n-1}|f'(e^{i\theta})|\,d\theta \leqslant 1+Bn^{-1}$$

where B depends only on f. Hence

$$\|g\|^n \leqslant (1+Bn^{-1})^n \leqslant e^B$$

proving the assertion.

Proof of theorem. Consider first the polynomial

$$P(z) = \big(1+(z/2)\big)\big(1-(z/3)\big). \tag{3}$$

Observe that it has no zeroes in U^-, and $\|P\| = 4/3$. Moreover $\|P\|_\infty$ is obviously less than 4/3 (computation shows it is around 1.19). Hence $p = (3/4)P$ satisfies $\|p\| = 1$, $\|p\|_\infty < 0.9$. Choose an integer k such that $k+1$ is a power of 2, say $k+1 = 2^r$, and define $g_k \in A$ by

$$g_k(z) = [p(z)p(z^3)p(z^9)\dots p(z^{3^k})]^{1/(k+1)}.$$

Observe that the A-norm of the product inside brackets equals the product of the norms of the factors, hence

$$\|g_k^{k+1}\| = 1$$

and so

$$1 = \|g_k^{2^r}\| \leqslant \|g_k^{2^{r-1}}\|^2 \leqslant \|g_k^{2^{r-2}}\|^4 \leqslant \dots \|g_k\|^{2^r}$$

showing that

(i) *all the elements $g_k, g_k^2, g_k^4, \dots g_k^{2^r}$ have norm at least* 1.

Moreover, we have

$$\|g_k\|_\infty \leqslant 0.9$$

and so

(ii) *the element $h_{N,k}$ of A defined by*

$$h_{N,k}(z) = 1-z^N g_k(z) \tag{4}$$

satisfies

$$\min_{z\in U^-}|h_{N,k}(z)| \geqslant 0.1. \tag{5}$$

Also,

$$\|g_k\| \leqslant \|p^{1/(k+1)}\|^{k+1} = (3/4)\|P^{1/(k+1)}\|^{k+1}$$

and, by Lemma 2, the last expression is bounded by an absolute constant. This establishes

(iii) *the estimate*

$$\|h_{N,k}\| \leqslant K \tag{6}$$

where K is an absolute constant.

Combining (i), (ii) and (iii) we can easily construct our counterexample. We have elements $h_{N,k}$ of A with bounded norms and such that (5) holds; the theorem will be proved (apart from a notational change) if we can show that *for suitably chosen k, N the norm of $h_{N,k}^{-1}$ exceeds any preassigned quantity.* Now, observe that $h_{N,k}^{-1}$ has the Taylor expansion obtained by combining like terms in the series $1+z^N g_k+z^{2N}g_k^2+z^{3N}g_k^3+\dots$ It is easy to see that, for fixed k, and $N = N(k)$ chosen large enough, the norm of this expression is larger than

$$\|g_k\|+\|g_k^2\|+\dots+\|g_k^{2^r}\|$$

which, by the above, is at least $r+1$. (This step becomes more obvious if we first approximate g_k by a polynomial.) Thus, fixing first $k = 2^r-1$ large enough, and then N large enough, we get $h_{N,k}$ whose inverse has norm as large as we please. This proves the theorem.

Remarks. 1. Although, as we have shown, a bound for x^{-1} depending only on B and δ does not in general exist, a method for estimating x^{-1} in terms of other parameters has been sketched by Cohen [1].

2. Y. Katznelson, upon receipt of the present manuscript, kindly communicated to the author a very simple alternative construction, in the framework of the algebra $A(T)$ of absolutely convergent Fourier series, purely elementary and not requiring the Hardy–Littlewood lemma. Katznelson's construction allows δ in the above theorem to be any number less than 1/2.

3. Both Katznelson and D. J. Newman (independently) pointed out a simple argument to show that the problem in our opening paragraph, in the case of $B = A(T)$, has an *affirmative* answer if $\delta > \sqrt{2}/2$, and raised the question (*) of the infimum δ_0 of the set of δ for which this is the case, suggesting that possibly $\delta_0 = \frac{1}{2}$. Katznelson's construction, cited above, shows $\delta_0 \geqslant \frac{1}{2}$.

