SYMPLECTIC SINGULARITIES AND GEOMETRY OF GAUGE FIELDS
BANACH CENTER PUBLICATIONS, VOLUME 39
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

SEIBERG-WITTEN THEORY

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Abstract. We give an introduction into and exposition of Seiberg-Witten theory.

1. Introduction. Let $Au = 0$ (D) be a partial differential equation on a manifold M, $\mathcal{S}$ the set of all solutions, $\mathcal{G}$ the automorphism group of (D), $\mathcal{M} = \mathcal{S}/\mathcal{G}$ the moduli space. It is one of the most striking achievements and insights of modern global analysis that $\mathcal{M}$ contains, reflects many (hidden) properties of M. Probably the most famous example until October 1994 was the instanton equation $*\Omega^\omega = \Omega^\omega$ or $\Omega^\omega_- = 0$ and Donaldson's moduli spaces. As well known, Donaldson's moduli spaces are rather complicated stratified spaces. At October 26, 1994, Ed Witten gave during a lecture at the MIT a hint that the equations established by him and Nathan Seiberg contain more or less the same information about the underlying manifold M as Donaldson's theory. Up to a great part this has been proven until now. But the Seiberg-Witten equations are much simpler than instanton equations. In the most interesting cases their moduli space is zero-dimensional and produces a $\mathbb{Z}_2$– or integer invariant.

In this paper, we attempt to give a comprehensive representation of Seiberg-Witten theory and further developments given by Taubes, LeBrun, Kronheimer, Mrowka and others. The main goal is to present this subject to a broader audience. Therefore we start with simple facts from Clifford theory and proceed step by step by honest calculations. In the second part, we present and discuss with much less calculations the achievements of the theory known to us until now.

The paper is organized as follows. In Section 2, we define $\mathrm{Spin}^{\mathbb{C}}$ structures and discuss

1991 *Mathematics Subject Classification*: 58G03, 58G10.

The paper is in final form and no version of it will be published elsewhere.

their existence. Section 3 is devoted to the Seiberg-Witten equations. In the next two sections, we discuss the configuration space and moduli space of Seiberg-Witten theory and define in Section 6 the Seiberg-Witten invariant in a little more general context. The following sections are devoted to vanishing theorems, the case $\dim \mathcal{M}_L(g) = 0$, Kähler and symplectic manifolds, in particular to Taubes' result concerning the coincidence of Seiberg-Witten and Gromov invariants.

This paper arose essentially from the preprints of the main contributors, an exposition of the second author written in Berlin, January 1995, and from 6 lectures given by the first author in May 1995 in Warsaw.

2. $\mathrm{Spin}^{\mathbb{C}}$ **structures.** In this section, we recall the main definitions concerning $\mathrm{Spin}^{\mathbb{C}}$ structures and discuss their existence.

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, $\mathrm{Cl}(n)$ the $\mathbb{R}$-algebra generated by $e_1, \ldots, e_n$ with relations $e_i e_j + e_j e_i = -2\delta_{ij}$, $\mathrm{Cl}^{\mathbb{C}}(n) = \mathrm{Cl}(n) \otimes \mathbb{C}$. Then

$$\mathrm{Cl}^{\mathbb{C}}(n) \cong \begin{cases} \mathbb{C}(2^k), & n = 2k \\ \mathbb{C}(2^k) \oplus \mathbb{C}(2^k), & n = 2k+1, \end{cases}$$

where $\mathbb{C}(l)$ denotes the algebra of complex $l \times l$-matrices. Set

$$g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\alpha(j) = \begin{cases} 1, & j \\ 2, & j \text{ even.} \end{cases}$$

Then a concrete isomorphism $\mathrm{Cl}^{\mathbb{C}}(2k) \xrightarrow{\cong} \mathbb{C}(2^k)$ is given by

$$e_j \mapsto E \otimes \cdots \otimes E \otimes g_{\alpha(j)} \otimes \underbrace{T \otimes \cdots \otimes T}_{[(j-1)/2] \text{ times}},$$

similarly the isomorphism $\mathrm{Cl}^{\mathbb{C}}(2k+1) \xrightarrow{\cong} \mathbb{C}(2^k) \otimes \mathbb{C}(2^k)$ is given by

$$e_j \mapsto (E \otimes \cdots \otimes E \otimes g_{\alpha(j)} \otimes \underbrace{T \otimes \cdots \otimes T}_{[(j-1)/2] \text{ times}}, E \otimes \cdots \otimes E \otimes g_{\alpha(j)} \otimes \underbrace{T \otimes \cdots \otimes T}_{[(j-1)/2] \text{ times}})$$

for $1 \le j \le 2k$ and

$$e_{2k+1} \mapsto (iT \otimes \cdots \otimes T, -iT \otimes \cdots \otimes T).$$

We have $\mathbb{R}^n \subset \mathrm{Cl}(n)$. If $x \in \mathbb{R}^n$, $x \neq 0$ then $x^{-1} \in \mathrm{Cl}(n)$. Hence $S^{n-1} \subset \mathrm{Cl}(n)$ generates a group $\mathrm{Pin}(n)$. Set

$$\mathrm{Spin}(n) = \{u \in \mathrm{Pin}(n) \mid u = x_1 \cdots x_m,\ x_i \in S^{n-1},\ m \text{ even}\}.$$

Then there exists a 2-fold covering $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$, $\lambda(u)y = uyu^*$, $y \in \mathbb{R}^n$, $(x_1 \cdots x_m)^* = x_m \cdots x_1$. Next we define the Spin modules Δ_n by presenting concrete bases. Set $u_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, $u_{-1} = \begin{pmatrix} 1 \\ i \end{pmatrix}$,

$$u(\varepsilon_1, \ldots, \varepsilon_k) := u_{\varepsilon_1} \otimes \cdots \otimes u_{\varepsilon_k}, \qquad \varepsilon_i = \pm 1.$$

Then we define

$$\begin{aligned} \Delta_{2k+1} &:= \langle u(\varepsilon_1, \ldots, \varepsilon_k) \mid \varepsilon_i = \pm 1 \rangle \\ \Delta_{2k}^{+} &:= \langle u(\varepsilon_1, \ldots, \varepsilon_k) \mid \varepsilon_i \cdots \varepsilon_k = +1 \rangle \end{aligned}$$

$$\Delta_{2k}^- := \langle u(\varepsilon_1, \dots, \varepsilon_k) \mid \varepsilon_i \cdots \varepsilon_k = -1 \rangle$$

A straightforward calculation shows

$$Tu_\varepsilon = -\varepsilon u_\varepsilon, \quad g_{\alpha(j)} u_\varepsilon = (-1)^{j-1} i^{\alpha(j)} \varepsilon^{\alpha(j+1)} u_{-\varepsilon}. \tag{2.1}$$

PROPOSITION 2.1.

a. *For* $n = 2k+1$, Δ_n *is an irreducible* $\mathrm{Spin}(n)$ *module.*

b. *For* $n = 2k$, $\Delta_n^\pm$ *are irreducible* $\mathrm{Spin}(n)$ *modules.*

c. $\nu \in \mathbb{R}^n \subset \mathrm{Cl}^{\mathbb{C}}(n)$, $\psi \in \Delta_n$ *imply* $\nu \cdot \psi \in \Delta_n$; *if* $n = 2k$ *then* $\mathbb{R}^n \otimes \Delta_n^+ \overset{\cdot}{\to} \Delta_n^-$, $\mathbb{R}^n \otimes \Delta_n^- \overset{\cdot}{\to} \Delta_n^+$.

PROPOSITION 2.2. Δ_n *has a Hermitian scalar product* $(\cdot,\cdot)$ *such that* $(x\cdot\psi, \psi') + (\psi, x\cdot\psi') = 0$.

Propositions 2.1c. and 2.2 essentially follow from (2.1). ■

PROPOSITION 2.3. *The representation* $\kappa : \mathrm{Spin}(n) \to \mathrm{Gl}(\Delta_n)$ *is unitary.*

Proof. $(x \cdot y \cdot \psi, x \cdot y \cdot \psi') = (\psi, y \cdot x \cdot x \cdot y \cdot \psi') = (\psi, \psi')$. ■

Now we define

$$\mathrm{Spin}^{\mathbb{C}}(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}_2} S^1 \subset \mathrm{Cl}^{\mathbb{C}}(n).$$

There are canonical maps

$$\begin{aligned}
&\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n), \quad \text{2-fold covering},\\
&\lambda : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \quad , \lambda[g,z] := \lambda(g),\\
&i : \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathbb{C}}(n), \quad i(g) := [g,1],\\
&k : \mathrm{Spin}^{\mathbb{C}}(n) \to S^1, \quad k[g,z] := z^2,\\
&\pi : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \times S^1, \quad \pi[g,z] := (\lambda(g), z^2), \quad \text{2-fold covering},\\
&\kappa : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{U}(\Delta_n), \quad \kappa[g,z](\psi) = z \cdot g(\psi).
\end{aligned}$$

PROPOSITION 2.4. $\det(\kappa[g,z]) = z^{\dim \Delta_n} = z^{2^{[n/2]}}$.

Proof. This is a consequence of $\kappa : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SU}(\Delta_n)$ and the latter follows from $\mathrm{tr}(e_i \cdot e_j | \Delta_n) = 0$. ■

COROLLARY 2.5. *If* $n = 2k$, $\kappa^\pm : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SU}(\Delta_n^\pm)$, *then* $\det \kappa^\pm[g,z] = z^{\dim \Delta^\pm} = z^{2^{k-1}}$, *in particular for* $n = 4$ $\det \kappa^\pm[g,z] = z^2$.

DEFINITION. Let X be a manifold, $P = P(X, \mathrm{SO}(n))$ an $\mathrm{SO}(n)$-principal fibre bundle. A $\mathrm{Spin}^{\mathbb{C}}(n)$ *structure* Q for P is a $\mathrm{Spin}^{\mathbb{C}}(n)$-principal fibre bundle $Q(X, \mathrm{Spin}^{\mathbb{C}}(n))$ and a commutative diagram

$$\begin{array}{ccccccc}
Q & \times & \mathrm{Spin}^{\mathbb{C}}(n) & \longrightarrow & Q \searrow & & \\
& \Lambda \downarrow \lambda & & & \Lambda \downarrow & & X \quad , \\
P & \times & \mathrm{SO}(n) & \longrightarrow & P \nearrow & &
\end{array}$$

i. e. $\Lambda(q \cdot x) = \Lambda(q) \cdot \lambda(x)$, $q \in Q$, $x \in \mathrm{Spin}^{\mathbb{C}}(n)$. The exact sequences $1 \to S^1 \to \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \to 1$, $1 \to \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathbb{C}}(n) \xrightarrow{k} S^1 \to 1$ imply $\mathrm{Spin}^{\mathbb{C}}(n)/S^1 = \mathrm{SO}(n)$, $\mathrm{Spin}^{\mathbb{C}}(n)/\mathrm{Spin}(n) = S^1$ and general compatibility properties yield $Q/S^1 = P$ and

$$P_{\mathrm{U}} := Q/\mathrm{Spin}(n)$$

is a S^1-principal fibre bundle, hence

$$L := Q \times_k \mathbb{C} = P_{\mathrm{U}} \times_{\mathrm{U}} \mathbb{C}$$

is a complex line bundle.

PROPOSITION 2.6. *$r_* c_1(L) = w_2(P)$ in $H^2(X;\mathbb{Z}_2)$, where $r_* : H^2(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z}_2)$ comes from $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \to 0$.*

Proof. Consider $E = P \times_{\mathrm{SO}} \mathbb{R}^n = Q \times_\lambda \mathbb{R}^n$, $E \oplus L = Q \times_{i\pi} (\mathbb{R}^n \oplus \mathbb{R}^2)$, where $i\pi : \mathrm{Spin}^{\mathbb{C}}(n) \xrightarrow{\pi} \mathrm{SO}(n) \times \mathrm{SO}(2) \xrightarrow{i} \mathrm{SO}(n+2)$. Let $\pi_1(\mathrm{Spin}^{\mathbb{C}}(n)) = \langle x \rangle \cong \mathbb{Z}$, $\pi_1(\mathrm{SO}(n)) = \langle y \rangle \cong \mathbb{Z}_2$, $n > 2$, $\pi_1(\mathrm{SO}(2)) = \langle z \rangle \cong \mathbb{Z}$. The diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & S^1 \arrow[d, "j"'] \arrow[dr, "z \to z^2"] & & \\
1 \arrow[r] & \mathrm{Spin}(n) \arrow[r, "i"] \arrow[dr, "\lambda"'] & \mathrm{Spin}^{\mathbb{C}}(n) \arrow[r, "k"] \arrow[d, "\lambda"] & S^1 \arrow[r] & 1 \\
 & & \mathrm{SO}(n) \arrow[d] & & \\
 & & 1 & &
\end{tikzcd}$$

induces on π_1-level

$$\begin{tikzcd}
 & 1 \arrow[d] & & \\
 & \pi_1(S^1) \arrow[d, "j_*"'] \arrow[dr, "z \to 2z"] & & \\
1 \arrow[r] & \pi_1(\mathrm{Spin}^{\mathbb{C}}(n)) \arrow[r, "k_*"] \arrow[d, "\lambda_*"] & \pi_1(S^1) \arrow[r] & 1 \\
 & \pi_1(\mathrm{SO}(n)) \arrow[d] & & \\
 & 1 & &
\end{tikzcd}$$

Hence $k_* : \pi_1(\mathrm{Spin}^{\mathbb{C}}(n)) \xrightarrow{\cong} \pi_1(S^1)$.

