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THE UPPER BOUND OF THE NORMS OF ORTHOGONAL PROJECTIONS ONTO SUBSPACES OF POLYGONALS

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Introduction

Let C stand for the space of all real-valued continuous functions f on $[0, 1]$, with the usual Chebyshev norm $||f|| = \{\max|f(x)|: x \in [0, 1]\}$. Given an arbitrary partition π_n: $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$ of $[0, 1]$ consider the subspace $S(\pi_n)$ of C, consisting of polygonals with the knots at the points of π_n, linear in each $[x_j, x_{j+1}]$ $(j = 0, \ldots, n)$. For a function f in C, define $P(\pi_n, f)$ as the best $S(\pi_n)$-approximant to f in the sense of L_2-distance on $[0, 1]$, i.e.,

$$P(\pi_n, f) \in S(\pi_n), \qquad ||f-P(\pi_n, f)||_2 = \min\{||f-g||_2 : g \in S(\pi_n)\}.$$

In this definition we used the conventional notation $||\cdot||_2$ for the norm in L_2, induced by the usual scalar product

$$(\varphi, \psi) = \int_0^1 \varphi(x)\psi(x)\,dx \qquad (\varphi, \psi \in L_2); \qquad ||\varphi||_2 = (\varphi, \varphi)^{1/2}.$$

It is routine to call operators like P *orthogonal projections*. Further, define the Chebyshev norm of the orthogonal projection $P(\pi_n)$ as follows:

$$||P(\pi_n)|| = \sup\{||P(\pi_n, f)|| : ||f|| \leqslant 1\}.$$

The quantity on the left depends on the partition π_n under consideration, i.e., it is a function of the distribution of knots $\{x_j\}$. It was shown by Z. Ciesielski [3] that, in fact,

$$||P(\pi_n)|| \leqslant 3, \tag{1}$$

whatever partition π_n one might take.

The aim of this paper is to show that the value 3 of the constant on the right-hand side of (1) is best possible for the class of all partitions π_n and all n, and thus to give an affirmative answer to a question raised in Ciesielski [4].

The lucky fact (1) and other advanced properties of orthogonal projections onto subspaces $S(\pi_n)$ and also properties of higher order splines have recently found some interesting applications to various problems of approximation theory and functional analysis (cf. [2], [3], [5], [6], [8]). Here we mention only that uniform boundedness of the norms $\|P(\pi_n)\|$ implies instantly that Franklin orthonormal system (see [7], [3]) forms a basis in the space C.

Exact values of the least upper bounds of the norms $\|P(\pi_n)\|$ for a special class of dyadic partitions of $[0, 1]$, generating Franklin system, and also, for the class of uniform partitions, were computed by Ciesielski (see [4]). It turned out that, for the first of the classes mentioned above, this exact value is equal to $2+(2-\sqrt{3})^2$, while for the second it is just 2.

Here we show that the value 3 of the constant in (1) cannot be decreased. It may also be seen from what follows that the *worst* partition, i.e., partition which serves to prove that 3 is *best* possible, is one containing a point of one-sided accumulation of the knots.

In Section 1 below, we carry out the proof of Ciesielski result (1). This section does not contain anything new and might be omitted; we have inserted it into this paper for completeness sake only. Section 2 deals with the proof of exactness of (1).

The author uses this occasion to express his sincere gratitude to Professor Z. Ciesielski for many useful advise and also to Miss J. S. Marsden for checking the language of the manuscript.

Section 1

THEOREM 1 (Z. Ciesielski [3]). *For an arbitrary $n \geqslant 0$ and partition π_n the following inequality holds*

$$\|P(\pi_n)\| \leqslant 3. \tag{2}$$

Proof. For a given partititon π_n, denote by $L_j(x)$ $(j = 0, 1, \ldots, n+1)$ the polygonal from $S(\pi_n)$ defined by the conditions:

$$L_j(x_i) = 0 \quad (i \neq j); \quad L_j(x_j) = 1.$$

It is plain that an arbitrary polygonal $L(x)$ from $S(\pi_n)$ may be uniquely written in the form

$$L(x) = \sum_{j=0}^{n+1} \xi_j L_j(x),$$

where $\xi_j = L(x_j)$ $(j = 0, 1, \ldots, n+1)$. Therefore, it follows that the orthogonal projection $P(\pi_n, f)(x)$ may be represented as

$$P(\pi_n, f)(x) = \sum_{j=0}^{n+1} \xi_j L_j(x),$$

where the coefficients $\{\xi_j\}$ satisfy the system of linear equations

$$\sum_{j=0}^{n+1} \xi_j (L_j, L_i) = (f, L_i) \quad (i = 0, 1, \ldots, n+1). \tag{3}$$

As the functions L_j are linearly independent, the Gramm matrix (L_j, L_i) of this system is non-degenerate by a well-known theorem from linear algebra, and thus (3) has a unique solution.