4. Yngve Domar has observed that the theorem of the present note can also be deduced by combining a general theorem of Jan-Erik Björk (*On the spectral radius formula in Banach algebras*, Pacific J. Math. 40 (1972), pp. 279–284) with known norm estimates for certain special unimodular functions.

5. Colin Graham and Carruth Mc Geehee have pointed out to me that the theorem can be deduced fairly easily from the "Wiener–Pitt phenomenon", i.e. from the existence of a bounded measure μ on the circle T such that $|\hat{\mu}(n)| \geqslant 1$ for all integers n, while $\{\hat{\mu}(n)^{-1}\}$ are not Fourier coefficients of any bounded measure on T.

(*) As remarked to us by Bell, when B is the algebra of absolutely convergent *Taylor* series, the affirmative result holds if $\delta > 1/2$, so in this case no problem remains.

However, this method and that in Remark 4 resort to ideas rather deeper than are needed for the problem, so perhaps the simple counterexample of the present note, the need for which was suggested by Ehrling, is not altogether superfluous.

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Presented to the Semester
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APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

ON BANACH SPACES IN WHICH EVERY M-BASIS IS A GENERALIZED SUMMATION BASIS

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We recall that if E is a Banach space, a (countable) biorthogonal system (x_n, f_n) in (E, E^*) is called an M-*basis* (*Markushevich basis*) for E if $\{x_n\}$ is complete in E (i.e., the closed linear span$[x_n]$ of $\{x_n\}$ is E, so E must be separable) and $\{f_n\}$ is total over E (i.e., $\{x \in E \mid f_n(x) = 0 \ (n = 1, 2, \ldots)\} = \{0\}$); it is well known that M-basis exist in every separable space E. Following M. I. Kadec [7], an M-basis (x_n, f_n) is called a *generalized summation basis* (g.s.b.) for E, if there exists a sequence $\{t_n\}$ of linear operators with $t_n\colon [x_i]_1^n \to [x_i]_1^n$ $(n = 1, 2, \ldots)$, such that $x = \lim_{n\to\infty} t_n s_n(x)$, for all $x \in E$, where $s_n(x) = \sum_{i=1}^{n} f_i(x) x_i$ $(x \in E, n = 1, 2, \ldots)$. A separable Banach space E is said to have the *bounded approximation property* (b.a.p.), if there exists on E a sequence of continuous linear operators $\{u_n\}$ of finite rank (i.e., $\dim u_n(E) < \infty$), such that $x = \lim_{n\to\infty} u_n(x)$, for all $x \in E$. Thus, if E has a g.s.b., then E is separable and has the b.a.p.; W. B. Johnson has proved that the converse is also true ([5], Theorem IV.1).

M. I. Kadec has shown ([7], Theorem 4) that if a reflexive space E has a g.s.b., then every M-basis for E is a g.s.b. By the above-mentioned result of Johnson, this is equivalent to the fact that, *in a separable reflexive space E with the* b.a.p., *every* M-*basis for E is a* g.s.b. ([5], Corollary IV.2). Therefore it is natural to raise the problem of characterizing the (separable) Banach spaces E with this property. Of course, a necessary condition is that such a space *E must have the* b.a.p. Furthermore, another necessary condition is that *E must be quasi-reflexive* (i.e., $\dim E^{**}/\pi(E) < \infty$, where $\pi\colon E \to E^{**}$ is the canonical isometrical embedding); indeed, for every separable non-quasi reflexive space E, the dual E^* contains a separable total subspace V of characteristic zero [2], and then E has an M-basis (x_n, f_n) with $[f_n] = V$ (see e.g. [5], Theorem III.1), but, as was observed by Kadec ([7], Theorem

* Prepared partially during the author's visit at the Stefan Banach International Mathematical Center, Warsaw, Semester on Approximation Theory, November–December 1975. We wish to express our thanks to T. Figiel for reading the manuscript and making valuable remarks.