ASSERTION.

$$i_*\pi_*(x) = i_*(y+z) = 0. \tag{2.2}$$

P r o o f. Consider $\lambda_*(x)$. Assume $\lambda_*(x) = 0$. Then $x = j_*(\tilde z)$ with some $\tilde z \in \pi_1(S^1)$ and $k_*(x) = k_* j_*(\tilde z) = 2\tilde z$. This contradicts the fact that $k_*(x)$ generates $\pi_1(S^1)$. Hence $\pi_*(x) = y + z$. Considering the inclusions $\mathrm{SO}(2) \hookrightarrow \mathrm{SO}(n+2)$, $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+2)$ at π_1-level, we see immediately $i_*(y+z) = 0$, $(i\pi)_*(x) = 0$.

The transition functions of $E \oplus L$ map into $\mathrm{SO}(n) \times \mathrm{SO}(2)$. We conclude from

$$\begin{array}{ccc} \mathrm{Spin}^{\mathbb{C}}(n) & \rightarrow & \mathrm{Spin}(n+2) \\ \downarrow \pi & & \downarrow \\ \mathrm{SO}(n) \times \mathrm{SO}(2) & \xrightarrow{i} & \mathrm{SO}(n+2) \end{array} \tag{2.3}$$

and (2.2) that they lift into $\mathrm{Spin}(n+2)$. Hence $E \oplus L$ admits a Spin structure, $0 = w_2(E \oplus L) = w_2(L) + w_2(E)$, $w_2(L) = w_2(E) = w_2(P)$. But $r_* c_1(L) = w_2(L)$. ∎

PROPOSITION 2.7. *$P(X, \mathrm{SO}(n))$ admits a $\mathrm{Spin}^{\mathbb{C}}$ structure if and only if $w_2(P) \in H^2(X; \mathbb{Z}_2)$ is $\mathbb{Z}_2$-reduction of an integer cohomology class $\in H^2(X; \mathbb{Z})$.*

P r o o f. We have just proven that the condition is necessary. Assume now $w_2(P) = r_*\alpha$, $\alpha \in H^2(X; \mathbb{Z})$. Choose L with $c_1(L) = \alpha$. Then $P \times L$ is a $\mathrm{SO}(n) \times \mathrm{SO}(2)$-bundle with $w_2 = 0$, i.e. with $\mathrm{Spin}(n+2)$ structure. Once again we conclude from (2.3) that we can lift transition functions to $\mathrm{Spin}^{\mathbb{C}}(n)$. ∎

Next we show that there is no obstruction against the existence of a $\mathrm{Spin}^{\mathbb{C}}(4)$ structure for the orthogonal frame bundle of an arbitrary closed oriented 4-manifold.

THEOREM 2.8 ([7]). *Let M^4 be an oriented closed 4-manifold, $P = L(M, \mathrm{SO}(4))$ its orthogonal frame bundle. Then P admits a $\mathrm{Spin}^{\mathbb{C}}(4)$ structure.*

P r o o f. The universal coefficient theorem yields

$$H^3(M^4; \mathbb{Z}) \cong H_3(M^4; \mathbb{Z}) / \operatorname{Tor} H_3(M^4; \mathbb{Z}) \quad \oplus \operatorname{Tor} H_2(M^4; \mathbb{Z}),$$

together with Poincaré duality

$$\begin{gathered} H_2(M^4; \mathbb{Z}) \cong H^2(M^4; \mathbb{Z}), \ \operatorname{Tor} H_2(M^4; \mathbb{Z}) \cong \operatorname{Tor} H^2(M^4; \mathbb{Z}), \\ \operatorname{Tor} H^3(M^4; \mathbb{Z}) \cong \operatorname{Tor} H^2(M^4; \mathbb{Z}) \equiv T. \end{gathered}$$

Consider $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \to 0$ and $\cdots \longrightarrow H^2(M^4; \mathbb{Z}) \xrightarrow{r_*} H^2(M^4; \mathbb{Z}_2) \xrightarrow{\beta_*} H^3(M^4; \mathbb{Z}) \xrightarrow{(\cdot 2)_*} H^3(M^4; \mathbb{Z}_2) \longrightarrow \cdots$ which implies $\operatorname{im} \beta_* \cong H^2(M^4; \mathbb{Z}_2) / \operatorname{im} r_*$. Moreover,

$$\begin{aligned} \operatorname{im} \beta_* = \{\alpha \in H^3(M^4; \mathbb{Z}) \mid 2\alpha = 0\} = \{\alpha \in \operatorname{Tor} H^3(M^4; \mathbb{Z}) \mid 2\alpha = 0\} \cong \\ \cong \{\gamma \in \operatorname{Tor} H^2(M^4; \mathbb{Z}) \mid 2\gamma = 0\} = \{\gamma \in T \mid 2\gamma = 0\}. \end{aligned}$$

We infer from the $\mathbb{Z}_2$-exact sequence

$$\begin{gathered} \{\gamma \in T \mid 2\gamma = 0\} \longrightarrow T \xrightarrow{\cdot 2} T \longrightarrow T/2T \longrightarrow 0 \\ \dim_{\mathbb{Z}_2} T/2T = \dim_{\mathbb{Z}_2} \{\gamma \in T \mid 2\gamma = 0\}. \end{gathered}$$

This and $r_*(T) = T/2T$ yield

$$\dim_{\mathbb{Z}_2} H^2(M^4;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} \operatorname{im} r_* + \dim_{\mathbb{Z}_2} \beta_* = \dim_{\mathbb{Z}_2} \operatorname{im} r_* + \dim_{\mathbb{Z}_2} r_*(T)$$

and moreover $H^2(M^4;\mathbb{Z}_2) \supset \operatorname{im} r_* \supset r_*(T)$. For $x \in \operatorname{im} r_*$, $x = r_*(\alpha)$, $y \in r_*(T)$, $y = r_*(\beta)$, $x \cup y = 0$ since $\alpha \cup \beta = 0$ in $H^4(M^4;\mathbb{Z})$ (β is a torsion element). Hence

$$\operatorname{im} r_* \subset Z = \{z \in H^2(M^4;\mathbb{Z}_2) \mid z \cup y = 0 \text{ for all } y \in r_*(T)\}.$$

Z is the orthogonal complement of $r_*(T)$ in $H^2(M^4 : \mathbb{Z}_2)$. This implies

$$\dim_{\mathbb{Z}_2} Z = \dim_{\mathbb{Z}_2} H^2(M^4;\mathbb{Z}_2) - \dim r_*(T).$$

On the other hand,

$$\dim_{\mathbb{Z}_2} H^2(M^4;\mathbb{Z}_2) - \dim_{\mathbb{Z}_2} r_*(T) = \dim \operatorname{im} r_*,$$

hence

$$\operatorname{im} r_* = Z. \tag{2.4}$$

According to Wu, for $x \in H^{n-k}(M^4;\mathbb{Z}_2)$,

$$S_q^k(x) = v_k \cup x$$

and

$$w_k = \sum_{i+j=k} S_q^i(v_j),$$

i.e.

$$\begin{aligned} w_1 &= S_q^0(v_1) + S_q^1(v_0) = v_1 + 0 = v_1 \\ w_2 &= S_q^0(v_2) + S_q^2(v_0) = v_2 + S_q^1(v_1), \end{aligned}$$

for $w_1 = 0$, $\dim M = 4$

$$w_2 = v_2$$

and $w_2 = v_2$ is characterized by

$$x^2 = S_q^2(x) = v_2 \cup x = w_2 \cup x \text{ for all } x \in H^2(M^4;\mathbb{Z}_2).$$

This means for all $x \in r_*(T)$ $0 = x^2 = w_2 \cup x$, and according to (2.4), $w_2 \in \operatorname{im} r_*$. The assertion now follows from Proposition 2.7. ■

For M^4 most of the steps of the proof of Theorem 2.8 are not available. Nevertheless we have

PROPOSITION 2.9. *Assume M^4 open, oriented, $H^3(M^4;\mathbb{Z})$ without 2-torsion. Then M^4 admits a $\mathrm{Spin}^{\mathbb{C}}(4)$ structure.*

Proof. Let $V_{n,k}$ be the Stiefel manifold of orthonormal k-frames in n-space. M^4 admits a 1-frame = vector field on the 3-dimensional skeleton. Since M^4 is open, it is possible to shift singularities on the 4-skeleton to infinity, i.e. M^4 admits a vector field, $TM^4 = \theta^1 \oplus \tau^3$, $w_2(TM^4) = w_2(\tau^3)$. All local systems of coefficients appearing here in obstruction theory are usual coefficients since τ^3 is oriented. Consider the obstruction classes $\sigma^2(\tau) \in H^2(M^4;\pi_1(V_{3,2}))$, $\sigma^2 = w_2(\tau^3)$, and $\sigma^3 \in H^3(M^4;\pi_2(V_{3,1})) = H^3(M^4;\mathbb{Z})$. Then $\beta_*\sigma^2 = \sigma^3$, $\beta_* w_2 = \sigma^3$, where β_* is the Bockstein operator. If $H^3(M^4;\mathbb{Z})$ has no 2-torsion then $\sigma^3 = 0$ and $w_2 \in \operatorname{im} r_*$. ■

3. The Seiberg-Witten equations. Let

$$\begin{array}{ccccc} Q \times \mathrm{Spin}^{\mathbb{C}}(n) & \longrightarrow & Q & \searrow & \\ \downarrow & & \downarrow & & M \\ P \times \mathrm{SO}(n) & \longrightarrow & P & \nearrow & \end{array}$$

be a $\mathrm{Spin}^{\mathbb{C}}(n)$ structure for $P(M, \mathrm{SO}(n))$, $P_{\mathrm{U}(1)} = Q/\mathrm{Spin}(n)$. Then $\pi : Q \to P \times P_{\mathrm{U}(1)}$ is a 2-fold covering. If $A \in \mathcal{C}_p =$ connection space of $P_{\mathrm{U}(1)}$ is a connection then A and the Levi-Civita connection of P generate (by lifting) a connection on Q and hence a covariant derivative

$$\nabla^A : \Omega(S) \to \Omega^1(S),$$

where $S = Q \times_{\mathrm{Spin}^{\mathbb{C}}(n)} \Delta$.

Remark. This S is not the classical Spin-bundle S_{Cl} of a Spin manifold. It is defined by means of $\times_{\mathrm{Spin}^{\mathbb{C}}(n)}$, $\kappa : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{U}(\Delta_n)$, $\kappa[g,z](\psi) = z \cdot g(\psi)$. Nevertheless it can be decomposed globally into S^+ and S^-, where locally $S^{\pm} = S^{\pm}_{\mathrm{Cl}} \otimes L^{1/2}$. Here $S^{\pm}_{\mathrm{Cl}}$ is a spinor bundle with respect to a local Spin structure on M, $L = P_{\mathrm{U}(1)} \times_{\mathrm{U}(1)} \mathbb{C}$.

We define the associated Dirac operator D_A as usual by $(D_A\Phi)(x) = \sum\limits_{i=1}^{n} e_i \cdot \nabla_{e_i}\Phi$, $e_1, \ldots, e_n$ an orthonormal basis in T_xM. D_A is elliptic. For $A : TP_{\mathrm{U}(1)} \to \mathrm{u}(1) = i\mathbb{R}$ the curvature $\Omega_A \in \Omega^2(i\mathbb{R})$ is defined by $\Omega_A = dA$ and there holds

$$c_1(L) = \left[-\frac{1}{2\pi i}\Omega_A\right] = \left[\frac{i}{2\pi}\Omega_A\right].$$

In a local cobasis e^α the curvature Ω_A can be presented by

$$\Omega_A = i \sum_{\alpha<\beta} \omega_{\alpha\beta} e^\alpha e^\beta,$$

and we define for $\Phi \in \Omega(S)$

$$\Omega_A \Phi := i \sum_{\alpha<\beta} \omega_{\alpha\beta} e_\alpha \cdot e_\beta \cdot \Phi.$$

PROPOSITION 3.1. *D_A^2 is given by*

$$D_A^2\Phi = (\nabla^A)^*\nabla^A\Phi + \frac{\tau}{4}\Phi - \frac{1}{2}\Omega_A\Phi, \tag{3.1}$$

where τ is the scalar curvature of M.

Consider for $n = 4$ the decompositions $\Lambda^2(\mathbb{R}^4) \equiv \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ and $\Delta_4 = \Delta_4^+ \oplus \Delta_4^-$. We have $\Lambda^2(\mathbb{R}^4) \subseteq \mathrm{Cl}(4)$ as vector spaces. Hence each $\mathrm{Cl}(4)$-module can be considered as Λ^2-module. If $e_1, \ldots, e_4$ is an orthonormal basis of $\mathbb{R}^4$ then

$$e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3$$

is a basis of Λ^2_-.