Let us estimate the value of

$$\|\xi\| = \max\{|\xi_i|: i = 0, 1, \ldots, n+1\}. \tag{4}$$

It is clear that

$$\|P(\pi_n, f)\| = \|\xi\|, \tag{5}$$

and thus, if we get an estimate for $\|\xi\|$ from above, this gives us an estimate for the norm of orthogonal projection P. To fulfill it, denote by k a number i, for which the max in (4) is attained, and put $\varepsilon_k = \operatorname{sign}\xi_k$. We now use the kth equation only from the system (3) and obtain

$$\|\xi\|(L_k, L_k) + \varepsilon_k\Big(\sum_{j=0}^{k-1} \xi_j (L_j, L_k) + \sum_{j=k+1}^{n+1} \xi_j (L_j, L_k)\Big) = \varepsilon_k (f, L_k). \tag{6}$$

Furthermore, using (4) and the fact that all the functions L_j are nonnegative, we get from (5) and (6) that

$$\|P(\pi_n, f)\|\Big((L_k, L_k) - \sum_{j\neq k} (L_j, L_k)\Big) \leqslant \|f\| (L_k, 1). \tag{7}$$

But it is obvious that the functions L_j form a partition of unity, i.e.,

$$\sum_{j=0}^{n+1} L_j(x) \equiv 1 \quad (x \in [0, 1]),$$

and so it follows from (7) that for an arbitrary $f \in C$,

$$\|P(\pi_n, f)\| \leqslant \frac{(L_k, 1)}{2(L_k, L_k) - (L_k, 1)} \|f\|. \tag{8}$$

An easy computation of integrals shows that in our case,

$$(L_k, 1) = \frac{\delta_{k-1}+\delta_k}{2}; \quad (L_k, L_k) = \frac{\delta_{k-1}+\delta_k}{3} \quad (k = 1, \ldots, n);$$

$$(L_0, 1) = \frac{\delta_0}{2}; \quad (L_0, L_0) = \frac{\delta_0}{3}; \quad (L_{n+1}, 1) = \frac{\delta_n}{2}; \quad (L_{n+1}, L_{n+1}) = \frac{\delta_n}{3},$$

where we used the notation δ_k for $x_{k+1}-x_k$ $(k = 0, 1, \ldots, n)$. Thus, the first factor on the right-hand side of (8) equals 3 for all k, and we are done.

Section 2

THEOREM 2. *For $n = 0, 1, \ldots$ the following inequalities hold*

$$\sup\|P(\pi_n)\| \geqslant 3 - \frac{4}{3\cdot 2^n}, \tag{9}$$

the sup *being taken over all the partitions* π_n: $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$ *of* $[0, 1]$ *with n distinct knots.*

Proof. For a fixed partition π_n define

$$X = X(\pi_n) = \sup\{\xi_0\colon \text{(3) holds, } \|f\| \leqslant 1\}. \tag{10}$$

The quantity X is a function of the partition π_n, that is, a function of $n+1$ variables $\delta_0, \delta_1, \ldots, \delta_n$, where $\delta_i = x_{i+1} - x_i > 0$ $(i = 0, \ldots, n)$, and $\sum_{i=0}^{n} \delta_i = 1$. This last restriction may be ignored in what follows, since all our speculations might be done for an arbitrary bounded segment $[a, b]$, not necessarily $[0, 1]$.

