SYMPLECTIC SINGULARITIES AND GEOMETRY OF GAUGE FIELDS
BANACH CENTER PUBLICATIONS, VOLUME 39
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

THE L^2 METRIC IN GAUGE THEORY: AN INTRODUCTION AND SOME APPLICATIONS

DAVID GROISSER

Department of Mathematics, University of Florida
Gainesville, Florida 32611, USA
E-mail: groisser@math.ufl.edu

Abstract. We discuss the geometry of the Yang-Mills configuration spaces and moduli spaces with respect to the L^2 metric. We also consider an application to a de Rham-theoretic version of Donaldson's μ-map.

1. Introduction. In classical four-dimensional Yang-Mills theory, the moduli spaces of self-dual (SD) or anti-self-dual (ASD) connections over a Riemannian manifold carry a natural metric known as the L^2 metric. Thus one can study the intrinsic geometry of the moduli spaces themselves as concrete Riemannian manifolds. The L^2 metric relates to several other aspects of gauge theory, some of primarily mathematical interest (such as Donaldson's μ-map, discussed below in §5.2) and some of primarily physical interest (such as the semi-classical measure discussed in [GP3]). These talks will provide a qualitative introduction to the basic features of the L^2 metric and to some of the mathematical questions it has been used to approach.

The ambient setting for the problems I will discuss is infinite-dimensional: the space of connections on a principal bundle. Consequently there are many technical issues—such as definitions of "manifold" and "smooth", and proofs that objects live in the proper category—that, while essential for complete proofs, can obscure the purely geometric ideas. As my purpose in these talks is more qualitative, I will ignore most of these technical issues; complete proofs of the theorems below are too long to present here in any case. I will generally speak as if the connections, gauge transformations, etc., to which I refer are all smooth, even though to make certain statements literally correct one must take various Sobolev completions. The underlying analysis of these completions has been presented excellently in many sources (e.g. [MV], [FU §3], [MM §6.4]) that can be consulted for details.

Research supported in part by NSF grant DMS-9307648.
1991 *Mathematics Subject Classification*: Primary 53C07; Secondary 58D27.
The paper is in final form and no version of it will be published elsewhere.

The basic objects in our discussion will be:

- (M, g_0), a compact four-dimensional Riemannian manifold;
- G, a compact semisimple Lie group (often $SU(2)$); Z, the center of G (a finite group); and $\hat{G} := G/Z$;
- $\mathfrak{g}$, the Lie algebra of G, equipped with an Ad-invariant inner product;
- P, a principal G-bundle over M; $\mathrm{Ad}\,P := P \times_{Ad} \mathfrak{g}$, the adjoint bundle; and $\Omega^k(\mathrm{Ad}\,P) := \Gamma(\Lambda^k(T^*M) \otimes \mathrm{Ad}\,P)$, the space of $\mathrm{Ad}\,P$-valued k-forms on M ($k = 0, \ldots, 4$);
- $\mathcal{A}$, the space of connections on P; and $\mathcal{A}^* \subset \mathcal{A}$, the subspace of irreducible connections;
- $\mathcal{G}$, the group of gauge transformations of P (automorphisms of P covering the identity); $\mathcal{Z}$, the center of $\mathcal{G}$, isomorphic to Z; and $\hat{\mathcal{G}} := \mathcal{G}/\mathcal{Z}$;
- $\mathcal{B} := \mathcal{A}/\mathcal{G}$, the "configuration space"; and $B^* := \mathcal{A}^*/\mathcal{G}$;
- $\mathcal{SD} \subset \mathcal{A}$, the subspace of self-dual connections (the reader can make the appropriate sign changes below for ASD connections);
- $\mathcal{M} := \mathcal{SD}/G \subset \mathcal{B}$, the *moduli space*, whose points are called *instantons*; and $\mathcal{M}^* = \mathcal{M} \cap \mathcal{B}^*$, the subspace of irreducible instantons.

$\mathrm{Ad}\,P$-valued differential forms inherit a pointwise inner product $(\cdot\,,\cdot)$ from the metrics on M and $\mathfrak{g}$. Integration then defines the L^2 metric on these forms:

$$\mathbf{g}(\alpha, \beta) := \langle \alpha, \beta \rangle := \int_M (\alpha, \beta) d\mathrm{vol}_{g_0}, \quad \forall \alpha, \beta \in \Omega^k(\mathrm{Ad}\,P), \quad k = 0, \ldots, 4. \tag{1}$$

As is well-known, $\mathcal{A}$ is an affine space whose tangent space at any point is canonically isomorphic to $\Omega^1(\mathrm{Ad}\,P)$. In particular, therefore, (1) *defines a flat Riemannian metric on* $\mathcal{A}$ (technically, only a weak metric, but this turns out not to be a serious problem for our purposes; see [GP1]). Furthermore, the Ad-invariance of the inner product on $\mathfrak{g}$ makes the metric $\mathbf{g}$ *gauge-invariant.*

The full quotient space $\mathcal{B}$ has a complicated, stratified structure due to the presence of reducible connections. However, on the open dense subspace $\mathcal{A}^* \subset \mathcal{A}$ the action of $\mathcal{G}$ is almost free: the stabilizer of every irreducible connection is precisely the finite group $\mathcal{Z}$. Thus $\hat{G}$ acts freely on $\mathcal{A}^*$, and one can prove the following.

PROPOSITION. *$\mathcal{B}^*$ is a Hilbert manifold, and the action of $\mathcal{G}$ on $\mathcal{A}^*$ induces a principal $\hat{\mathcal{G}}$-fibration $\mathcal{A}^* \to \mathcal{B}^*$ (where $\hat{\mathcal{G}} = \mathcal{G}/\mathcal{Z}$).*

This proposition is literally true only after completing $\mathcal{G}$ and $\mathcal{A}$ in appropriate Sobolev norms (see [FU §3]).

In Sections 2–4 below we discuss (without detailed proofs) the induced geometry of $\mathcal{B}^*$ and $\mathcal{M}^*$. In Section 5 we describe applications of a key "localization principle" to the proofs of several of the theorems of Section 4, and to a differential-form version of Donaldson's μ-map.

2. The geometry of $\mathcal{A}^*/\mathcal{G}$ with the L^2 metric.

2.1 *The connection on $\mathcal{A}^* \xrightarrow{\hat{\mathcal{G}}} \mathcal{B}^*$ and its curvature.* It is worthwhile first to consider a finite-dimensional "toy model" that captures all of the essential geometry.