PROPOSITION 3.2. *Λ^2_- acts trivially on Δ_4^+, i.e. if $\omega \in \Lambda^2_-$ and $\varphi \in \Delta_4^+$ then $\omega \cdot \varphi = 0$.*

Proof. With respect to the explicit description of Δ_4^+ and the $e_i \in \mathrm{Cl}(4)$ above the endomorphisms $e_ie_j : \Delta_4^+ \to \Delta_4$ are given by

$$e_1e_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_1e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1e_4 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$
$$e_2e_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_2e_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3e_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

which implies $e_1e_2 - e_3e_4 = 0$, $e_1e_3 + e_2e_4 = 0$, $e_1e_4 - e_2e_3 = 0$. ∎

Define for $\varphi \in \Delta_4^+$

$$\omega^\varphi(X, Y) := (X \cdot Y \cdot \varphi, \varphi) + (X, Y)|\varphi|^2.$$

PROPOSITION 3.3. *$\omega^\varphi \in \Lambda^2_+(\mathbb{R}^4) \otimes i\mathbb{R}$ and*

$$(\omega^\varphi \cdot \varphi, \varphi) = -2|\varphi|^4, \quad |\omega^\varphi|^2 = 2|\varphi|^4. \tag{3.2}$$

Proof.

$$\begin{aligned}\omega^\varphi(X, Y) = (XY\varphi, \varphi) + (X, Y)|\varphi|^2 = ((-YX - 2(X, Y))\varphi, \varphi) + (XY)|\varphi|^2 = \\ = -(YX\varphi, \varphi) - (X, Y)|\varphi|^2 = -\omega^\varphi(Y, X),\end{aligned}$$

i.e. ω^φ is skewsymmetric.

$$\begin{aligned}\overline{\omega^\varphi(X, Y)} = \overline{(XY\varphi, \varphi)} + (X, Y)|\varphi|^2 = (\varphi, XY\varphi) + (X, Y)|\varphi|^2 = \\ = (YX\varphi, \varphi) + (X, Y)|\varphi|^2 = \omega^\varphi(Y, X) = -\omega^\varphi(X, Y), \quad \omega^\varphi(X, Y) \in i\mathbb{R}.\end{aligned}$$

For the second assertion, we set $\varphi = \left(\begin{smallmatrix}\varphi_1 \\ \varphi_2\end{smallmatrix}\right)$. Then

$$\begin{aligned}\omega^\varphi(e_1, e_2) = (e_1e_2\varphi, \varphi) = \left(\left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)\left(\begin{smallmatrix}\varphi_1 \\ \varphi_2\end{smallmatrix}\right), \left(\begin{smallmatrix}\varphi_1 \\ \varphi_2\end{smallmatrix}\right)\right) = i\left(|\varphi_1|^2 - |\varphi_2|^2\right) = \omega^\varphi(e_3, e_4), \\ \omega^\varphi(e_1, e_3) = -\varphi_2\overline{\varphi_1} + \varphi_1\overline{\varphi_2} = \omega^\varphi(e_2, e_4), \\ \omega^\varphi(e_1, e_4) = -i\varphi_2\overline{\varphi_1} - i\varphi_1\overline{\varphi_2} = \omega^\varphi(e_2, e_3)\end{aligned}$$

and

$$\begin{aligned}(\omega^\varphi \cdot \varphi, \varphi) = \sum_{i<j}\omega^\varphi(e_i, e_j)(e_i \cdot e_j \cdot \varphi, \varphi) = \sum_{i<j}\omega^\varphi(e_i, e_j)^2 = \\ = 2\left\{-(|\varphi_1|^2 - |\varphi_2|^2)^2 + (-\varphi_2\overline{\varphi_1} + \varphi_1\overline{\varphi_2})^2 - (\varphi_2\overline{\varphi_1} + \varphi_1\overline{\varphi_2})^2\right\} = \\ = 2\left\{-|\varphi_1|^4 + 2|\varphi_1|^2|\varphi_2|^2 - |\varphi_2|^4 + \varphi_2^2\overline{\varphi_1}^2 - 2|\varphi_1|^2|\varphi_2|^2 + \right. \\ \left. + \varphi_1^2\overline{\varphi_2}^2 - \varphi_2^2\overline{\varphi_1}^2 - 2|\varphi_1|^2|\varphi_2|^2 - \varphi_1^2\overline{\varphi_2}^2\right\} = \\ = 2\left\{-|\varphi_1|^4 - 2|\varphi_1|^2|\varphi_2|^2 - |\varphi_2|^4\right\} = -2(|\varphi_1|^2 + |\varphi_2|^2)^2 = -2|\varphi|^4.\end{aligned}$$

Similarly, $|\omega^\varphi|^2 = 2|\varphi|^4$. ∎

Now we are able to define the Seiberg-Witten equations. Let (M^4, g) be an oriented closed Riemannian 4-manifold, $P = L(M^4, \mathrm{SO}(4))$, $c \in H^2(M^4; \mathbb{Z})$ a $\mathrm{Spin}^{\mathbb{C}}(4)$ structure, i.e. $r_*c = w_2$. Then we define

$$D_A\Phi = 0, \qquad \Omega_A^+ = \frac{1}{4}\omega^\Phi \tag{SW}$$

as equations for $A \in \mathcal{C}_{P_{\mathrm{U}(1)}}$ and $\Phi \in \Omega(S^+)$. (SW) are the Seiberg-Witten equations.

Remark. These equations are similar to the Landau-Ginzburg model of super conductivity: (M^2, g) a Riemannian surface, $P \to M$ an U(1)-principal fibre bundle, $L = P \times_{\mathrm{U}(1)} \mathbb{C}$, $A \in \mathcal{C}_P$, $\Phi \in \Omega(L)$

$$\partial''_A \Phi = 0, \qquad \Omega_A = \frac{1}{2} * (1 - |\Phi|^2).$$

Solutions (Φ, A) of (SW) are exactly the zeros of the Seiberg-Witten functional

$$\int_{M^4} |\Omega_A^+ - \frac{1}{4}\omega^\Phi|^2 + |D_A\Phi|^2. \tag{3.3}$$

The latter can be reformulated as follows.

$$\begin{aligned} |\Omega_A^+ - \frac{1}{4}\omega^\Phi|^2 &= -\sum_{i<j} [\Omega_A^+(e_i, e_j) - \frac{1}{4}\omega^\Phi(e_i, e_j)]^2 = \\ &= |\Omega_A^+|^2 + \frac{1}{16}|\omega^\Phi|^2 + \frac{1}{2}\sum_{i<j} \Omega_A^+(e_i, e_j\Phi, \Phi) = \\ &= |\Omega_A^+|^2 + \frac{1}{16}|\omega^\Phi|^2 + \frac{1}{2}(\Omega_A^+\Phi, \Phi) = |\Omega_A^+|^2 + \frac{1}{8}|\Phi|^4 + \frac{1}{2}(\Omega_A^+\Phi, \Phi). \end{aligned}$$

According to the Weitzenboeck formula (3.1),

$$\int_{M^4} |D_A\Phi|^2 = \int_{M^4} |\nabla^A\Phi|^2 + \frac{\tau}{4}|\Phi|^2 - \frac{1}{2}(\Omega_A^+\Phi, \Phi).$$

Hence $\int_{M^4} |\Omega_A^+ - \frac{1}{4}\omega^\Phi|^2 + |D_A\Phi|^2 = \int_{M^4} |\Omega_A^+|^2 + |\nabla^A\Phi|^2 + \frac{\tau}{4}|\Phi|^2 + \frac{1}{8}|\Phi|^4$, the functional

$$\mathrm{SW}(\Phi, A) = \int_{M^4} |\Omega_A^+|^2 + |\nabla^A\Phi|^2 + \frac{\tau}{4}|\Phi|^2 + \frac{1}{8}|\Phi|^4 \tag{3.4}$$

is nonnegative and its zeros are exactly the solutions of (SW).

A more general approach is give by Jost/Peng/Wang in [10]. They do not study only the absolute minima of the functional $\mathrm{SW}(\Phi, A)$, i.e. solutions of (SW), but more general the stationary points of $\mathrm{SW}(\Phi, A)$ which are given by the Euler-Lagrange equation of (3.3)

$$\begin{aligned} &D_A^* D_A\Phi + \frac{1}{2}\Omega_A^+\Phi + \frac{1}{4}|\Phi|^2 \cdot \Phi = 0, \\ &\mathrm{d}^*(\Omega_A^+ - \frac{1}{4}\omega^\Phi) + \Im(D_A\Phi, e_j \cdot \Phi)e^j = 0, \end{aligned} \tag{3.5}$$

which can rewritten, according to (3.4), as

$$\begin{aligned} &\Delta_A\Phi + \frac{\tau}{4}\Phi + \frac{1}{4}|\Phi|^2 \cdot \Phi = 0, \\ &\mathrm{d}^*\Omega_A^+ + \Im(\nabla_j\Phi, \Phi)e^j = 0. \end{aligned} \tag{3.6}$$

Here Δ_A is endowed with the sign that the spectrum is nonnegative.

PROPOSITION 3.4. *Let* $\tau_0 = \min_{x \in M} \tau(x)$ *the minimum of the scalar curvature,* (Φ, A) *a solution of* (3.5) *or* (3.6). *Then* $|\Phi(x)|^2 \le \max\{-\tau_0, 0\}$.

Proof. Let $x_{\max} \in M$ be a maximum point of $|\Phi(x)|$. Then at this point

$$0 \le \Delta|\Phi|^2 = 2((\nabla^A)^*\nabla^A\Phi, \Phi) - 2(\nabla^A\Phi, \nabla^A\Phi) \le 2((\nabla^A)^*\nabla^A\Phi, \Phi) = = 2(-\frac{\tau}{4}|\Phi|^2 - \frac{1}{4}|\Phi|^4) = -\frac{\tau}{2}|\Phi|^2 - \frac{1}{2}|\Phi|^4.$$

If $|\Phi(x_{\max}| = 0$ then the assertion is established. If $|\Phi(x_{\max}| > 0$ then $|\Phi(x_{\max}|^2 < -\tau(x_{\max})$. This is possible only for $\tau \le 0$. Hence $|\Phi(x)|^2 \le -\tau_0$ for all $x \in M$. ∎

COROLLARY 3.5. *If* $\tau \ge 0$, (Φ, A) *is a solution of* (3.5) *or* (3.6) *then* $\Phi \equiv 0$. *In particular, this holds for* (Φ, A) *being a solution of* (SW).

4. The configuration space of Seiberg-Witten theory. Let $P_{\mathrm{U}(1)} \to M$ be an U(1)-principal fibre bundle. The gauge group $\hat{\mathcal{G}}$ is given by $\hat{\mathcal{G}} = \{\hat{f} : P \to \mathrm{U}(1) \mid \hat{f}(p \cdot a) = a^{-1}\hat{f}(p)a\}$. Since U(1) is Abelian $\hat{f}$ descends to $f : M \to \mathrm{U}(1) = S^1$, i.e. $\hat{\mathcal{G}} = \mathcal{G} = \mathrm{Map}(M, S^1)$, in our case $\mathcal{G} = \mathrm{Map}(M^4, S^1)$. $\mathcal{G}$ acts on $\mathcal{C}_{p_{\mathrm{U}(1)}}$ by

$$A \mapsto f^*A = A + \pi^* f^*\theta, \tag{4.1}$$

where $\theta = \frac{\mathrm{d}z}{z} = \bar{z}\mathrm{d}z : TS^1 \to u(1) = i\mathbb{R}$ is the Maurer-Cartan form. The curvature satisfies

$$\Omega_A = \mathrm{d}A, \quad \Omega_{f^*A} = \Omega_A.$$

The explicit calculation for (4.1) yields for $f : M^4 \to S^1$

$$f^*A = A + \pi^*\frac{\mathrm{d}f}{f}.$$

Let LC denote the Levi-Civita connection on P. Then

$$LC \oplus A \mapsto LC \oplus (A + \pi^*\frac{\mathrm{d}f}{f}) : T(P \times P_{\mathrm{U}(1)}) \to so(4) \oplus u(1).$$

$LC \oplus (A + \pi^*\frac{\mathrm{d}f}{f}) - LC \oplus A$ is a 1-form on $P \times P_{\mathrm{U}(1)}$ with values in $u(1)$ which equals to zero on TP. Lifting both connections to the $\mathrm{Spin}^{\mathbb{C}}(4)$ structure, we obtain for the covariant derivatives

$$\nabla^{f^*A}_{\vec{t}}\Phi - \nabla^A_{\vec{t}}\Phi = \frac{\mathrm{d}f((\vec{t}))}{f} \cdot \Phi$$

and hence for the Dirac operators $D_A, D_{f^*A} : \Omega(S) \to \Omega(S)$

$$D_{f^*A}\Phi - D_A\Phi = \frac{1}{f}\operatorname{grad} f \cdot \Phi.$$

Additionally, we define an action $\mathcal{G} = \mathrm{Map}(M^4, S^1)$ on $\Omega(S) \times \mathcal{C}_{P_{\mathrm{U}(1)}}$ by $f \cdot (\Phi, A) := (\frac{1}{f}\Phi, f^*A)$. It would be better to write $(\Phi, A) \cdot f$. We obtain

$$\begin{aligned} D_{f^*A}(\frac{1}{f}\Phi) &= D_A(\frac{1}{f}\Phi) + \frac{1}{f}\operatorname{grad} f \cdot (\frac{1}{f}\Phi) = \\ &= \frac{1}{f}D_A\Phi - \frac{1}{f^2}\operatorname{grad} f \cdot \Phi + \frac{1}{f^2}\operatorname{grad} f \cdot \Phi = \frac{1}{f}D_A\Phi. \end{aligned} \tag{4.2}$$

PROPOSITION 4.1. *If* (Φ, A) *is a solution of* (SW) *then the same holds for* $(\Phi, A) \cdot f$.