Lemma. *For any fixed positive numbers* $\delta_1, \delta_2, \ldots, \delta_n$,

$$\lim_{\delta_0 \to 0} X(\delta_0, \delta_1, \ldots, \delta_n) = \tfrac{3}{2} + \tfrac{1}{2} X(\delta_1, \delta_2, \ldots, \delta_n). \tag{11}$$

Iterated application of (11) shows that

$$\lim_{\delta_{n-1}\to 0} \lim_{\delta_{n-2}\to 0} \ldots \lim_{\delta_0 \to 0} X(\delta_0, \delta_1, \ldots, \delta_n) = 3 - \frac{3 - X(\delta_n)}{2^n},$$

and it can easily be checked, that $X(\delta_n) = X(\pi_0) = 5/3$. Thus taking into account (4), (5), and also (10), we deduce from (11) estimates (9), and so to prove Theorem 2 it is sufficient to establish the validity of Lemma.

For the purpose of the latter, an explicite form of Gramm matrix $G = \{(L_i, L_j)\}_{i,j}$ is needed (see [3]). Calculation of integrals, involved in its definition, shows that this form is as follows:

$$\begin{bmatrix} \frac{\delta_0}{3} & \frac{\delta_0}{6} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{\delta_0}{6} & \frac{\delta_0+\delta_1}{3} & \frac{\delta_1}{6} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{\delta_1}{6} & \frac{\delta_1+\delta_2}{3} & \frac{\delta_2}{6} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\delta_{n-1}}{6} & \frac{\delta_{n-1}+\delta_n}{3} & \frac{\delta_n}{6} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{\delta_n}{6} & \frac{\delta_n}{3} \end{bmatrix}. \tag{12}$$

Denote by $D(\delta_0, \delta_1, \ldots, \delta_n)$ the determinant of this matrix, and let $\Delta_i(\delta_0, \delta_1, \ldots, \delta_n)$ stand for the partial determinant corresponding to the ith entry of the 0th column of D $(i = 0, \ldots, n+1)$. For abbreviation sake, we shall use notation $(\delta, n+1)$ for $(\delta_0, \delta_1, \ldots, \delta_n)$ and (δ, n) for $(\delta_1, \delta_2, \ldots, \delta_n)$. Next we express $D(\delta, n+1)$ and $\Delta(\delta, n+1)$ through $D(\delta, n)$ and $\Delta(\delta, n)$:

$$\Delta_0(\delta, n+1) = D(\delta, n) + \frac{\delta_0}{3}\Delta_0(\delta, n);$$

$$\begin{aligned} D(\delta, n+1) &= \frac{\delta_0}{3}\Delta_0(\delta, n+1) - \frac{\delta_0}{6}\cdot\frac{\delta_0}{6}\Delta_0(\delta, n) \\ &= \frac{\delta_0}{3}\left(D(\delta, n) + \frac{\delta_0}{4}\Delta_0(\delta, n)\right); \end{aligned} \tag{13}$$

$$\Delta_{i+1}(\delta, n+1) = \frac{\delta_0}{6}\Delta_i(\delta, n) \qquad (i = 0, 1, \ldots, n).$$

These formulas imply in particular, that all the D and Δ are positive. Furthermore, we get (see (3))

$$\xi_0 = \frac{1}{D}\sum_{i=0}^{n+1}(-1)^i \Delta_i(f, L_i) = \frac{1}{D}\int_0^1 f(x)\left(\sum_{i=0}^{n+1}(-1)^i \Delta_i L_i(x)\right)dx,$$

and thus

$$X(\delta, n+1) = \sup\{\xi_0\colon \|f\| \leqslant 1\} = \frac{1}{D}\int_0^1 |K(x)|\,dx, \tag{14}$$

where

$$K(x) = \sum_{i=0}^{n+1}(-1)^i \Delta_i L_i(x).$$

Using the previous remark, we see that the polygonal $K(x)$ has the following properties, usefull for computation of the integral in (14):

$$K(y_i) = 0, \quad \text{where} \quad y_i = \frac{x_i\Delta_{i+1} + x_{i+1}\Delta_i}{\Delta_{i+1} + \Delta_i} \in (x_i, x_{i+1});$$

$$|K(x_i)| = \Delta_i \qquad (i = 0, 1, \ldots, n+1).$$

Therefore, we get from (14) that

$$X(\delta, n+1) = \frac{1}{2D(\delta, n+1)}\sum_{i=0}^{n}\delta_i \frac{[\Delta_i(\delta, n+1)]^2 + [\Delta_{i+1}(\delta, n+1)]^2}{\Delta_i(\delta, n+1) + \Delta_{i+1}(\delta, n+1)} \tag{15}$$