Suppose K is a Lie group with Lie algebra $\mathfrak{k}$, N a finite-dimensional manifold, and $\pi : Q \to N$ a principal K-bundle. We call the tangent space to the K-orbit through $q \in Q$ the *vertical space* $V_q \subset T_qQ$. Suppose in addition that Q carries a Riemannian metric $(\cdot\,,\cdot)_Q$ invariant under the action of K. Then these structures determine a connection on $Q \to N$ by defining (at each $q \in Q$) the *horizontal space* $H_q = (V_q)^\perp \subset T_qQ$. The horizontal distribution $\{H_q\}_{q\in Q}$ is K-invariant, hence a connection, which we shall call the *canonical connection.*

Recall that any connection on Q defines a *connection form* $\omega \in \Omega^1(Q,\mathfrak{k})$ as follows. For each $q \in Q$ the right K-action on Q defines a map

$$\begin{aligned} \iota_q : \mathfrak{k} &\longrightarrow T_qQ \\ v &\longmapsto \frac{d}{dt}(q\cdot\exp(tv))\Big|_{t=0} \end{aligned} \tag{2}$$

carrying $\mathfrak{k}$ isomorphically to V_q. The splitting of T_qQ given by the connection determines projections $\mathrm{hor}_q : T_qQ \to H_q$ and $\mathrm{vert}_q : T_qQ \to V_q$, and ω is defined by

$$\omega(X) := \iota_q^{-1}\big|_{V_q}(\mathrm{vert}_q(X)) \quad \forall X \in T_qQ. \tag{3}$$

Thus $H_q = \ker(\omega_q)$. For the canonical connection we can be even more explicit. Since both $\mathfrak{k}$ and T_qQ are Hilbert spaces, ι_q has an adjoint $\iota_q^* : T_qQ \to \mathfrak{k}$. Since $V_q = \mathrm{im}(\iota_q)$, we have $H_q = \ker(\iota_q^*)$ and $\mathrm{vert}_q = \iota_q(\iota_q^*\iota_q)^{-1}\iota_q^*$, and the subexpression $(\iota_q^*\iota_q)^{-1}\iota_q^*$ inverts ι_q on $\mathrm{im}(\iota_q)$. Hence for the canonical connection, we have $\omega_q = (\iota_q^*\iota_q)^{-1}\iota_q^*$.

Now assume further that the total space Q an open subset of a flat affine space, so that there is a fixed Hilbert space W and a trivialization $j_* : TQ \cong Q \times W$ (induced by a global chart) such that for each $q \in Q$, the isomorphism $j_{*q} : T_qQ \to W$ is an isometry. If we set $\tilde\iota_q = j_{*q} \circ \iota_q$, we can then write the canonical connection form as a pullback $\omega = j^*\tilde\omega$, where

$$\tilde\omega_q = (\tilde\iota_q^*\tilde\iota_q)^{-1}\tilde\iota_q^*.$$

In this context, we wish to compute the curvature two-form $F \in \Omega^2(Q;\mathfrak{k})$ of the canonical connection. For a general connection, given $X_0, Y_0 \in H_q$, and horizontal local extensions X, Y, one has

$$F(X_0,Y_0) = -\omega_q([X,Y]), \tag{4}$$

independent of the choice of extensions. In our case there is a particularly simple way to choose X, Y: writing $X_0 = j_{*q}^{-1}\tilde X_0$ (etc. for Y), for arbitrary p set

$$X_p = \mathrm{hor}_p(j_{*p}^{-1}\tilde X_0) = j_{*p}^{-1}\left(\tilde X_0 - \tilde\iota_p(\tilde\iota_p^*\tilde\iota_p)^{-1}\tilde\iota_p^*\tilde X_0\right).$$

Note that $\tilde\iota$ and $\tilde\iota^*$ are now simply functions on Q with values in *fixed* vector spaces: $\tilde\iota : Q \to \mathrm{Hom}(\mathfrak{k},W)$ and $\tilde\iota^* : Q \to \mathrm{Hom}(W,\mathfrak{k})$. Hence the Lie bracket above reduces to directional derivatives of $\tilde\iota, \tilde\iota^*$ (written $X_0(\tilde\iota)$ etc.), and we find

$$F(X_0,Y_0) = -2(\tilde\iota_q^*\tilde\iota_q)^{-1}\{X_0,Y_0\} \tag{5}$$

where

$$\{X_0,Y_0\} := -\frac{1}{2}\left(X_0(\tilde\iota^*)\tilde Y_0 - Y_0(\tilde\iota^*)\tilde X_0\right). \tag{6}$$

(Here $X_0(\iota^*) \in \mathrm{Hom}(W,\mathfrak{k})$ is the directional derivative; thus $X_0(\iota^*)\tilde Y_0$ lives in $\mathfrak{k}$.)

Now let us return to gauge theory, replacing $Q \xrightarrow{K} N$ by $\mathcal{A}^* \xrightarrow{\hat{\mathcal{G}}} \mathcal{B}^*$. In this case $W = \Omega^1(\operatorname{Ad} P)$ (with the metric (1)) and, for each $A \in \mathcal{A}^*$, j_{*A} is simply the natural identification of $T_A\mathcal{A}^*$ with $\Omega^1(\operatorname{Ad} P)$. The Lie algebra of $\mathcal{G}$ is $\Omega^0(\operatorname{Ad} P)$, and the map $\tilde{\iota}_A$ is simply covariant derivative:

$$\tilde{\iota}_A = d^A : \Omega^0(\operatorname{Ad} P) \to \Omega^1(\operatorname{Ad} P).$$

Furthermore $\tilde{\iota}_A^* = (d^A)^*$, the formal L^2 adjoint of d^A, so that the vertical and horizontal spaces at A are

$$\mathcal{V}_A = \operatorname{im}(d^A) \subset \Omega^1(\operatorname{Ad} P), \quad \mathcal{H}_A = \ker(d^A)^* \subset \Omega^1(\operatorname{Ad} P). \tag{7}$$

In addition,

$$(\tilde{\iota}_A^* \tilde{\iota}_A)^{-1} = (\Delta_0^A)^{-1} := G_0^A.$$

(This covariant Green operator on $\Omega^0(\operatorname{Ad} P)$ exists since A is irreducible.) After proper attention to analytic details (see [GP1]), the formal calculation (5) gives precisely the right answer. Replacing X_0, Y_0 by $\alpha, \beta \in \Omega^1(\operatorname{Ad} P)$, one finds

$$\{\alpha, \beta\} = -\alpha(\tilde{\iota}^*)\tilde{\beta} = \beta(\tilde{\iota}^*)\tilde{\alpha} = \sum_i [\alpha_i, \beta_i]. \tag{8}$$