Proof. For the first equation this is just (4.2). The assertion for the second equation follows from $\Omega_{f^*A} = \Omega_A$ and $\omega^{\frac{1}{f}\Phi} = \omega^{\Phi}$. The latter follows from $|f| \equiv 1$. ∎

As usual, the configuration space of (SW) is given by $(\Omega(S) \times \mathcal{C}_{P_{\mathrm{U}(1)}})/\mathcal{G}$. At the first glance, $(\Omega(S) \times \mathcal{C}_{P_{\mathrm{U}(1)}})/\mathcal{G}$ is an absolutely senseless object, no topology is defined, the properties of the action are totally unclear. One has to define suitable topologies, completions and to establish good properties of the action. Then it is possible to show that the completed configuration space has the structure of a stratified space or even of a manifold. If M^4 is compact this causes no principal troubles. For gauge theory this has been performed by Kondracki/Rogulski. For Seiberg-Witten theory part of this has been done partially by Jost/Peng/Wang in [10]. They show that $\mathcal{G}$ is after completion a Hilbert-Lie group and that the Seiberg-Witten functional satisfies a Palais-Smale condition. For M^4 open one has to develop a framework along [2], [3], [5]. This shall be done in a forthcoming paper.

5. The moduli space of Seiberg-Witten theory. As usual, the moduli space of Seiberg-Witten theory is the space of all solutions of (SW) factorized by the gauge group. At irreducible solutions of (SW) and for generic metrics g it is a finite-dimensional manifold. Its dimension can be calculated by means of an elliptic complex which arises from linearization and projection transversal to the orbits. In comparison to Donaldson's theory, the moduli space of (SW) has an important convenient feature, it is compact. This shall now be established. For reasons of brevity and technical simplicity, we omit the whole Sobolev calculus. On compact manifolds this is absolutely standard.

Assume $(M^4; g)$ closed, oriented with $\mathrm{Spin}^{\mathbb{C}}(4)$ structure $c \in H^2(M^4, \mathbb{Z})$, $L = P_{\mathrm{U}(1)} \times \mathbb{C}$, $c_1(L) = c$, L endowed with a Hermitian scalar product,

$$\mathcal{M}_L = \{(\Phi, A) \in \Omega(S^+) \times \mathcal{C}_{P_{\mathrm{U}(1)}} \mid D_A\Phi = 0, \Omega_A^+ = \frac{1}{4}\omega^{\Phi}\}/\mathcal{G}$$

is the moduli space of (SW). Here we tacitly assume $\mathcal{M}_L$ endowed with the L_2-topology coming from $\Omega(S^+) \times \mathcal{C}_{P_{\mathrm{U}(1)}}$.

THEOREM 5.1. *$\mathcal{M}_L$ is compact.*

Proof. Consider

$$F(L) = \{\omega \in \Omega^2(M^4) \mid \mathrm{d}\omega = 0, [\frac{i}{2\pi}\omega] = c_1(L)\} = \frac{2\pi}{i} \cdot c_1(L).$$

Endow $F(L)$ with the induced L_2-topology. $F(L)$ has a unique harmonic representative ω_h. We obtain a map $P : \mathcal{M}_L \to F(L)$, $P[\Phi, A] := \Omega_A$.

LEMMA 5.2. *$P(\mathcal{M}_L) \subset F(L)$ is a compact subset of $F(L)$.*

Proof. Assume $D_A\Phi = 0$, $\Omega_A^+ = \frac{1}{4}\omega^{\Phi}$. $\Omega_A = \Omega_A^+ + \Omega_A^-$ has a representation as $\Omega_A = \omega_h + \mathrm{d}\eta$, where $\eta \perp_{L_2} \mathcal{H}^1(M) \oplus \mathrm{im}(\mathrm{d}^0 : \Omega^0 \to \Omega^1)$, $\mathcal{H}^1 =$ harmonic 1-forms.

$$\Delta|\Phi|^2 = 2((\nabla^A)^*\nabla^A\Phi, \Phi) - 2(\nabla^A\Phi, \Phi) = -\frac{\tau}{2}|\Phi|^2 - \frac{1}{2}|\Phi|^4 - 2(\nabla^A\Phi, \nabla^A\Phi)$$

and

$$\int\limits_{M^4} \Delta|\Phi|^2 = 0$$

imply

$$2|\nabla^A\Phi|^2_{L_2} = \int\limits_{M^4} -\frac{\tau}{4}|\Phi|^2 - \frac{1}{2}|\Phi|^4 \leq \int\limits_{\tau<0} -\frac{\tau}{2}|\Phi|^2$$
$$\leq \int\limits_{\tau<0} \left(-\frac{\tau}{2}\right)\cdot(-\tau_{\min}) = C_1 = C_1(\tau),$$

hence $|\nabla^A\Phi|^2_{L_2} \leq C_1$.

Now $\Omega_A^+ = \frac{1+*}{2}\Omega_A$, hence

$$\delta\Omega_A^+ = *\mathrm{d} * \Omega_A^+ = *\mathrm{d} * \left(\frac{1+*}{2}\right)\Omega_A = \frac{1}{2} * \mathrm{d} * \Omega_A = \frac{1}{2}\delta\Omega_A,$$

$$\delta\Omega_A = 2\delta\Omega_A^+ = \frac{1}{2}\delta\omega^\Phi. \tag{5.1}$$

We want to estimate $|\delta\Omega_A|_{L_2}$ and conclude from this $|\Delta\eta|^2_{L_2} \leq C_2$.

By definition, assuming $\nabla_{e_i}e_j = \nabla_{e_i}\vec{t} = 0$ at the point under consideration,

$$(\delta\omega^\Phi)(\vec{t}) = \sum_{i=1}^4(\nabla_{e_i}\omega^\Phi)(e_i,\vec{t}) = \sum_i \nabla_{e_i}(\omega^\Phi(e_i,\vec{t})) =$$
$$= \sum_i\{(e_i\cdot\vec{t}\cdot\nabla_{e_i}\Phi,\Phi) + (e_i\cdot\vec{t}\Phi,\nabla_{e_i}\Phi) + (e_i,\vec{t})\nabla_{e_i}|\Phi|^2\}.$$

Using $\left|\mathrm{d}|\Phi|^2\right| \leq 2|\Phi|\cdot|\nabla\Phi|$ and the Schwarz inequality several times, it is easy to conclude

$$|(\delta\Omega_A)(\vec{t})| \leq C\cdot|\vec{t}|\cdot|\Phi|\cdot|\nabla^A\Phi|,$$
$$|\delta\Omega_A|^2_{L_2} \leq C_2^*|\Phi|^2_{L_2}\cdot|\nabla\Phi|^2_{L_2},$$

together with the C^∞-bound for Φ coming from Proposition 3.4

$$|\delta\Omega_A|^2_{L_2} \leq C_2. \tag{5.2}$$

(5.2) and $\Omega_A = \omega_h + \mathrm{d}\eta$ imply

$$|\delta\mathrm{d}\eta|^2_{L_2} = |\Delta\eta|^2_{L_2} \leq C_2,$$

since $\eta \perp \mathcal{H}^1(M^4)$, $\eta \perp \mathrm{im}(\mathrm{d}^0 : \Omega \to \Omega^1)$. M^4 is closed, the spectrum $\sigma(\Delta)$ purely discrete, hence

$$|\eta|^2_{L_2} \leq C_3^*|\nabla\eta|^2_{L_2}$$

and therefore

$$|\eta|^2_{L_2} \leq C_3,\ |\nabla\eta|^2_{L_2} \leq C_2.$$

We have shown that the map

$$P : \mathcal{M}_L \to F(L)$$
$$[\Phi, A] \mapsto \omega_h + \mathrm{d}\eta, \quad \eta = \eta(A),$$

has bounded image in the Sobolev space of second order and the image in $L_2(\Lambda^2(M^4))$ is therefore compact.

Lemma 5.3. *Consider $P_1 : \mathcal{M}_L \to \mathcal{C}_{P_{U(1)}}/\mathcal{G}$, $P_1[\Phi, A] = [A]$. Then $P_1(\mathcal{M}_L)$ is compact.*

P r o o f. The map $P_2 : \mathcal{C}_{P_{U(1)}}/G \to F(L)$, $[A] \mapsto \Omega_A$, is a fibering with the compact fibre $P_{ic}(M^4) = H^1(M^4;\mathbb{R})/H^1(M^4;\mathbb{Z}) \cong T^{b_1(M^4)}$.

The diagram

$$\begin{array}{ccc} \mathcal{M}_L & \xrightarrow{P_1} & \mathcal{C}_{P_{U(1)}}/\mathcal{G} \\ {\scriptstyle P} \searrow & & \swarrow {\scriptstyle P_2} \\ & F(L) & \end{array}$$

commutes. Hence $P_2|_{\operatorname{im} P_1} : \operatorname{im} P_1 \to F(L)$ has compact image and compact fibre, $\operatorname{im} P_1$ is compact. Consider finally for fixed $[A]$ the set of all Φ such that $[\Phi, A] \in \mathcal{M}_L$. They are, according to Proposition 3.4, contained in a bounded ball in a finite-dimensional vector space, contained in a Sobolev space of arbitrary high order. It is easy to see that this set is closed. We obtain together with Lemma 5.2 that $\mathcal{M}_L$ is compact. ■

6. The Seiberg-Witten invariant. The Seiberg-Witten equations

$$D_A\Phi = 0, \quad \Omega_A^+ = \frac{1}{4}\omega^\Phi$$

define an operator

$$\begin{aligned} \Omega(S^+) \times \mathcal{C}_{P_{\mathrm{U}(1)}} &\to \Omega(S^-) \times \Omega^2_+(i\mathbb{R}) \\ (\Phi, A) &\mapsto (D_A\Phi, \Omega_A^+ - \frac{1}{4}\omega^\Phi). \end{aligned} \tag{6.1}$$

As usual, one tries to calculate the dimension of the tangent space to $\mathcal{M}_L$ at irreducible solutions ($\Phi \not\equiv 0$) by means of the linearization of (6.1), projection transversal to the orbits under $\mathcal{G}$ and an elliptic complex. Consider the linearization at the point (Φ, A). This yields an operator

$$P^1_{(\Phi,A)} : \Omega(S^+) \oplus \Omega^1 \to \Omega(S^-) \oplus \Omega^2.$$

We obtain with $\Psi \in \Omega(S^+)$, $\eta \in \Omega^1$, $A_t = A + t\eta$, $\Phi_t = \Phi + t\Psi$, $\Omega_{A_t} = \mathrm{d}A + t\mathrm{d}\eta$, $\frac{\mathrm{d}}{\mathrm{d}t}(\Omega^+_{A_t})|_{t=0} = (\mathrm{d}\eta)^+$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\omega^{\Phi_t}|_{t=0}(X,Y) = \\ &\quad = (XY\frac{\mathrm{d}\Phi_t}{\mathrm{d}t}|_{t=0}, \Phi) + (XY\Phi, \frac{\mathrm{d}\Phi_t}{\mathrm{d}t}|_{t=0}) + (X,Y)(\Psi,\Phi) + (X,Y)(\Phi,\Psi) = \\ &\quad = (XY\Psi,\Phi) + (XY\Phi,\Psi) + (X,Y)(\Psi,\Phi) + (X,Y)(\Phi,\Psi) = \\ &\quad = (XY\Psi,\Phi) + (\Phi, YX\Psi) + (X,Y)(\Psi,\Phi) + (X,Y)(\Phi,\Psi) = \\ &\quad = (XY\Psi,\Phi) + \overline{(YX\Phi,\Psi)} + (X,Y)(\Psi,\Phi) + (X,Y)(\Phi,\Psi) = \\ &\quad = (XY\Psi,\Phi) + \overline{(\{-XY - 2(X,Y)\}\Psi,\Phi)} + (X,Y)(\Psi,\Phi) + (X,Y)(\Phi,\Psi) = \\ &\quad = (XY\Psi,\Phi) - \overline{(XY\Psi,\Phi)} - 2(XY)\overline{(\Psi,\Phi)} + (X,Y)(\Psi,\Phi) + (X,Y)(\Phi,\Psi) = \\ &\quad = (XY\Psi,\Phi) - \overline{(XY\Psi,\Phi)} + (XY)\{(\Psi,\Phi) - \overline{(\Psi,\Phi)}\} \equiv \frac{1}{4}\omega^{\Phi,\Psi}. \end{aligned}$$

Hence the linearization of $\Omega_A^+ - \frac{1}{4}\omega^\Phi$ equals to

$$(\mathrm{d}\eta)^+ - \frac{1}{4}\omega^{\Phi,\Psi}. \tag{6.2}$$

Still easier,

$$D_{A_t}\Phi_t = D_{A_t}\Phi + tD_{A_t}\Psi,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(D_{A_t}\Phi_t)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(D_{A_t}\Phi)|_{t=0} + D_A\Psi,$$
$$D_{A_t}\Phi = \sum_i e_i \cdot \nabla^{A_t}_{e_i}\Phi = \sum_i e_i\left(\nabla^A_{e_i}\Phi + t\eta(e_i)\Phi\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(D_{A_t}\Phi_t\right)|_{t=0} = \eta\Phi.$$

$D_A\Phi$ linearizes to

$$\eta\Phi + D_A\Psi. \tag{6.3}$$

Therefore $P^1 : \Omega(S^+)\oplus\Omega^1 \to \Omega(S^-)\oplus\Omega^2_+$ is given by

$$P^1_{\Phi,A}(\Psi,\eta) = (\eta\Phi + D_A\Psi, (\mathrm{d}\eta)^+ - \frac{1}{4}\omega^{\Phi,\Psi}). \tag{6.4}$$

Finally, we calculate the tangent space to an orbit through (Φ, A). Let $f_t : M^4 \to S^1$ be a family of gauge transformations, $f_0 \equiv 1$, $\frac{\mathrm{d}}{\mathrm{d}t}f_t|_{t=0} = h$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{f_t}\Phi\right)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\overline{f_t}\Phi)|_{t=0} = \overline{h}\Phi = -h\Phi,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(f_t^*A - A)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}f_t}{f_t}\right)|_{t=0} = \mathrm{d}h.$$

Hence

$$\begin{aligned} &P^0_{(\Phi,A)} : \Omega^0 \to \Omega(S^+)\oplus\Omega^1 \\ &P^0_{(\Phi,A)}(h) = (-h\Phi, \mathrm{d}h) \end{aligned} \tag{6.5}$$

projects to the tangent vectors of the orbit.