(cf. [4]). If we insert in (15), instead of $D(\delta, n+1)$ and $\Delta(\delta, n+1)$, their representations given by (13), we obtain

$$X(\delta, n+1) = \frac{3}{2[D(\delta, n) + O(\delta_0)]}\,\frac{D^2(\delta, n) + O(\delta_0)}{D(\delta, n) + O(\delta_0)} + $$
$$+ \frac{1}{2}\,\frac{1}{2[D(\delta, n) + O(\delta_0)]}\sum_{i=0}^{n-1}\delta_{i+1}\frac{[\Delta_{i-1}(\delta, n)]^2 + [\Delta_i(\delta, n)]^2}{\Delta_{i-1}(\delta, n) + \Delta_i(\delta, n)},$$

the factors in O depending on $\delta_1, \delta_2, \ldots, \delta_n$. The first term on the right in this estimate tends to $3/2$ as $\delta_0 \to 0$, and the second one to $(1/2)X(\delta, n)$, according to (15). This proves (11) and Theorem 2 with it.

Remarks. 1. Let $n \geqslant 0$ be fixed and let

$$X_{j,n} = \sup\{\xi_j : \pi_n, ||f|| \leqslant 1, \ \xi_j \text{ satisfies } (3)\}.$$

Then arguments analogous to just used above show that

$$X_{j,n} \geqslant 3 - c2^{-\max(j, n-j)},$$

c being an absolute positive constant.

2. Let $\mathring{C}$ stand for the space of all real-valued continuous functions on the real line, periodic with period 1, with usual Chebyshev norm, and let $\mathring{S}(\pi_n)$ denote the subspace of $\mathring{C}$, containing all 1-periodic polygonals with the knots at the points, which coincide mod 1 with some x_j from the system

$$\pi_n : 0 \leqslant x_0 < x_1 < \ldots < x_n < 1.$$

Denote by $\mathring{P}(\pi_n)$ the orthogonal projection onto $\mathring{S}(\pi_n)$. For convenience sake, we take $x_0 = 0$ and denote $x_0 + 1 = 1$ by x_{n+1}.

An arbitrary polygonal $L \in \mathring{S}(\pi_n)$ may be uniquely written in the form

$$L(x) = \sum_{j=0}^{n} \xi_j L_j(x),$$

where $L_j \in \mathring{S}(\pi_n)$, $L_j(x_i) = 0$ if $i \neq j$ and $L_j(x_j) = 1$ $(j = 0, \ldots, n)$. In this periodic case, the order of system (3) is decreased by 1, and the corresponding Gramm matrix of this system is now $(n+1) \times (n+1)$-matrix, which differs in its shape from $(n+2) \times \times (n+2)$-matrix in (12) by 0th and nth rows only. These last mentioned are now correspondingly

$$\frac{\delta_0 + \delta_n}{3} \ \ \frac{\delta_0}{6} \ \ 0 \ \ldots \ 0 \ \ 0 \ \ \frac{\delta_n}{6}$$

and

$$\frac{\delta_n}{6} \ \ 0 \ \ 0 \ \ldots \ 0 \ \ \frac{\delta_{n-1}}{6} \ \ \frac{\delta_{n-1} + \delta_n}{3}.$$

Thus, if we take $\delta_n \to 0$, while $\delta_0, \delta_1, \ldots, \delta_{n-1}$ are fixed and positive, we return instantly to the non-periodic case, with the distances $\delta_0, \delta_1, \ldots, \delta_{n-1}$ between the $n+1$ consecutive knots. After it, Theorem 2 may be applied and we arrive at the following statement.

THEOREM 3. *For $n = 0, 1, \ldots$ the following inequalities hold*

$$\sup ||\mathring{P}(\pi_n)|| \geqslant 3 - \frac{8}{3 \cdot 2^n},$$

the sup *being taken over all the partitions π_n of the real axis which contain exactly $n+1$ points, pairwise distinct* mod 1.

This theorem shows in particular, that the value 3 is exact also in the case of orthogonal projections onto subspaces of periodic polygonals.

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* In the meantime the same result has been obtained independently by P. Oswald in Mat. Zametki 21.4 (1977), pp. 495–502 (*editor*).

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