In the last expression α_i, β_i are the local $\operatorname{Ad} P$-valued components of α, β relative to a local orthonormal frame $\{\theta^i\}$ of T^*M (i.e. $\alpha = \sum_i \alpha_i \otimes \theta^i, \beta = \sum_i \beta_i \otimes \theta^i$), and $[\cdot\,,\cdot]$ denotes the pointwise bracket in $\operatorname{Ad} P$ inherited from $\mathfrak{g}$. Thus the curvature $\mathcal{F}$ at a point $A \in \mathcal{A}^*$ is given by

$$\mathcal{F}(\alpha, \beta) = -2G_0^A(\{\alpha, \beta\}). \tag{9}$$

2.2 *The Riemannian structure of* $\mathcal{B}^*$. The data of the "toy model" $\pi : Q \xrightarrow{K} N$ described above also determine a Riemannian metric on N, as follows. Given two vectors $X_x, Y_x \in T_xN$, lift them horizontally to horizontal vectors $X', Y' \in H_q$ (where $q \in \pi^{-1}(x)$ is arbitrary), and define $(X_x, Y_x)_N = (X', Y')_Q$; the choice of q is immaterial because of the K-invariance of $(\cdot\,,\cdot)_Q$ and the equivariance of horizontal lifts. With this definition of $(\cdot\,,\cdot)_N$, such a setup is called a (*principal*) *Riemannian submersion*.

Since the canonical connection on $Q \to N$ and the metric on N are determined by the same data, the Riemann tensor of N is closely related to the bundle curvature of $Q \to N$. The relation can be derived from O'Neill's formula for the sectional curvature σ of general Riemannian submersions (see [CE §3]):

$$\sigma_N(X_x, Y_x) = \sigma_Q(X', Y') + \frac{3}{4}\left\|\operatorname{vert}_q[X', Y']\right\|_2^2. \tag{10}$$

(Here we take $\{X_x, Y_x\}$ to be an orthonormal pair, and on the right, take arbitrary horizontal extensions of X', Y' to define the bracket.) In our situation, $\sigma_Q \equiv 0$, and the vertical part of the bracket in (10) is (minus) the image under ι_q of the bundle curvature $F(X, Y)$. Hence for orthonormal $\{X_x, Y_x\}$, (5) gives

$$\sigma_N(X_x, Y_x) = 3\left\|\tilde{\iota}_q(\tilde{\iota}_q^*\tilde{\iota}_q)^{-1}\{X', Y'\}\right\|_W^2 = 3(\{X', Y'\}, (\tilde{\iota}_q^*\tilde{\iota}_q)^{-1}\{X', Y'\})_Q. \tag{11}$$

Returning to gauge theory, after due attention to analysis (see [GP1]), once again the answer given by the finite-dimensional model is correct: for $A \in \mathcal{A}^*$, and L^2-orthonormal

$\alpha, \beta \in \ker(d^A)^* = \mathcal{H}_A$ representing tangent vectors $\overline{\alpha}, \overline{\beta} \in T_{[A]}\mathcal{B}^*$,

(12) $$\sigma_{\mathcal{B}^*}(\overline{\alpha}_{[A]}, \overline{\beta}_{[A]}) = 3\langle\{\alpha, \beta\}, G_0^A\{\alpha, \beta\}\rangle$$

This formula and (9) were first written down by Singer [S].

3. The Riemannian structure of the moduli space. The moduli space $\mathcal{M} = \mathcal{SD}/\mathcal{G} \subset \mathcal{B}$ is in general not a manifold. For a given point $[A] \in \mathcal{M}$, there are two "obstructions" already visible in the deformation complex

(13) $$\Omega^0(\operatorname{Ad} P) \xrightarrow{d^A} \Omega^1(\operatorname{Ad} P) \xrightarrow{d^A_-} \Omega^2_-(\operatorname{Ad} P).$$

(Here d^A is always covariant exterior derivative, and the subscript "$-$" denotes anti-self-dual components or the projection onto these components.) The formal tangent space $T_{[A]}\mathcal{M}$ is

(14) $$T_A(\mathcal{SD})/\mathcal{V}_A \cong T_A(\mathcal{SD}) \cap \mathcal{H}_A \cong \ker(d^A)^* \cap \ker(d^A_-) := \mathcal{T}_A \subset \Omega^1(\operatorname{Ad} P).$$

If $\ker(d^A) : \Omega^0 \to \Omega^1$ and $\ker(d^A_-)^* : \Omega^1 \to \Omega^2_-$ are both zero then $\mathcal{M}$ is in fact a manifold in a neighborhood of $[A]$ (see [FU]). These are both open, gauge-invariant conditions on the connection, the first of which is satisfied by all irreducible connections. The second—the "h^2-condition", equivalent to surjectivity of d^A_- and to the existence of $G^A_- := \left(d^A_-(d^A_-)^*\right)^{-1}$—can be shown to hold for all $[A] \in \mathcal{M}$ for a generic choice of metric g_0 on M ([FU]), as well as for certain other special metrics. Even when the h^2 condition fails for some $[A] \in \mathcal{M}$, it is often satisfied for $[A]$ near the "boundary" of $\mathcal{M}$ (see Section 4 below). This will be the region of greatest interest to us later, so for now we will not assume anything special about g_0, but instead will write

(15) $$\mathcal{M}^{**} = \left\{[A] \in \mathcal{M} \mid \ker d^A = \{0\} \text{ and } \ker(d^A_-)^* = \{0\}\right\}.$$

Thus $\mathcal{M}^{**}$ is a finite-dimensional submanifold of $\mathcal{B}^*$, of dimension equal to the index of (13). As a submanifold of a Riemannian manifold, $\mathcal{M}^{**}$ inherits a metric (automatically strong, by finite-dimensionality) by restriction.