LEMMA 6.1. *For* $[\Phi, A] \in \mathcal{M}_L$, $P^1_{(\Phi,A)} \circ P^0_{(\Phi,A)} = 0$, *i.e.* $P^1_{(\Phi,A)}$ *really acts transversal to the orbits.*

Proof. Let $\Psi = -h\Phi$, $\eta = \mathrm{d}h$, h purely imaginary. Then $\eta\Phi + D_A\Psi = \mathrm{d}h\Phi - D_A(h\Phi) = \mathrm{d}h\Phi - \mathrm{d}h\Phi + hD_A\Phi = hD_A\Phi = 0$, $\mathrm{d}\eta = 0$, and $\omega^{\Phi,\Psi}(X,Y) = (XY(-h)\Phi,\Phi) - \overline{h}\,\overline{(XY\Phi,\Phi)} + (X,Y)[(-h)(\Phi,\Phi) - \overline{h}(\Phi,\Phi)] = 0$. ■

Consider now the elliptic complex defined by (6.4), (6.5), (6.1)

$$\Omega^0 \xrightarrow{P^0_{(\Phi,A)}} \Omega(S^+)\oplus\Omega^1 \xrightarrow{P^1_{(\Phi,A)}} \Omega(S^-)\oplus\Omega^2_+. \tag{$\mathfrak{C}_1$}$$

We want to calculate its index over $\mathbb{R}$. Since the index can be calculated using the leading symbols only the index of $(\mathfrak{C}_1)$ equals to the index of

$$\Omega^0 \xrightarrow{(0,\mathrm{d})} \Omega(S^+)\oplus\Omega^1 \xrightarrow{(D_A, pr_+\circ\mathrm{d})} \Omega(S^-)\oplus\Omega^2_+. \tag{$\mathfrak{C}_2$}$$

$(\mathfrak{C}_2)$ is the sum of

$$\Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{pr_+\circ\mathrm{d}} \Omega^2_+ \tag{$\mathfrak{C}_3$}$$

and

$$0 \to \Omega(S^+) \xrightarrow{D_A} \Omega(S^-). \tag{$\mathfrak{C}_4$}$$

We obtain

$$\operatorname{ind}_{\mathbb{R}}(\mathfrak{C}_1) = \operatorname{ind}_{\mathbb{R}}(\mathfrak{C}_3) - 2\ \operatorname{ind}_{\mathbb{C}} D_A^+. \tag{6.6}$$

It is well known that

$$\operatorname{ind}_{\mathbb{R}}(\mathfrak{C}_3) = \frac{1}{2}\chi\left(M^4\right) + \frac{1}{2}\sigma\left(M^4\right). \tag{6.7}$$

LEMMA 6.2.

$$\operatorname{ind}_{\mathbb{C}}(D_A^+) = \frac{1}{8}c^2 - \frac{1}{8}\sigma. \tag{6.8}$$

Proof. For M^{2k}, $D_A^+ : \Omega(S^+) \to \Omega(S^-)$, $c = c_1(L) \in H^2\left(M^{2k}, \mathbb{R}\right)$, $r_* c = w_2$ we have

$$\operatorname{ind} D_A^+ = \left\langle e^{\frac{1}{2}c}\hat{A}\left(M^{2k}\right), [M]\right\rangle.$$

If $\dim M = 4$ then $\hat{A}\left(M^4\right) = 1 - \frac{1}{24}p_1\left(M^4\right)$,

$$\left\langle e^{\frac{1}{2}c}\hat{A}\left(M^4\right), \left[M^4\right]\right\rangle = \left\langle \left(1 + \frac{1}{2}c + \frac{1}{8}c^2\right)\left(1 - \frac{1}{24}p_1\right), [M]\right\rangle =$$
$$= \frac{1}{8}c^2 - \frac{1}{24}p_1 = \frac{1}{8}c^2 - \frac{1}{8}\sigma \quad \text{since } \sigma = \frac{1}{3}\langle p_1, [M]\rangle. \ \blacksquare$$

Here and in the sequel we denote characteristic classes and numbers by the same symbol since the meaning is clear from the context. (6.6)-(6.8) imply

THEOREM 6.3. $\operatorname{ind}(\mathfrak{C}_1) = \frac{1}{4}(2\chi + 3\sigma) - \frac{1}{4}c^2$.

By definition, $\operatorname{ind}_{\mathbb{R}}(\mathfrak{C}_1) = \sum\limits_{i=0}^{2}(-1)^i \dim_{\mathbb{R}} H^i(\mathfrak{C}_1)$, and $H^i(\mathfrak{C}_1) = \mathcal{H}^i :=$ kernel of the i-th Laplace operator of $\mathfrak{C}_1$. As usual, at "good" points of $[\Phi, A] \in \mathcal{M}_L$

$$\dim \mathcal{M}_L = \dim T\mathcal{M}_L = \dim \mathcal{H}^1 = -\operatorname{ind}(\mathfrak{C}_1)$$

if $\mathcal{H}^0 = \mathcal{H}^2 = 0$. Therefore we have to check in which cases $\mathcal{H}^0 = \mathcal{H}^2 = 0$.

DEFINITION. We call a solution (Φ, A) of (SW) *reducible* if $\Phi \equiv 0$. Then $\Omega_A^+ \equiv 0$. In the other case (Φ, A) is called *irreducible*. If (Φ, A) is irreducible then

$$H^0(\mathfrak{C}_1) \equiv \ker P^0_{(\Phi,A)} = 0$$

since

$$P^0_{(\Phi,A)}(h) = (-h\Phi, \mathrm{d}h).$$

PROPOSITION 6.4. *For a generic set of metrics* $g \in \operatorname{Met}\left(M^4\right)$

$$H^2(\mathfrak{C}_1) = 0$$

holds at any irreducible solution (Φ, A) *of* (SW).

Proof. To perform the proof in detail, we had to introduce a suitable Sobolev topology in the space $\operatorname{Met}\left(M^4\right)$ of metrics and to complete. This is done in [4] for open manifolds which includes closed manifolds as a very simple case. We omit here the details. It comes out that the tangent space to $\operatorname{Met}\left(M^4\right)$ is given by a Sobolev space of all symmetric twofold covariant tensors h. A small variation of a metric g can be assumed to be of the form $g_t = g + th$. We consider the Seiberg-Witten equations additionally depending on the metric g,

$$\text{(SW)}(g) \qquad D^g_A\Phi = 0, \quad \Omega^{+(g)}_A = \frac{1}{4}\omega^\Phi.$$

Then the linearization of (SW)(g) defines an operator

$$P_{(\Phi,A,g)} : \Omega(S^+) \oplus \Omega^1 \oplus \Omega(S^2T^*M) \to \Omega(S^-) \oplus \Omega^2,$$

$$P_{(\Phi,A,g)}(\Psi,\eta,h) = \\ = \Big(\eta\Phi + D^g_A\Psi + \frac{\mathrm{d}}{\mathrm{d}t}D^{g+th}_A\Psi|_{t=0}, \mathrm{d}\eta^{+(g)} - \frac{1}{4}\omega^{\Phi,\Psi} + \frac{\mathrm{d}}{\mathrm{d}t}\Omega^{+(g+th)}_A|_{t=0}\Big).$$

It is sufficient to consider variations of g only transversal to the action of the diffeomorphism group Diff $\big(M^4\big)$, i.e. we can assume $\delta^g(h) = 0$. We are done if we can show $\operatorname{coker} P = 0$ or that any pair (χ,ζ) which is orthogonal to $\operatorname{im} P$ equals to zero. The variation of g enters into $\nabla^g_A \mapsto \nabla^{g+th}_A$, $\bullet_g \mapsto \bullet_{g+th}$, $e_i(g) = e_i(g+th)$, $*_g \mapsto *_{g+th}$. A straightforward but lengthy calculation leads to the conditions

$$\langle \mathrm{d}\operatorname{tr}(h)\cdot\Phi, \chi\rangle = 0$$

and

$$\langle \varrho(h), \zeta\rangle = 0,$$

where $\varrho(h)$ describes to variation of the $*$-operator, $\langle\ ,\ \rangle = \int(\ ,\ )$. We conclude from the second equation immediately $\varrho = 0$. As nontrivial solution of $D^g_A\Phi = 0$, Φ cannot vanish on a dense set which yields $\chi = 0$. ∎

Definition. We define the *virtual dimension* v-$\dim_{\mathbb{R}}\mathcal{M}_L$ by

$$\text{v-}\dim_{\mathbb{R}}\mathcal{M}_L := \frac{1}{4}[c^2 - (2\chi + 3\sigma)].$$

Corollary 6.5. *Let $c \in H^2(M^4;\mathbb{Z})$ with $r_*c = w_2$. Then*

$$\text{v-}\dim_{\mathbb{R}}\mathcal{M}_L = \frac{1}{4}[c^2 - 2(\chi+3\sigma)] \equiv -1 + b_1 - b_2^+ \bmod 4.$$

Proof. $\frac{1}{4}[c^2-(2\chi+3\sigma)] = -\frac{1}{2}(\chi+\sigma)+2\dim_{\mathbb{C}}\ker D^+_A = -\frac{1}{2}(\chi+\sigma)+2\dim_{\mathbb{C}}\ker D^+_A \equiv -\frac{1}{2}(\chi+\sigma) = \frac{1}{2}(1-b_1+b_2^++b_2^--b_3+1+b_2^+-b_2^-) = 1+b_1-b_2^+$ since $\mathrm{Spin}(4) = \mathrm{Sp}(1)$, $\dim_{\mathbb{C}}(D^+_A) \equiv 0 \bmod 2$. ∎

Definition. Let $L \to M^4$ be a complex line bundle, $c_1(L) \in H^2_{\mathrm{dR}}(M^4;\mathbb{R})$ the first Chern class. A metric g on M^4 is called *L-good* if $c_1(L)$ has no harmonic anti-self-dual representative, i.e. there does not exist an $\omega \in \Omega^2$ s.t. $\Delta\omega = 0$, $*\omega = -\omega$, $[\omega] = c_1(L)$.

For given g let $*_g : H^2_{\mathrm{dR}}(M^4) \to H^2_{\mathrm{dR}}(M^4)$ be the Hodge $*$-operator and $E^\pm(g)$ be the ±1 eigenspace of $*$, $\dim E^\pm(g) = b_2^\pm$. There exist 4 cases.

Case 1. $b_2^+ = 0$, $c_1(L) \neq 0$. Then there does not exist an L-good metric g.

Case 2. $b_2^+ = 1$. Then H^2_{dR} is a pseudo-Euclidean space of index $(1, b_2^-)$. If $b_2^- = 0$ then every metric g is good. If $b_2^- > 0$ then $(H^2_{\mathrm{dR}}, \wedge) \cong (\mathbb{R}^{b_2}, z^2 - x_1^2 - \cdots - x^2_{b_2-1})$. It follows from transversality reasons that it is possible to choose an L-good metric g. But it is not always possible to connect such two metrics by an L-good arc. Hence if $b_2^+ = 1$, $b_2^- > 0$

and $c_1(L) < 0$ then the space of L-good metrics is not connected. If $c_1(L) \wedge c_1(L) > 0$ then any metric is L-good.

Case 3 and 4. $b_2^+ \geq 2$. Then for $c_1(L) \wedge c_1(L) > 0$ each metric is L-good and for $c_1(L) \wedge c_1(L) < 0$ the space of L-good metrics is connected.

PROPOSITION 6.6. *Let (M, g) be closed, oriented, c — a* $\mathrm{Spin}^{\mathbb{C}}(4)$ *structure,* $L = P_{U_1} \times_{U_1} \mathbb{C}$*, g be L-good. Then there do not exist reducible solutions of* (SW).

Proof. $D_A\Phi = 0$, $\Omega_A^+ = \frac{1}{4}\omega^\Phi$, $\Phi \equiv 0$ imply $\Omega_A^+ = 0$ and $*\Omega_A = -\Omega_A$, $\Delta\Omega_A = 0$ since $\mathrm{d}\Omega_A = 0$, and we obtain from $[\frac{i}{2\varpi}\Omega_A] = c_1(L)$, a contradiction. ∎

We now start our definition of the Seiberg-Witten invariant.