Were the ambient manifold $\mathcal{B}^*$ finite-dimensional, this curvature of the Riemannian manifold $(\mathcal{M}^{**}, \mathbf{g})$ could now be computed from the Gauss equation:

$$\begin{aligned}\langle R_{\text{submanifold}}(X,Y)Z, W\rangle = \langle R_{\text{ambient}}&(X,Y)Z, W\rangle \\ &+ \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle.\end{aligned}$$

Here h is the second fundamental form h of the submanifold. Once again, after doing the necessary analysis, this finite-dimensional model gives the right answer for our gauge-theory example (see [GP1]). The second fundamental form of the embedding $M^{**} \hookrightarrow \mathcal{B}^*$ at $[A] \in \mathcal{M}^{**}$ is given by

(16) $$h(\alpha, \beta) = -d^A_- G^A_-([\alpha, \beta]_-), \quad \alpha, \beta \in \mathcal{T}_A,$$

where, in the notation following (8), $[\alpha, \beta] = \sum_{i,j} [\alpha_i, \beta_j]\theta^i \wedge \theta^j$ (the subscript "$-$" again denotes ASD projection). Combining this with (12), we find that the sectional curvature of $\mathcal{M}^{**}$ at $[A] \in \mathcal{M}^{**}$ is given by

(17) $$\sigma_{\mathcal{M}}(\alpha, \beta) = 3\langle\{\alpha, \beta\}, G_0^A\{\alpha, \beta\}\rangle + \langle[\alpha, \alpha]_-, G^A_-[\beta, \beta]_-\rangle - \langle[\alpha, \beta]_-, G^A_-[\alpha, \beta]_-\rangle.$$

(Again α, β are assumed L^2-orthonormal here.) This is a pretty formula, incorporating all the data encoded in (13), but what does it tell us? In general it is hard even to determine the sign of the sectional curvature from (17). The term $G_-^A[\beta,\beta]_-$, for example, is doubly non-local: the Green operator acts non-locally on its argument, and the harmonic 1-forms α, β themselves involve non-local information. However, in the next section we shall see that near the boundary of certain moduli spaces, one can extract useful information from (17).

4. Special case: five-dimensional moduli spaces. In this section (except as indicated in §4.2) we will assume that

(i) P is an $SU(2)$-bundle of Pontryagin index ("instanton number") 1,
(ii) M is simply connected,
(iii) the intersection form (the quadratic form on $H^2(M;\mathbf{R})$ given by cup product) is positive-definite.

Under these conditions, $\dim(\mathcal{M}^{**}) = 5$, and there is a "collar region" in $\mathcal{M}^{**}$ diffeomorphic to $(0,1)\times M$. The collar consists of instantons whose curvatures are sharply peaked in a small region in M and are small elsewhere. One can introduce gauge-invariant parameters $\lambda_D(A), p(A)$, the *scale* and *center* of a concentrated connection (essentially the width and center of the peak; see [D1] or [FU] for a careful definition) and thereby obtain a diffeomorphism, for λ_0 sufficiently small,

$$\begin{array}{rccl} \Psi_D : \mathcal{M}_{\lambda_0} & \xrightarrow{\cong} & (0,\lambda_0)\times M \\ [A] & \longmapsto & (\lambda_D(A), p(A)), \end{array} \tag{18}$$

on some subset $\mathcal{M}_{\lambda_0}$ of the collar whose complement in $\mathcal{M}$ is compact. The cited definitions of Ψ_D are non-canonical, involving a choice of a smooth cut-off function; later we will discuss a more canonical definition. Below, we write λ for λ_D.

It is for regions of the form $\mathcal{M}_{\lambda_0}$ that one can pry something tangible out of (17). The reason is the following

Localization Principle. *For $[A]\in\mathcal{M}_{\lambda_0}$, formulas of interest should reduce to local formulas as $\lambda_D(A)\to 0$.*

Theorems concerning the geometry of the collar are based on attempting to force this principle to be true.

4.1 *Asymptotic properties of the metric in the collar.* The first theorems on the geometry of the collar were proven in [GP2]:

Theorem 1. *In the notation above, as $\lambda\to 0$ the metric $\mathbf{g}$ on $\mathcal{M}_{\lambda_0}$ behaves asymptotically like a product (in a C^0 sense):*

$$(\Psi_D^{-1})^*\mathbf{g} \sim 4\pi^2(2\,d\lambda^2 \oplus g_0). \tag{19}$$

Consequently, if we define $\overline{\mathcal{M}_{\lambda_0}}$ to be the Cauchy completion of $(\mathcal{M}_{\lambda_0},\mathbf{g})$, then

(i) *$\overline{\mathcal{M}_{\lambda_0}}$ is a Riemannian manifold-with-boundary, and Ψ_D extends to a diffeomorphism $\overline{\mathcal{M}_{\lambda_0}} \xrightarrow{\cong} [0,\lambda_0]\times M$.*

(ii) *The induced topological, smooth, and Riemannian structures on the boundary* $\partial\overline{\mathcal{M}} := \Psi_D^{-1}(\{0\}\times M)$ *are independent of* Ψ_D *or* (*sufficiently small*) λ_0, *and there is an isometry of Riemannian manifolds*

$$(\partial\overline{\mathcal{M}}, \mathbf{g}|_{\partial\overline{\mathcal{M}}}) \cong (M, 4\pi^2 g_0). \tag{20}$$

(iii) *If we define* $\overline{\lambda} :=$ (distance to $\partial\overline{\mathcal{M}})/\sqrt{8\pi^2}$ *then* $\overline{\lambda}/\lambda \to 1$ *as* $\lambda \to 0$.

Remarks.

(a) Statement (i) implies that *the* L^2 *completion implements Donaldson's compactification scheme*, attaching a boundary of "delta-connections" of zero scale.

(b) Statement (iii) implies that in the collar there is a *canonical* definition of scale, namely the (normalized) distance-to-boundary $\overline{\lambda}$, that is asymptotic to any of the non-canonical scales λ_D.

Theorem 1 involves only C^0 properties of the metric $\mathbf{g}$. To see that the asymptotic product relation (19) fails already at the level of second derivatives, it is worthwhile to look at the two examples in which $\mathbf{g}$ has been computed explicitly: $M = S^4$ ([GP1], [DMM], [H]) and $M = \mathbf{CP}^2$ ([G1], [K]), both with their standard metrics g_0. In the first case the moduli space is a (smooth) cone on a point (i.e. a ball), while in the second it is a cone on $\mathbf{CP}^2$. In either case there is rotational symmetry, and on the complement of the vertex the metric takes the form

$$(\Psi^{-1})^*\mathbf{g} = 4\pi^2(2f(\lambda)d\lambda^2 \oplus h(\lambda)g_0) \tag{21}$$

where the functions f, h have the asymptotic behavior (as $\lambda \to 0$) indicated in Table 1.

Table 1

	S^4	$\mathbf{CP}^2$
$f(\lambda)$	$1 + 3\lambda^2\log\lambda + O(\lambda^2)$	$1 + 6\lambda^2\log\lambda + O(\lambda^2)$
$h(\lambda)$	$1 - \frac{3}{2}\lambda^2 - \frac{3}{2}\lambda^4\log\lambda + O(\lambda^4)$	$1 - 3\lambda^2 - 6\lambda^4\log\lambda + O(\lambda^4)$

(In these examples, the definition of λ used is the radius of the smallest ball containing half the total Yang-Mills action.) From the precise formulas for f, h given in the references above, the formulas in Table 1 can be extrapolated to the level of second derivatives, and one finds that in both cases

(i) $\partial\overline{\mathcal{M}}$ is a totally geodesic submanifold of $\overline{\mathcal{M}_{\lambda_0}}$, and
(ii) the Riemann tensor of $\mathcal{M}_{\lambda_0}$ extends continuously to $\overline{\mathcal{M}_{\lambda_0}}$.

At first glance (ii) is surprising, since Table 1 seems to imply that $\mathbf{g}$ is not C^2 at $\lambda = 0$ ($f'' \sim \log\lambda$). However, this is an artifact of a bad coordinate system. Instead, note that the map

$$\begin{array}{rcl} \Psi_{\mathrm{nat}} : \mathcal{M}_{\lambda_0} & \to & (0,\lambda_0)\times M \\ [A] & \mapsto & (\overline{\lambda}(A), \overline{p}(A) := \text{unique closest point to } [A] \text{ in } \partial\overline{\mathcal{M}}) \end{array} \tag{22}$$

is well-defined in the two examples above (for λ_0 less than the "radius" of the punctured cone), and we have the following theorem ([G2]):

THEOREM 2. *For $M = S^4$ or $\mathbf{CP^2}$ (with their standard metrics), Ψ_{nat} is C^∞ in the collar, and hence can be used to define a smooth structure on $\overline{\mathcal{M}_{\lambda_0}}$ compatible with the original smooth structure on $\mathcal{M}_{\lambda_0}$. With respect to this smooth structure on $\overline{\mathcal{M}_{\lambda_0}}$, the metric $\mathbf{g}$ is smooth on the interior, C^5 on $\overline{\mathcal{M}_{\lambda_0}}$, but not C^6 on $\overline{\mathcal{M}_{\lambda_0}}$.*

This begs the question: what is the optimal regularity of $\mathbf{g}$? Theorem 2 implies that there is no coordinate system in which $\mathbf{g}$ is C^∞, for if there were, then $\mathbf{g}$ would also be C^∞ in the coordinates used above (given by the inverse of the normal exponential map from the boundary).

In all likelihood, the high degree of symmetry in the two examples above leads to extra regularity. More generally, since one cannot write down explicit formulas for the metric, and experimenting with changes of coordinates can be quite messy, it seems difficult to obtain information about optimal regularity. However, with some effort, first derivatives of the metric can be dealt with directly, and we have the following theorem (see [G3]).

THEOREM 3. *With the hypotheses and notation as in Theorem 1, endow $\overline{\mathcal{M}_{\lambda_0}}$ with the C^∞ structure induced by Ψ_D. With respect to this smooth structure, the extension of $\mathbf{g}$ to $\overline{\mathcal{M}_{\lambda_0}}$ is C^1 (in fact $C^{1,\alpha}$ for small α). Furthermore, the boundary $\partial\overline{\mathcal{M}}$ is always a totally geodesic submanifold of $\overline{\mathcal{M}_{\lambda_0}}$.*

The proof of Theorem 3 given in [G3] is too computational to be useful for higher derivatives. To obtain such information, it seems reasonable to attempt to use curvature invariants. Such an approach yielded the following in [G2]:

THEOREM 4. *With the hypotheses and notation as in Theorem 1, the Riemann tensor $R_{\mathcal{M}}$ of $(\overline{\mathcal{M}_{\lambda_0}}, \mathbf{g})$ is bounded for λ_0 sufficiently small. The restriction $R^{\tan}_{\mathcal{M}}$ of $R_{\mathcal{M}}$ to any "tangential foliation" with leaves of the form $\{\lambda_D = \mathrm{const.}\}$ extends continuously to $\partial\overline{\mathcal{M}}$, and*

$$R^{\tan}_{\mathcal{M}}\big|_{\partial\overline{\mathcal{M}}} = (4\pi^2)^{-1} R_M, \tag{23}$$

where R_M is the Riemann tensor of (M, g_0).

It is likely that the entire tensor $R_{\mathcal{M}}$ extends continuously to $\overline{\mathcal{M}_{\lambda_0}}$, though the methods of [G2] do not establish this (at issue is rather detailed information on the eigensections of Δ_0^A with small eigenvalue). Note also that (23) is exactly what one would obtain from the Gauss equations, Theorem 3, and (20), if one knew that the metric on $\overline{\mathcal{M}_{\lambda_0}}$ were C^2. In light of these theorems, the following conjecture seems plausible:

CONJECTURE 1. *With hypotheses and notation as above, for λ_0 sufficiently small Ψ_{nat} is always well-defined and C^3 in the collar. With respect to the C^3 structure induced on $\overline{\mathcal{M}_{\lambda_0}}$ by Ψ_{nat}, the extension of $\mathbf{g}$ to $\overline{\mathcal{M}_{\lambda_0}}$ is C^2.*

The proof of Theorem 4 in [G2] relies on a surprising cancellation phenomenon. In (17), as $\lambda \to 0$ the eigenvalues of G^A_- are uniformly bounded, while G^A_0 has three eigenvalues growing as λ^{-2}. Thus one might expect that the last two terms in (17) are bounded as $\lambda \to 0$ and therefore (knowing that the sectional curvature is bounded) that the first term is bounded as well; that somehow the quantities $\{\alpha, \beta\}$ must be almost perpendicular to the λ^{-2}-eigenspaces.

What actually happens, however, is that as $\lambda \to 0$, all three terms in (17) diverge as λ^{-2}, but the divergences cancel, leaving behind a finite remainder (see §4.3). This conspiratorial cancellation is all the more surprising because of the geometrically independent sources of the terms in (17)—the first term arising from the ambient space $\mathcal{B}^*$, the other two terms arising from the second fundamental form of the embedding $\mathcal{M}_{\lambda_0} \hookrightarrow \mathcal{B}^*$. (The flaw in the naive analysis above is that although α, β are bounded in L^2 as $\lambda \to 0$, $\{\alpha, \beta\}$ and $[\alpha, \beta]_-$ are not.)

We mention in passing that the $\mathcal{M}$'s above have another geometrically interesting feature: cone singularities in the interior. It turns out that near these singularities $\mathbf{g}$ is asymptotic to a "linear" cone metric $dr^2 \oplus r^2 g_{\mathbf{CP}^2}$; see [GP2].