Case $b_2^+(M^4) \geq 2$. Then for a generic metric g, g is L-good, $\mathcal{H}^2(\mathfrak{C}_1) = 0$, and the space of such metrics is connected, $\mathcal{M}_L(g) = \emptyset$ or is a smooth closed manifold of dimension

$$\dim \mathcal{M}_L(g) = \frac{1}{4}[c^2 - (2\chi + 3\sigma)].$$

The bordism class $[\mathcal{M}_L(g)]$ of $\mathcal{M}_L(g)$ in the unoriented bordism ring $\mathcal{N}_* = \Omega_*^0$ is independent of g. We define

$$\mathrm{SW}(M^4, c) := [\mathcal{M}_L(g)] \in \mathcal{N}_*.$$

Case $b_2^+(M^4) = 1$, $c_1(L) \wedge c_1(L) > 0$. Once again any metric g is L-good and generically $\mathcal{H}^2 = 0$, $[\mathcal{M}_L(g)] \in \mathcal{N}_*$ is uniquely determined. We define

$$\mathrm{SW}(M^4, c) := [\mathcal{M}_L(g)] \in \mathcal{N}_*.$$

Case $b_2^+(M^4) = 1$, $c_1(L) \wedge c_1(L) < 0$. A generic metric is L-good and $\mathcal{H}^2 = 0$. We define after a choice of a component in the space of L-good metrics

$$\mathrm{SW}(M^4, c) := [\mathcal{M}_L(g)] \in \mathcal{N}_*.$$

Case $b_2^+(M^4) = 0$. Then generically $\mathcal{H}^2 = 0$, but there exist reducible solutions. If $\Phi \equiv 0$ then $\Omega_A^+ = 0$ and Ω_A is the single g-harmonic form in $c_1(L)$. According to Weyl,

$$\{(\Phi \equiv 0, A) \mid [\frac{i}{2\pi}\Omega_A] = c_1(L)\}/\mathcal{G}$$

is diffeomorphic to $P_{ic}(M^4) = H^2(M^4; \mathbb{R})/H^2(M^4; \mathbb{Z}) \cong T^{b_1(M^4)}$. Hence $\mathcal{M}_L(g)$ is empty or a compact manifold with singularity set $P_{ic}(M^4) = T^{b_1(M^4)}$. Considering bordism with singularities, one could define $\mathrm{SW}(M^4, c)$ as the corresponding bordism class.

Remarks.

1. The "rough" definitions above can step by step rapidly refined.
2. Let $x_0 \in M^4$ be fixed and consider $\mathcal{G}_0 \subset \mathcal{G}_{P_{U(1)}}$,

$$\mathcal{G}_0 := \{f : M^4 \to S^1 \mid f(x_0) = 1\}.$$

Then

$$\mathcal{M}_L^0(g) := \{(\Phi, A) \mid D_A\Phi = 0, \Omega_A^+ = \frac{1}{4}\omega^\Phi\}/\mathcal{G}_0$$

is a $\mathcal{G}/\mathcal{G}_0 = S^1$-principal fibre bundle over $\mathcal{M}_L(g)$. Hence $\mathcal{M}_L(g)$ has a canonical cohomology class $c \in H^2(\mathcal{M}_L(g); \mathbb{Z})$. It is possible to understand $\mathrm{SW}(M^4, c)$ as

$$([\mathcal{M}_L(g), c]).$$

If $\mathcal{M}_L(g)$ can be given an orientation (see below) then one can also define $\mathrm{SW}(M^4, c)$ as the pairing between the maximal cup product of c and fundamental cycle of $\mathcal{M}_L(g)$.

3. In many cases $\mathcal{M}_L(g)$ can be oriented.

Roughly spoken, we can identify $T_{[\Phi,A]}\mathcal{M}_L(g)$ for a generic metric with

$$\{\Psi \in \Omega(S^+) \mid D_A\Psi = 0\} \oplus \{\eta \in \Omega^1 \mid \delta\eta = 0,\, (\mathrm{d}\eta)^+ = 0\}.$$

$\{\Psi \in \Omega(S^+) \mid D_A\Psi = 0\}$ is a complex vector space and therefore has a canonical orientation. The determinant of the second space is given by that of the complex

$$\Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}^+} \Omega^2_+$$

or the operator

$$\delta \oplus \mathrm{d}^+ : \Omega^1 \to \Omega^0 \oplus \Omega^2_+.$$

We have

$$\det(\delta \oplus \mathrm{d}^+) = \det\ker(\delta \oplus \mathrm{d}^+) \otimes \det\operatorname{coker}(\delta \oplus \mathrm{d}^+) = \det H^1 \oplus \det H^2_+.$$

Hence an orientation of $H^1(M^4;\mathbb{R}) \oplus H^2_+(M^4;\mathbb{R})$ induces an orientation of $\mathcal{M}_L(g)$.

Given M^4 closed, oriented, $c \in H^2(M^4;\mathbb{Z})$ a $\mathrm{Spin}^{\mathbb{C}}(4)$ structure and an orientation of $H^1(M^4;\mathbb{R}) \oplus H^2_+(M^4;\mathbb{R})$, then in the cases $b_2^+(M^4) \geq 2$ or $b_2^+(M^4) = 1$ and $c^2 < 0$ the Seiberg-Witten invariant is defined as an element of the oriented bordism ring Ω_*^{SO},

$$\mathrm{SW}(M^4, c) \in \Omega_*^{SO}.$$

For $\text{v-dim}_{\mathbb{R}}(\mathcal{M}_L(g)) = d > 0$, the other possible definition is, as indicated above,

$$\mathrm{SW}(M^4, c) = \int\limits_{\mathcal{M}_L(g)} c^{d/2}.$$

7. Vanishing theorems. We show that $\mathcal{M}_L(g) = \emptyset$, $\mathrm{SW}(M^4, c) = 0$ for almost all $\mathrm{Spin}^{\mathbb{C}}(4)$ structures $c \in H^2(M^4;\mathbb{Z})$. Assume $b_2^+(M^4) \geq 1$ and $\tau(g) > 0$. By an arbitrary small perturbation, g is L-good. Then

$$\begin{aligned}\mathcal{M}_L(g) = \{(\phi \equiv 0, A) \mid \Omega_a^+ = 0\}/\mathcal{G} = \{A \in \mathcal{C}_{P_{\mathrm{U}(1)}} \mid \Omega_A = 0\}/\mathcal{G} = \\ = \text{space of all flat connections.}\end{aligned}$$

Proposition 7.1. *Assume (M^4, g) with $\tau(g) > 0$ and $c_1(L) \in H^2(M^4;\mathbb{Z})$ not a torsion element. Then $\mathcal{M}_L(g) = \emptyset$ and*

$$\mathrm{SW}(M^4, c) = 0.$$

Proof. $[0, A] \in \mathcal{M}_L(g)$ would imply $c_1(L) = \left[\frac{i}{2\pi}\Omega_A\right] = 0$ which contradicts $c_1(L) \neq 0$ in $H^2(M;\mathbb{R})$. ∎

Theorem 7.2. *Assume (M^4, g) closed, oriented. Then for almost all* $\mathrm{Spin}^{\mathbb{C}}(4)$ *structures $c \in H^2(M;\mathbb{Z})$ $\mathcal{M}_L(g) = \emptyset$.*

Proof. We are done if we could show that only finitely many $c_1(L)$ with $\mathcal{M}_L(g) \neq \emptyset$ are possible. This would follow if we could show that there exists a constant C independent of A such that for all irreducible solutions (Φ, A) of (SW)

$$\int_{M^4} |\Omega_A^+|^2, \qquad \int_{M^4} |\Omega_A^-|^2 \leq C. \tag{7.1}$$

We know that for any irreducible solution (Φ, A) of (SW)

$$|\Phi(x)|^2 \leq -\min_x \tau(g,x).$$

The Weitzenboeck formula

$$\Delta|\Phi|^2 = -\frac{\tau}{2}|\Phi|^2 - \frac{1}{2}|\Phi|^4 - 2|\nabla^A\Phi|^2$$

then yields

$$\int_{M^4} |\Phi|^4 \leq \int_{M^4} (-\tau)|\Phi|^2 \leq \tau_{\min}^2 \cdot \operatorname{vol}(M^4) = C_1.$$

Hence, according to (3.3),

$$\int_{M^4} |\Omega_A^+|^2 = \frac{1}{16}\int_{M^4} |\omega^\Phi|^2 \leq \frac{1}{8}\cdot \tau_{\min}^2 \cdot \operatorname{vol}(M^4). \tag{7.2}$$

On the other hand,

$$c_1(L)^2 = \frac{1}{4\pi^2}\int |\Omega_A^+|^2 - |\Omega_A^-|^2.$$

If $\mathcal{M}_L(g) \neq \emptyset$ then $c_1(L)^2 - (2\chi + 3\sigma) \geq 0$,

$$2\chi + 3\sigma \leq c_1(L)^2 = \frac{1}{4\pi^2}\int_{M^4} |\Omega_A^+|^2 - |\Omega_A^-|^2,$$

$$\int_{M^4} |\Omega_A^-|^2 \leq \int_{M^4} |\Omega_A^+|^2 - 4\pi^2(2\chi+3\sigma) \leq C_1 - 4\pi^2(2\chi + 3\sigma) = C_2. \tag{7.3}$$

(7.3) and (7.2) imply (7.1). The assertion now follows from Proposition 7.1 and (7.1). ∎

8. The case $\dim \mathcal{M}_L = 0$**.** If $\text{v-dim}_{\mathbb{R}}\, \mathcal{M}_L(g) = 0$ then $\mathrm{SW}(M^4, c, g) \in \mathcal{N}_*$ is a numerical invariant, i.e. if $c^2 - (2\chi + 3\sigma) = 0$ then we define the $\mathbb{Z}_2$-invariant

$$n_L(g) := \#\mathcal{M}_L(g) \bmod 2.$$

For $b_2^+ \geq 2$ $n_L(g) \in \mathbb{Z}_2$ is generically uniquely determined, i.e. independent of g. For $b_2^+ = 1$ one has still to fix a component in the space of L-good metrics. If additionally $H^1(M^4;\mathbb{R}) \oplus H^2_+(M^4;\mathbb{R})$ is endowed with an orientation then we have a numerical invariant

$$\mathrm{SW}(L,g) = n_L^0(g) = \text{sum of signed points} \in \mathbb{Z}.$$

This invariant does not exist for any manifold M^4. Necessary is the existence of an element $c \in H^2(M^4;\mathbb{Z})$ such that

$$c^2 = 2\chi + 3\sigma \text{ and } r_* c = w_2 \in H^2(M^4;\mathbb{Z}_2),$$

i.e. in particular that the number

$$2\chi + 3\sigma = \langle (2E + p_1), [M^4]\rangle$$

($E =$ Euler class) lies in the image of the quadratic intersection form

$$H^2(M^4;\mathbb{R}) \ni x \to x^2[M].$$

Assume TM^4 is a complex vector bundle of rank 2 and has Chern classes $c_1 \in H^2(M^4;\mathbb{R})$, $c_2 \in H^4(M^4;\mathbb{R})$. There holds

$$E = c_2, \quad r_* c_1 = w_2, \quad p_1 = c_1^2 - 2c_2.$$

This implies

$$2E + p_1 = c_1^2.$$

The converse statement is also true.

PROPOSITION 8.1. *Let M^4 be closed, oriented. There exists an almost complex structure with the same orientation if and only if there exists $c \in H^2(M^4;\mathbb{R})$ such that $c^2 = 2E + p_1$.*

For the proof we refer to [7]. ■

9. The Seiberg-Witten invariant for Kähler manifolds. Kähler and symplectic manifolds belong to the basic classes for the application of Seiberg-Witten theory to 4-manifolds. In this section we are concerned with Kähler manifolds. Consider the inclusion $j : \mathrm{U}(n) \to \mathrm{SO}(2n)$ and $j \times \det : \mathrm{U}(n) \to \mathrm{SO}(2n) \times S^1$. Looking at the π_1-level immediately yields a lifting l,

$$\begin{array}{ccc} & & \mathrm{Spin}^{\mathbb{C}}(2n) \\ & l \nearrow & \downarrow \pi \\ \mathrm{U}(n) & \longrightarrow & \mathrm{SO}(2n) \times S^1, \end{array}$$

and $k \circ l : \mathrm{U}(n) \to \mathrm{Spin}^{\mathbb{C}}(2n) \to S^1$ coincides with $\det : \mathrm{U}(n) \to S^1$. $l : \mathrm{U}(n) \to \mathrm{Spin}^{\mathbb{C}}(2n)$ maps the maximal torus $\begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \subset \mathrm{U}(n)$ into

$$\left[\underbrace{\prod_{i=1}^{n} \left(\cos\frac{\theta i}{2} + e_{2i-1}e_{2i}\sin\frac{\theta i}{2} \right)}_{\in \mathrm{Spin}(2n)}, \underbrace{e^{\frac{i}{2}(\theta_1 + \cdots + \theta_n)}}_{\in S^1} \right].$$

Let (M^{2n}, g, J) be Kählerian (or at least Hermitian) and denote by $P_J \subset L(M, \mathrm{SO}(2n))$ the corresponding $\mathrm{U}(n)$ principal bundle. Then M^{2n} has the canonical $\mathrm{Spin}^{\mathbb{C}}(2n)$-structure

$$Q = P_J \times_l \mathrm{Spin}^{\mathbb{C}}(2n)$$

with the associated line bundle

$$L = Q \times_k \mathbb{C} = P_J \times_{k \circ l} \mathbb{C} = P_J \times_{\det} \mathbb{C} = \Lambda^n(TM^{2n}).$$

Hence

$$c_1(L) = c_1(M^{2n}).$$

For a Kählerian 4-manifold M^4

$$\sigma = \frac{1}{3}(c_1^2 - 2c_2), \quad \chi = c_2,$$

which implies

$$\text{v-dim}\,\mathcal{M}_L(g) = \frac{1}{4}(c_1(L)^2 - (2\chi + 3\sigma)) = \frac{1}{4}(c_1^2 - (2c_2 + c_1^2 - 2c_2)) = 0.$$

Remark. The set of all almost complex structures J, $J^2 = -\operatorname{id}$, $\det J = 1$, splits into two components: (X, JX, Y, JY) defines the given orientation or not. Let $A^+(M)$ or $A^-(M)$ be the corresponding bundles. If $J \in \Omega(A^+(M^4))$ then (TM^4, J) is a complex vector bundle and $c_2 = e$. We obtain with $c = c_1(TM^4, J) \in H^2(M^4; \mathbb{Z})$

$$\begin{aligned}\text{v-dim}\,\mathcal{M}_L(g) &= \frac{1}{4}(c^2 - (2\chi + 3\sigma)) = \frac{1}{4}(c_1^2 - (2c_2 + p_1)) = \\ &= \frac{1}{4}(c_1^2 - (2c_2 + c_1^2 - 2c_2)) = 0.\end{aligned}$$

where $p_1(TM^4) = c_1^2(TM^4, J) - 2c_2(TM^4, J)$.