4.2 *Further questions about the geometry of $\mathcal{M}$.* The hypotheses we placed on M and P to obtain the five-dimensional moduli cases considered above are very restrictive. The constraints on M are only satisfied if M is homeomorphic to a sphere or to a connected sum of (one or more) $\mathbf{CP}^2$'s. Furthermore, we required P to be an $SU(2)$-bundle of instanton number 1. What happens if we relax these requirements? At present, very little is known. However, there are several questions suggested by Theorems 1, 3, and 4.

For the first question, let $\mathcal{M}_k$ denote (in this subsection) the moduli space for the $SU(2)$-bundle of instanton number k over a given Riemannian manifold. In general this space is non-compact for the same reason the 5-dimensional spaces were: one can have a sequence of connections whose squared curvatures approach a delta-function, or more generally a sum of delta-functions. This leads to the "Donaldson/Uhlenbeck compactification" $\overline{\mathcal{M}_k}$, defined as the closure of $\mathcal{M}_k$ with respect to an appropriate topology on

$$\mathcal{M}_k \coprod (\mathcal{M}_{k-1} \times M) \coprod (M_{k-2} \times \Sigma^2 M) \coprod \dots \coprod \Sigma^k M,$$

where $\Sigma^j M$ is the j-fold symmetric product of M with itself (see [DK §4.4]). The stratum $\mathcal{M}_{k-j} \times \Sigma^j M$ corresponds, heuristically, to j units of "charge" (instanton number) bubbling off at points whose locations are labeled by $\Sigma^j M$, leaving behind a background connection of instanton number $k - j$. There is enough evidence to make the following conjecture.

Conjecture 2. *The L^2 completion of $\mathcal{M}_k$ is always the Donaldson/Uhlenbeck compactification.*

There are at least two pieces of supporting evidence. The first was provided in [D2], where (to circumvent technical difficulties) Donaldson defined ϵ-thickened moduli spaces

$$\mathcal{B}_\epsilon = \{[A] \in \mathcal{B} \mid \|F_-^A\|_{L^2} < \epsilon\};$$

thus $\bigcap_{\epsilon>0} \mathcal{B}_\epsilon = \mathcal{M}$. (We have dropped k for simplicity.) Let d_ϵ be the distance function on d_ϵ defined by taking the infimum of the L^2-lengths of connecting paths. Donaldson proved that for any $\epsilon > 0$, the completion of $\mathcal{M}$ in the metric $d_\epsilon|_{\mathcal{M}}$ is homeomorphic to the compactification above. Intuitively, as $\epsilon \to 0$, the metric d_ϵ ought to approach the path-length metric defined by $\mathbf{g}$, so Donaldson's result supports Conjecture 2.

The other piece of evidence was provided by P. Feehan [F], who proved that for arbitrary k, but for M of the restricted topological type considered earlier (simply connected and with definite intersection form), Conjecture 2 is true.

Conjecture 2 concerns only rather coarse properties of M, those that do not involve derivatives of the metric. There are analogues of Theorems 3–4 that come to mind as more general possibilities. In particular, we have the following

VAGUE QUESTION. Assume $\mathcal{M}_k$ is a moduli space for which Conjecture 2 is true. Are the boundary strata $\mathcal{M}_{k-j} \times \Sigma^j M$ totally geodesic subspaces of the completion $\overline{\mathcal{M}_k}$ (at least for large enough strata)?

It is premature to elevate this to the level of a conjecture, or even to attempt to state it precisely. But the question is not unreasonable. The analysis involved for general $\mathcal{M}$ has many similarities to the analysis in [G3] for the five-dimensional $\mathcal{M}$'s, and it is this analysis that drives Theorem 3. The analysis makes it plausible that if $\partial\overline{\mathcal{M}}$ contains an entire stratum $\mathcal{M}_{k-j} \times \Sigma^j M$, then the stratum will be totally geodesic away from the diagonals in the symmetric product. However, in some cases charge can only bubble off along certain subvarieties of M (for general M, k). In such a case one would not expect a totally geodesic stratum.

5. The localization principle at work: two applications.

5.1 *Localization and Theorems* 1, 3, *and* 4. The proofs of Theorems 1, 3, and 4 all rely on the localization principle described earlier. In order to put this principle to work one needs an approximation to the tangent space $\mathcal{T}_A$ (see (14) that uses only local information. To this end we define the *approximate tangent space*

$$\tilde{\mathcal{T}}_A := \{\tilde{X} := \iota_X F^A \mid X = \text{certain type of vector field on } M\}. \tag{24}$$

Here F^A is the curvature two-form of A, and ι_X denotes contraction, so that $\tilde{X} \in \Omega^1(\mathrm{Ad}\, P)$. Specifically, we take X to be a linear combination of vector fields of the form $\mathrm{grad}(\beta f)$, where β is a cut-off function centered at $p_D(A)$ (zero beyond, say, half the injectivity radius), and f either is linear in normal coordinates, or is squared distance to $p_D(A)$. Essentially, if f is of linear type then $\tilde{X}$ corresponds to an infinitesimal motion of the center point p_D, while if f is of distance-squared type then $\tilde{X}$ corresponds to an infinitesimal change of scale λ. For such X one finds that $(d^A)^*\tilde{X} = 0$ and that $d_-^A\tilde{X}$ is small, in several relevant norms, relative to $\|\tilde{X}\|_2$. In particular this implies that if $\pi : \tilde{\mathcal{T}}_A \to \mathcal{T}_A$ is the L^2-orthogonal projection, then $\mathrm{Id} - \pi$ is also small in relevant norms—i.e. that $\tilde{\mathcal{T}}_A$ is, in fact, close to $\mathcal{T}_A$ in a useful sense.

To put this approximation to use in the context of Theorem 4 requires another application of the localization principle: localizing objects of the form $G_0^A\{\pi\tilde{X}, \pi\tilde{Y}\}$ and $G_-^A[\pi\tilde{X}, \pi\tilde{Y}]_-$. This is accomplished by inverting the Weitzenböck identities for 1-forms and 2-forms (see [G2]):

$$G_0^A\{\tilde{X}, \tilde{Y}\} = -\frac{1}{2}F^A(X,Y) + G_0^A(\mathrm{Rem}_0(X,Y)), \tag{25}$$

$$G_-^A[\tilde{X}, \tilde{Y}]_- = (X^* \wedge \iota_Y F^A)_- + G_-^A(\mathrm{Rem}_-(X,Y)), \tag{26}$$

where $\mathrm{Rem}_i(X,Y)$ is a local expression involving X, Y, F^A and their derivatives. (In (26), X^* is the 1-form that is metric-dual to X.) It turns out that $G_i^A(\mathrm{Rem}_i(X,Y))$ is small in relevant norms. The purely local first terms in (25)-(26) are what lead to the cancellation in the sectional curvature formula discussed following Conjecture 1. (Of course, to prove the theorem one still has to deal with the effect of replacing $\tilde{X}, \tilde{Y}$ by $\pi\tilde{X}, \pi\tilde{Y}$ in (25)–(26).)

5.2 *Localization and the μ-map.* We conclude with a rather different application of the localization principle. Throughout this section, $G = SU(2)$ and $\hat{G} = SO(3)$.

The space $\mathcal{A}^* \times P$ carries two free, commuting, group actions: the diagonal action of $\mathcal{G}$, and the action of G on the right-hand factor. If we divide out first by the $\mathcal{G}$-action, defining $\mathcal{P} := \mathcal{A}^{\times}_{\mathcal{G}} P$, there remains an induced free $\hat{G}$-action on $\mathcal{P}$, with quotient $\mathcal{B}^* \times M$. Thus we obtain an $SO(3)$-bundle $\mathcal{P} \to \mathcal{B}^* \times M$. Donaldson defined a map $\mu : H_*(M; \mathbf{Q})) \to H^*(\mathcal{B}^*; \mathbf{Q})$ (see [DK, §5.1]) by

$$\mu([\Sigma]) = -\frac{1}{4} p_1(\mathcal{P})/[\Sigma]. \tag{27}$$

(Here $p_1(\mathcal{P}) \in H^4(\mathcal{B}^*; \mathbf{Z})$ is the first Pontryagin class, and / denotes slant product, i.e. "integration over fibers".)

For today, our interest is not directly in the topological invariants that arise by composing μ with the restriction map $H^*(\mathcal{B}^*) \to H^*(\mathcal{M}^*)$, but rather in the interplay between the topology of the μ-map and the L^2 metric. Specifically, we will discuss a de Rham-theoretic version of μ, by which we mean a map

$$\mu_{DR} : \Omega^i(M) \to \Omega^i(\mathcal{B}^*), \quad i = 0, \dots, 4, \tag{28}$$

commuting with exterior derivative, with the property that for a homology class $[\Sigma] \in H_*(M)$, the de Rham cohomology class of $\mu_{DR}(\text{P.D.}(\Sigma))$ (where P.D. denotes Poincaré dual) equals the image of $\mu(\Sigma)$ in de Rham cohomology.

Any connection on P gives us a candidate for μ_{DR}; simply replace $p_1(\mathcal{P})$ by the corresponding Chern-Weil representative $\xi := p_1^{DR}(\mathcal{P})$. The arbitrariness in the connection makes μ_{DR} non-canonical, the ambiguity disappearing in cohomology. However, we will see that by using the L^2 metric to define the connection (as in [DK, §5.2.3]), the map μ_{DR} we obtain behaves well even at the level of forms. This is perhaps surprising, since the role the base metric g_0 plays (via the definition of $\mathcal{M}$) of most Donaldson invariants is only incidental.

For simplicity we will only consider the case $i = 2$ in (28); for $i = 4$ see [GS], where an application of μ_{DR} to Kronheimer-Mrowka simple type is discussed. Note that ξ is an element of $\Omega^4(\mathcal{B}^* \times M) \cong \sum_{j=0}^4 \Omega^j(\mathcal{B}^*) \otimes \Omega^{4-j}(M)$. The slant product in (27) kills all but the component lying in $\Omega^2(\mathcal{B}^*) \otimes \Omega^2(M)$, and pairs the $\Omega^2(M)$ factor of the surviving component with an element $[\Sigma]$ of $H_2(M)$—or, equivalently, wedges the $\Omega^2(M)$ factor with P.D.$([\Sigma])$ and integrates the result over M, leaving an element of $\Omega^2(\mathcal{B}^*)$. Thus, for a closed form $\phi \in \Omega^2(M)$ and $\overline{\alpha}, \overline{\beta} \in T_{[A]}\mathcal{B}^*$,

$$\mu_{DR}(\phi)(\overline{\alpha}, \overline{\beta}) = \int_M (\iota_{\overline{\beta}} \iota_{\overline{\alpha}} \xi) \wedge \phi. \tag{29}$$

To define the connection on $\mathcal{P}$, let $\pi_1 : \mathcal{A}^* \times P \to \mathcal{P}$, $\pi_2 : \mathcal{P} \to \mathcal{B}^* \times M$, and $\pi_3 : \mathcal{A}^* \times P \to (\mathcal{A}^* \times P)/G = \mathcal{A}^* \times M$ be the natural projections. We then define the subspaces $\mathcal{V}_{A,p}, \mathcal{H}_{A,p}$ of $T_{A,p}(\mathcal{A}^* \times P) = T_A\mathcal{A}^* \oplus T_pP$ by $\mathcal{V}_{A,p} := \ker(\pi_{1*})$ and $\mathcal{H}_{A,p} = \mathcal{H}_A \oplus H_p^A$, where $\mathcal{H}^A$ is as in (7) and where $V_p^A, H_p^A \subset T_pP$ are the vertical and horizontal subspaces defined by the connection A. Thus $T_{(A,p)}(\mathcal{A}^* \times P) = \mathcal{V}_{A,p} \oplus (0, V_p^A) \oplus \mathcal{H}_{A,p}$ and this decomposition is invariant under both group actions. Note that the first summand is precisely the vertical space for π_1, while the second is the vertical space for π_3 (and hence can be naturally identified with the vertical space for π_2). Thus the $(\mathcal{G} \times G)$-invariant distribution $\{H_{A,p}\}$ induces a $\hat{G}$-invariant horizontal distribution on $\mathcal{P}$, i.e. a connection.

The curvature $\mathbf{F}$ of this connection can now be computed; we omit the details (see [DK, §5.2]). The result is that the pullback $\tilde{\mathbf{F}} = \pi_1^*\mathbf{F}$ satisfies

$$\tilde{\mathbf{F}}((\alpha, X), (\beta, Y)) = \left(\mathcal{F}^A(\alpha, \beta) + \alpha(Y) - \beta(X) + F^A(X, Y)\right)\Big|_p \tag{30}$$

where $\mathcal{F}^A$ is as in (9) and $(\alpha, X), (\beta, Y) \in \mathcal{H}_{A,p}$. According to Chern-Weil theory, $-p_1(\mathcal{P})/4$ is represented by $(8\pi^2)^{-1}\mathrm{tr}(\mathbf{F} \wedge \mathbf{F})$. Hence if $\overline{\alpha}, \overline{\beta} \in T_{[A]}\mathcal{B}^*$ are represented by $\alpha, \beta \in \mathcal{H}_A$, then combining (29) and (30) we have

$$\mu_{DR}(\phi)(\overline{\alpha}, \overline{\beta}) = -\frac{1}{4\pi^2} \int_M \mathrm{tr}(\alpha \wedge \beta + 2G_0^A\{\alpha, \beta\}F^A) \wedge \phi. \tag{31}$$

(Note: [DK, Proposition 5.2.18] omits an overall sign and a relative factor of 2 that are important below.)