Let $\Omega(X, Y) = g(X, JY)$ be the Kähler form of (M^4, g, J). Ω acts as an endomorphism $\Omega : S^+ \to S^+$ with eigenvalues $\pm 2i$: In a frame $\in P_J$, $\Omega = e_1 \wedge e_2 + e_3 \wedge e_4$, and according to Section 3, the endomorphisms $e_1 \wedge e_2 = e_3 \wedge e_4$ are given by $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$.

Let $S^+(\pm 2i) \subset S^+$ the corresponding subbundle, $\Phi \in S^+(2i)$, $\Omega\Phi = 2i\Phi$. With respect to a basis $\Delta^+ \cong \mathbb{C}^2$, Φ has a representation $\Phi = \left(\begin{smallmatrix} \Phi_1 \\ 0 \end{smallmatrix}\right)$. Then $\omega^\Phi(e_1, e_2) = \omega^\Phi(e_3, e_4) = i|\Phi|^2$, $\omega^\Phi(e_1, e_3) = -\omega^\Phi(e_2, e_4) = 0$, $\omega^\Phi(e_1, e_4) = \omega^\Phi(e_2, e_3) = 0$. Hence

$$\omega^\Phi = i|\Phi|^2\Omega \text{ for } \Phi \in S^+(2i).$$

Similarly,

$$\omega^\Phi = -i|\Phi|^2\Omega \text{ for } \Phi \in S^+(-2i).$$

It is a well known matter of fact that for a Hermitian manifold (M^{2n}, g, J) with its canonical $\mathrm{Spin}^{\mathbb{C}}(2n)$ structure the bundle S of $\mathrm{Spin}^{\mathbb{C}}(2n)$ spinors is isomorphic to $\Lambda^{0,*}$ (confer [8], [14]), where the splitting $S = S^+ \oplus S^-$ corresponds to $\Lambda^{0,*} = \Lambda^{0,\mathrm{even}} \oplus \Lambda^{0,\mathrm{odd}}$ and the Dirac operator can be identified with (const.) $\cdot\, \bar\partial \oplus \bar\partial^* : \Lambda^{0,\mathrm{even}} \to \Lambda^{0,\mathrm{odd}}$, $\bar\partial^* =$ Hermitian adjoint of the Dolbeaut operator $\bar\partial$. In our case (M^4, g, J),

$$S^+ \cong \Lambda^{0,0} \oplus \Lambda^{0,2}, \quad S^- \cong \Lambda^{0,1}$$

where additionally

$$S^+(2i) \cong \Lambda^{0,2}, \quad S^+(-2i) \cong \Lambda^{0,0}.$$

With respect to the Levi-Civita connection A_0,

$$D_{A_0} : \Omega(S^+) \to \Omega(S^-)$$

can be identified with

$$\sqrt{2}(\bar\partial_0 \oplus \bar\partial_2^*) : \Omega^{0,0} \oplus \Omega^{0,2} \to \Omega^{0,1}.$$

Proposition 9.1. *Let (M^4, g, J) be a Kähler manifold with constant negative scalar curvature $\tau = \tau(M^4, g, J) < 0$, Φ_0 the spinor in $\Omega(S^+(-2i)) \cong \Omega(\Lambda^{0,0}) \equiv \Omega^{0,0}$ which corresponds to the function identically equal to* 1. *Then $(\sqrt{-\tau}\,\Phi_0, A_0)$ is a solution of* (SW).

Proof. In our coordinates, $\Phi_0 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\omega^{\Phi_0} = -i\Omega$, with $\Phi = \sqrt{-\tau}\,\Phi_0 \in \Omega(S^+)$

$$D_{A_0}\Phi = 0 \tag{9.1}$$

$$\omega^\Phi = -|\Phi|^2 i\Omega = -(-\tau)i\Omega = \tau i\Omega. \tag{9.2}$$

On the other hand, the curvature Ω_{A_0} of the complex line bundle $L = \Lambda^2 T$ is given by the Ricci form ϱ,

$$\Omega_{A_0} = i\varrho, \tag{9.3}$$

$\varrho(X, Y) = g(X, J\,\mathrm{Ric}(Y))$, $\mathrm{Ric} : T \to T$ the Ricci tensor. With respect to an orthonormal basis $e_1, \ldots, e_4 \in T_x M^4$ $J : T \to T$ and Ric are given by matrices

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{Ric} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{pmatrix}.$$

Since J and Ric commute, we obtain

$$J \circ \mathrm{Ric} = \begin{pmatrix} 0 & -A & D & C \\ -A & 0 & C & D \\ -D & -C & 0 & -B \\ C & -D & B & 0 \end{pmatrix}, \quad A = R_{11} = R_{22},\ B = R_{33} = R_{44}.$$

An easy calculation yields

$$\varrho = A e_1 \wedge e_2 + B e_3 \wedge e_4 + (e_1 \wedge e_4 - e_2 \wedge e_3) - D(e_1 \wedge e_3 + e_2 \wedge e_4),$$

hence

$$\varrho^+ = \frac{A+B}{2}(e_1 \wedge e_2 + e_3 \wedge e_4),$$

where ϱ^+ is the projection of ϱ to Λ^2_+. In our frame the Kähler form Ω has a representation

$$\Omega = e_1 \wedge e_2 + e_3 \wedge e_4,$$

as we have already seen. Hence, in any 4-dimensional Kähler manifold

$$\varrho^+ = \frac{\tau}{4}\Omega, \tag{9.4}$$

since $A + B = R_{11} + R_{33} = \frac{1}{2}(R_{11} + R_{22} + R_{33} + R_{44})$. We conclude from (9.2)-(9.4)

$$\Omega^+_{A_0} = i\varrho^+ = i\frac{\tau}{4}\Omega = \frac{1}{4}\omega^\Phi. \tag{9.5}$$

(9.1) and (9.4) express that (Φ, A_0) is a solution of (SW). ∎

THEOREM 9.2. *Let (M^4, g, J) be a closed Kähler manifold with constant negative curvature $\tau < 0$ and let c be the canonical* $\mathrm{Spin}^{\mathbb{C}}(4)$ *structure. Then*

$$\mathrm{SW}(M^4, c, g) = 1 \text{ in } \mathbb{Z}_2.$$

Remarks.

1. To have $\mathrm{SW}(M, c)$ independent of g we require additionally $b_2^+ \geq 2$.
2. Theorem 9.2 has been proven by LeBrun in [15].

Proof. We are done if we can show that $\mathcal{M}_L$ consists of one orbit, i.e. that any solution of (SW) is gauge equivalent to the solution of (9.1). Let (Φ, A) be any solution of $D_A\Phi = 0$, $\Omega^+_A = \frac{1}{4}\omega^\Phi$. Then

$$|\Phi(x)|^2 \leq -\tau_{\min} = -\tau, |\Omega^+_A|^2 = \frac{1}{16}|\omega^\Phi|^2 = \frac{1}{8}|\Phi|^4 \leq \frac{\tau^2}{8}. \tag{9.6}$$

Using $\varrho^+ = \frac{\tau}{4}\Omega$, $|\Omega|^2 = 2$, we obtain

$$\int_{M^4} |\Omega_A^+|^2 \leq \int_{M^4} \frac{\tau^2}{8} = \int_{M^4} \frac{\tau^2|\Omega|^2}{16} = \int_{M^4} \left(\frac{\tau\Omega}{4}\right)^2 = \int_{M^4} |\varrho^+|^2. \tag{9.7}$$

On the other hand, ϱ is the curvature form of the Levi-Civita connection in $\Lambda^2 T$, hence

$$c_1^2(\Lambda^2 T) = \frac{1}{4\pi^2}\int_{M^4} |\varrho^+|^2 - |\varrho^-|^2,$$

by adding $\int\limits_{M^4} |\varrho^+|^2$ and dividing by 2,

$$\int_{M^4} |\varrho^+|^2 = 2\pi^2 c_1^2(\Lambda^2 T) + \frac{1}{2}\int_{M^4} |\varrho|^2.$$

Moreover, we conclude from $\Omega_{A_0} = i\varrho$, $\mathrm{d}\Omega_{A_0} = 0$, $\varrho = \varrho^+ + \varrho^-$, $\varrho^+ = \frac{\tau}{4}\Omega$, $\tau = \text{const.}$ that $\mathrm{d}\varrho^+ = \delta\varrho^+ = 0$, $\mathrm{d}\varrho^- = \delta\varrho^- = 0$ and finally

$$\Delta\varrho = 0.$$

Harmonic forms realize the L_2-minimum in any cohomology class. Hence for any connection A in $\Lambda^2 T$

$$\int_{M^4} |\varrho|^2 \leq \int_{M^4} |\Omega_A|^2$$

and therefore

$$\int_{M^4} |\varrho^+|^2 \leq 2\pi^2 c_1^2 2(L) + \frac{1}{2}\int_{M^4} |\Omega_A|^2 = \int_{M^4} |\Omega_A^+|^2. \tag{9.8}$$

(9.6), (9.7) and the uniqueness of harmonic representatives imply

$$\Omega_A = i\varrho. \tag{9.9}$$

Similarly, we infer from (9.5), (9.6) and (9.9)

$$|\Phi|^2 \equiv -\tau.$$

From the proof of $|\Phi|^2 \leq -\tau_{\min}$ we can conclude

$$\nabla^A\Phi \equiv 0,\ \ \Phi \text{ is parallel.}$$

We decompose Φ, according to $S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$, as

$$\Phi = \Phi^{0,0} \oplus \Phi^{0,2}.$$

Then $\nabla^A\Phi = 0$ if and only if $\nabla^A\Phi^{0,0} = \nabla^A\Phi^{0,2} = 0$. In particular, $\Phi^{0,2}$ is a ∇^A-parallel section of $\Lambda^{0,2}$. If $\Phi^{0,2}$ would be nontrivial then $c_1(\Lambda^{0,2}) = c_1(\Lambda^2 T) = 0$ which contradicts

$$\Omega_{A_0}^+ = i\varrho^+ = i\frac{\tau}{4}\Omega \neq 0$$

since $\tau < 0$. We conclude $\Phi^{0,2} \equiv 0$, $\Phi = f\Phi_0 \in \Omega(\Lambda^{0,0})$. $|\Phi|^2 = \text{const.}$ implies $|f|^2 = -\tau$. A has a representation

$$A = A_0 + \eta, \qquad \eta \in \Omega^1(i\mathbb{R}).$$

We have $\mathrm{d}\eta = 0$ since $\Omega_A = i\varrho = \Omega_{A_0} = \Omega_{A_0} + \mathrm{d}\eta$. Moreover

$$\operatorname{grad} f + \eta f = 0 \tag{9.10}$$

since $0 = D_A\Phi = D_{A_0}\Phi + \eta \cdot \Phi = \operatorname{grad} f \cdot \Phi_0 + \eta \cdot f \cdot \Phi_0$. Consider now the gauge transformation $g = \frac{f}{|f|} : M^4 \to S^1$. Locally, $f = \sqrt{-\tau} e^{iF}$, $\mathrm{d}g = \frac{1}{-\tau}\mathrm{d}f$, $g = e^{iF} = \frac{1}{-\tau}\mathrm{d}f$, together with (9.10)

$$\eta = -\frac{\mathrm{d}g}{g}.$$

We conclude $A = A_0 - \frac{\mathrm{d}g}{g}$, $\Phi = g \cdot (\sqrt{-\tau}\Phi_0)$, i.e. (Φ, A) is gauge equivalent to $(\sqrt{-\tau}\Phi_0, A_0)$. ∎

COROLLARY 9.3. *Let M^4 be closed, oriented, $b_2^+ \geq 2$. If M^4 admits a Kähler structure with constant negative scalar curvature then M^4 does not admit a metric with positive scalar curvature.*

Proof. $\mathrm{SW}(M^4, g_1, J) = 1$, $\mathrm{SW}(M^4, g_2) = 0$. ∎

Until now we considered the canonical $\mathrm{Spin}^{\mathbb{C}}(4)$ structure associated to (M^4, g, J). Now we admit other $\mathrm{Spin}^{\mathbb{C}}(4)$ structures $c \in H^2(M^4; \mathbb{Z})$ with corresponding line bundle L and describe the corresponding moduli space $\mathcal{M}_L(g)$.