Now return to the 5-dimensional moduli spaces considered in Section 4, with a slight modification: to make easier contact with the literature, consider *anti*-self-dual connections over manifolds with *negative*-definite intersection forms. Let $\lambda = \lambda_D$ be a scale function as in (18). For small λ the collar map gives an embedding $\tau_\lambda : M \to \mathcal{B}^*$ (factoring through $\mathcal{M}$). One of the first theorems of Donaldson concerning the μ-map was that in this context, the composition $\tau_\lambda^* \circ \mu : H_2(M) \to H^2(M)$ is precisely Poincaré duality (see [DK §5.3]).

Since μ_{DR} starts out by Poincaré-dualizing the argument of μ, a de Rham-theoretic version of Donaldson's theorem would simply assert that the composition $\tau_\lambda^* \circ \mu_{DR}$: $\Omega^2(M) \to \Omega^2(M)$ is the identity *on the level of cohomology.* But the localization discussed in §5.1 gives much more. If we write $\alpha = \pi\tilde{X}$, $\beta = \pi\tilde{Y}$ as in §5.1, and use (25), we find that the local terms in (31) dominate:

$$\begin{aligned} \mu_{DR}(\phi)(\overline{\pi\tilde{X}}, \overline{\pi\tilde{Y}}) &\approx -\frac{1}{4\pi^2} \int_M \mathrm{tr}(\iota_X F^A \wedge \iota_Y F^A - F^A(X, Y)F^A) \wedge \phi \\ &= \frac{1}{8\pi^2} \int_M |F^A|^2 \ \phi(X, Y) \ d\mathrm{vol}_{g_0}. \end{aligned} \tag{32}$$

Now let $\lambda \to 0$, holding the center point of A fixed at some $p \in M$, and holding X, Y fixed. Then $(8\pi^2)^{-1}|F^A|^2 d\mathrm{vol}$ approaches a delta-form centered at p, and the remainder implicit in (32) tends to zero as $\lambda \to 0$ ([GP4]). Thus

$$\mu_{DR}(\phi)(\overline{\pi\tilde{X}}, \overline{\pi\tilde{Y}}) \to \phi(X_p, Y_p) \quad \text{as } \lambda \to 0. \tag{33}$$

Finally, we invoke the fact that if the vector field X is of "linear type" centered at p (see §4.3), then $\pi\tilde{X} \approx -(\tau_\lambda)_* X_p$, the approximation becoming arbitrarily good as $\lambda \to 0$ (see [GP2]). Thus (33) implies that

$$\lim_{\lambda \to 0} \tau_\lambda^*(\mu_{DR}(\phi)) = \phi. \tag{34}$$

In other words, as $\lambda \to 0$, we recover Donaldson's Poincaré duality result, *on the level of forms*, not merely in cohomology.

References

[CE] J. Cheeger and D. G. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.

[DMM] H. Doi, Y. Matsumoto, and T. Matumoto, *An explicit formula of the metric on the moduli space of BPST-instantons over* S^4, in: A Fête of Topology, Academic Press, 1987, 543–556.

[D1] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. 18 (1983), 279–315.

[D2] S. K. Donaldson, *Compactification and completion of Yang-Mills moduli spaces*, in: Differential Geometry, Proc. Conf. Peniscola 1988, F. J. Carreras et al. (ed.), Lecture Notes in Math. 1410, Springer, Berlin, 1989, 145–160.

[DK] S. K. Donaldson, P. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press, New York, 1990.

[F] P. Feehan, *Geometry of the ends of the moduli space of anti-self-dual connections*, J. Differential Geom. 42 (1995), 465–553.

[FU] D. S. Freed and K. K. Uhlenbeck, *Instantons and Four-Manifolds*, second edition, Springer, New York, 1991.

[G1] D. Groisser, *The geometry of the moduli space of* $\mathbf{CP}^2$ *instantons*, Invent. Math. 99 (1990), 393–409.

[G2] D. Groisser, *Curvature of Yang-Mills moduli spaces near the boundary, I*, Comm. Anal. Geom. 1 (1993), 139–216.

[G3] D. Groisser, *Totally geodesic boundaries of Yang-Mills moduli spaces*, Preprint, 1996.

[GP1] D. Groisser and T. H. Parker, *The Riemannian geometry of the Yang-Mills Moduli Space*, Comm. Math. Phys. 112 (1987), 663–689.

[GP2] D. Groisser and T. H. Parker, *The geometry of the Yang-Mills moduli space for definite manifolds*, J. Differential Geom. 29 (1989), 499–544.

[GP3] D. Groisser and T. H. Parker, *Semiclassical Yang-Mills Theory I, Instantons*, Comm. Math. Phys. 135 (1990), 101–140.

[GP4] D. Groisser and T. H. Parker, *Differential forms on the Yang-Mills moduli space*, in preparation.

[GS] D. Groisser and L. Sadun, *Simple type and the boundary of moduli space*, in preparation.

[H] L. Habermann, *On the geometry of the space of* $Sp(1)$*-instantons with Pontrjagin index* 1 *on the* 4*-sphere*, Ann. Global Anal. Geom. 6 (1988), 3–29.

[K] K. Kobayashi, *Three Riemannian metrics on the moduli space of* 1*-instantons over* $\mathbf{CP}^2$, Hiroshima Math. J. 19 (1989), 243–249.

[MM] K. B. Marathe and G. Martucci, *The Mathematical Foundations of Gauge Theories*, North-Holland, Amsterdam, 1992.

[M] T. Matumoto, *Three Riemannian metrics on the moduli space of BPST-instantons over* S^4, Hiroshima Math. J. 19 (1989), 221–224.

[MV] P. K. Mitter and C. M. Viallet, *On the bundle of connections and the gauge orbit manifold in Yang-Mills theory*, Comm. Math. Phys. 79 (1981), 457–472.

[S] I. M. Singer, *The Geometry of the Orbit Space for Nonabelian Gauge Theories*, Phys. Scripta 24 (1981), 817–820.