THEOREM 9.4. *$\mathcal{M}_L(g)$ consists of pairs (holomorphic structure on $S^+(\pm 2i)$, $[\Phi_\pm]$) where $[\Phi_+] \in \mathbb{P}H^0(M^4; S^+(2i))$ if $\Omega \wedge c < 0$ or $[\Phi_-] \in \mathbb{P}H^0(M^4; S^+(-2i))$ if $\Omega \wedge c > 0$, respectively. If $b_1(M^4) = 0$ then the holomorphic structure on $S^+(\pm 2i)$ is uniquely determined and $\mathcal{M}_L(g) \cong \mathbb{P}H^0(M^4; S^+(\pm 2i))$.*

Proof. The Kähler form Ω acts once again as endomorphism of the $\mathrm{Spin}^{\mathbb{C}}(4)$-spinor bundle S^+ with eigenvalues $\pm 2i$ and defines a decomposition

$$S^+ = S^+(2i) \oplus S^+(-2i).$$

This implies

$$L \cong \Lambda^2 S^+ = \Lambda^2(S^+(2i) \oplus S^+(-2i) \otimes S^+(-2i).$$

It follows immediately from $\nabla^A \Omega = [\nabla^A, \Omega]$ for $A \in \mathcal{C}_L$ that the decomposition $S^+ = S^+(2i) \oplus S^+(-2i)$ is ∇^A-parallel. Decomposing $\Phi = \Phi_+ + \Phi_-$, we can write the Seiberg-Witten functional as

$$\text{(9.11)} \quad \int_{M^4} |\Omega_A^+ - \frac{1}{4}\omega^\Phi|^2 + |D_A\Phi|^2 =$$

$$= \int_{M^4} \left(|\Omega_A^+|^2 + |\nabla^A\Phi_+|^2 + |\nabla^A\Phi_-|^2 + \frac{\tau}{4}(|\Phi_+|^2 + |\Phi_-|^2) + \frac{1}{8}(|\Phi_+|^2 + |\Phi_-|^2) \right)^2.$$

We infer from (9.11) that if $(\Phi = \Phi_+ + \Phi_-, A)$ is a solution of (SW) then $(\hat{\Phi} = \Phi_+ - \Phi_-, A)$ is too. Assume $(\Phi = \Phi_+ + \Phi_-, A)$ is a solution of (SW),

$$\begin{aligned} D_A\Phi &= 0 \\ \Omega_A^+ &= \frac{1}{4}\omega^\Phi, \\ \text{then} \qquad D_A\hat{\Phi} &= 0 \\ \Omega_A^+ &= \frac{1}{4}\omega^{\hat{\Phi}}. \end{aligned}$$

Let $e_1, \ldots, e_4$ be a local orthonormal basis. We can write

$$\Omega = e_1 \wedge e_2 + e_3 \wedge e_4.$$

An easy calculation shows

$$\begin{aligned} \omega^\Phi = {} & i(|\Phi_+|^2 - |\Phi_-|^2)(e_1 \wedge e_2 + e_3 \wedge e_4) + \\ & + (\Phi_+\overline{\Phi_-} - \overline{\Phi_+}\Phi_-)(e_1 \wedge e_3 - e_2 \wedge e_4) - \\ & - i(\Phi_+\overline{\Phi_-} + \overline{\Phi_+}\Phi_-)(e_1 \wedge e_4 + e_2 \wedge e_3) \end{aligned} \tag{9.12}$$

which implies immediately

$$\omega^\Phi + \omega^{\hat\Phi} = 2i(|\Phi_+|^2 - |\Phi_-|^2)\Omega. \tag{9.13}$$

(9.12), (9.13) and $\Omega_A^+ = \frac{1}{4}\omega^\Phi = \frac{1}{4}\omega^{\hat\Phi}$ yield

$$\Omega_A^+ = i(|\Phi_+|^2 - |\Phi_-|^2)\Omega \tag{9.14}$$

and $\Phi_+ \cdot \overline{\Phi_-} \equiv 0$. Hence, according to the Aronszajn theorem, either $\Phi_+ \equiv 0$ or $\Phi_- \equiv 0$.

Finally, we obtain from $\Lambda^{0,2} \cap \Lambda^2_- = \{0\} = \Lambda^{2,0} \cap \Lambda^2_-$ and (9.14) that

$$\Omega_A^{0,2} = 0 = \Omega_A^{2,0},$$

i.e. Ω_A is a $(1,1)$-form and therefore defines a holomorphic structure in L and $S^+(\pm 2i)$. Hence $D_A\Phi_+ = 0$ or $D_A\Phi_- = 0$, respectively, imply $\Phi_\pm$ holomorphic. Recall $c_1(L) = \frac{i}{2\pi}\Omega_A$, $c_1^+(L) = \frac{i}{2\pi}\Omega_A^+ = \frac{1}{2\pi}(|\Phi_-|^2 - |\Phi_+|^2)\Omega$, $\Omega \wedge c_1 = \Omega \wedge c_1^+$.

$$J := \int_{M^4} \Omega \wedge c_1(L) = \frac{1}{2\pi}\int (|\Phi_-|^2 - |\Phi_+|^2)\Omega \wedge \Omega$$

is a topological invariant. $J < 0$ implies $\Phi_- \equiv 0$, $J > 0$ implies $\Phi_+ \equiv 0$. The assertion now follows by taking into account $D_A = \bar\partial_A + \bar\partial_A^*$ and the action of the gauge group. ∎

Remark. It is not possible to determine $\mathrm{SW}(M^4, c) \in \Omega_*$ by means of g since the Kähler metric g in general is not generic.

10. The Thom conjecture.

Let $\mathbb{C}P^1 \subset \mathbb{C}P^2$ the hyperplane in $\mathbb{C}P^2$ with standard orientation, $H = PD[\mathbb{C}P^1]$ its Poincaré dual, $\Sigma \hookrightarrow \mathbb{C}P^2$ a smoothly embedded Riemann surface such that $PD(\Sigma) = d \cdot H$. Thom conjectured that the genus g of Σ is at least $(d-1)(d-2)/2$. The case $d = 1, 2$ is trivial. The case $d = 3$ has been solved by Kervaire/Milnor 1964. A proof for $d > 3$ now has been given by Kronheimer/Mrowka in [12] using Seiberg-Witten theory.

Theorem 10.1. *Let Σ be an oriented 2-manifold smoothly embedded in $\mathbb{C}P^2$ so as to present the same homology class as an algebraic curve of degree d. Then the genus g of Σ satisfies $g \geq (d-1)(d-2)/2$.*

Proof. Assume $d > 3$, $\Sigma \hookrightarrow \mathbb{C}P^2$, $PD(\Sigma) = d \cdot H$. Blowing up of d^2 points replaces $\mathbb{C}P^2$ by

$$\mathbb{C}P^2 \# \underbrace{\overline{\mathbb{C}P^2} \# \cdots \# \overline{\mathbb{C}P^2}}_{d^2}.$$

Let $E_i \subset \overline{\mathbb{C}P^2}$ the exceptional divisors which correspond to H in $\mathbb{C}P^2$. Then $E_i^2 = -1$ and E_i can be represented by $S_i^2 \hookrightarrow \overline{\mathbb{C}P^2}$. Form

$$\tilde{\Sigma} := \Sigma \# S_1^2 \# \cdots \# S_{d^2}^2 \hookrightarrow \mathbb{C}P^2 \# d^2\overline{\mathbb{C}P^2} \equiv X^4.$$

Then $g(\tilde{\Sigma}) = g(\Sigma) = g$, $PD(\tilde{\Sigma}) = dH - E$, $E = \sum_{i=1}^{d^2} E_i$, $[\tilde{\Sigma}] \cdot [\tilde{\Sigma}] = d^2 - d^2 = 0$, hence $\tilde{\Sigma}$ has trivial normal bundle.

We prove below the following two lemmas.

Lemma 10.2. *Set $c_1(L) = 3H - E$. Then there exists a translation invariant solution of* (SW) *in temporal gauge on $\mathbb{R} \times S^1 \times \tilde{\Sigma}$.*

Lemma 10.3. *If there exists a translation invariant solution in temporal gauge of* (SW) *on $X = \mathbb{R} \times S^1 \times \tilde{\Sigma}$ then*

$$|\langle c_1(L), [\tilde{\Sigma}]\rangle| \leq \max\{0, 2g(\tilde{\Sigma}) - 2\}.$$

Now $\langle c_1(L), [\tilde{\Sigma}]\rangle = (3H - E) \cup (dH - E) = 3d - d^2$. We assume $d > 3$, i.e. exclude $3d - d^2 \geq 0$, hence

$$\begin{aligned} 3d - d^2 &\geq 2 - 2g, \\ 2g &\geq (d-1)(d-2). \ \blacksquare \end{aligned}$$

To complete the proof of Theorem 10.1, we have to prove Lemmas 10.2, 10.3. Consider $X = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ which arises as the blow-up of the projective plane at n points. Hence it has a preferred $\mathrm{Spin}^{\mathbb{C}}(4)$ structure $c = c_1(L)$ and $c_1(L)$ is given by $c_1(L) = c_1(X) = 3H - E$, $E = \sum E_i$. Then, as we already know, v-dim $\mathcal{M}_L = 0$. But $\mathrm{SW}(X, L, g)$ is dependent on the metric g since $b_2^+ = 1$. $(H^2(X;\mathbb{R}), \text{intersection form})$ has signature $(1, n)$. Let C^+ be the component of the positive cone which contains H. For every Riemannian metric g on X there exists an uniquely determined (up to positive scaling) self-dual harmonic form $[\omega_g]$ whose cohomology lies in

$$C^+ = \{[\omega] \in H^2(X;\mathbb{R}) \mid [\omega]^2 > 0, H \cup [\omega] > 0\}.$$

It is clear that a metric g is L-good if and only if $c_1(L) \cup [\omega]_g \neq 0$. We state without proof (for a proof see [12])

Proposition 10.4. *Let g be L-good. Then*

$$\mathrm{SW}(X, L, g) \equiv \begin{cases} 0 \bmod 2 & \textit{if and only if } c_1(L) \cup [\omega_g] > 0 \\ 1 \bmod 2 & \textit{if and only if } c_1(L) \cup [\omega_g] < 0. \end{cases}$$

Corollary 10.5. *If $c_1(L) \cup [\omega_g] < 0$ then $\mathcal{M}_L \neq \emptyset$.*

Now we consider a more general situation. Let X^4 be a closed, oriented 4-manifold with $\mathrm{Spin}^{\mathbb{C}}(4)$ structure $c = c_1(L)$, $r_* c = w_2(X)$, $Y^3 \subset X^4$ an oriented 3-manifold with trivial normal bundle and local product metric. Then Y^3 inherits a $\mathrm{Spin}^{\mathbb{C}}(3)$ structure by means of $w_2(Y^3) = i^* w_2(X^4) = i^* r_* c = r_* i^* c$. Moreover, we receive back the $\mathrm{Spin}^{\mathbb{C}}(4)$ structure $Q(\mathrm{Spin}^{\mathbb{C}}(4), [-R, R] \times Y)$ of a tubular neighborhood as $\pi^* Q(\mathrm{Spin}^{\mathbb{C}}(3), Y) \times_{\mathrm{Spin}^{\mathbb{C}}(3)} \mathrm{Spin}^{\mathbb{C}}(4)$ and on $[-R, R] \times Y$ the $\mathrm{Spin}^{\mathbb{C}}(4)$ spinors S^+, S^- can be identified with $\pi^* S(Y)$. The same holds for $X^4 = \mathbb{R} \times Y^3$ with product metric $\mathrm{d}r^2 \oplus g_Y$.

Starting once again with closed X^4 and $Y \hookrightarrow X^4$ as above, $X = X_- \cup_Y X_+$, we consider the manifold

$$X^R := X_- \cup_Y [0,R] \times Y \cup_Y X_+$$

with metric g^R, $g^R|_{[0,R]\times Y} = \mathrm{d}t^2 + g_Y$. A connection A on L over $\mathbb{R} \times Y$ or $[0,R]\times Y$ is called temporal if the $\mathrm{d}t$ component of A vanishes. Such a connection can be considered as a family $(A_t)_t$ of connections on $L|_Y$. The Seiberg-Witten equations in temporal gauge are

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \partial\!\!\!/_{A_t}\Phi_t$$
$$\frac{\mathrm{d}A_t}{\mathrm{d}t}(X) = -*\Omega_{A_t}(X) - \frac{1}{2}(\Phi_t, X\cdot\Phi_t).$$

These are the gradient flow equations of the functional $\frac{1}{2}C$ on $\mathcal{C}(L|_Y) \times \Omega(S^+|_Y)$,

$$C(A,\Phi) := \int_Y (A-b)\wedge(\Omega_A + \Omega_B) + 2(\Phi_A, \partial\!\!\!/_{A_t}\Phi),$$

where B is a fixed reference connection on $L|_Y$. Let $h \in \mathrm{Map}(Y, S^1)$ be a gauge transformation. Then

$$C(h(A,\Phi)) = C(A,\Phi) + 8\pi^2[\frac{i}{2\pi}\mathrm{d}\log h] \cup c_1(L),$$

where $[\frac{i}{2\pi}\mathrm{d}\log h]$ is the mapping degree of h.

PROPOSITION 10.6. *Assume that for a sequence $(R_i)_i \to \infty$ $\mathcal{M}_L(X^{R_i}, g^{R_i})$ is nonempty. Then there exists on $\mathbb{R}\times Y$ a translation invariant solution of* (SW) *in temporal gauge.*

P r o o f. We infer from the assumption the existence of a solution (Φ^R, A^R) in temporal gauge on the cylinder $[0,R]\times Y$. We state without proof that there exists a bound K, independent of A, Φ and R, such that

$$0 \le C(A^R(R), \Phi^R(R)) - C(A^R(0), \Phi^R(0)) \le K. \tag{10.1}$$

The proof follows from a careful inspection of (10.1) under gauge transformations. Now we decompose the cylinder $[0,R]\times Y$ into pieces of length 1 and obtain

$$0 \le C(A^R(1), \Phi^R(1)) - C(A^R(0), \Phi(0)) \le \frac{K}{[R]}.$$

According to Proposition 3.4, $|\Phi^R|$ is bounded by the infimum of the scalar curvature on X^R which is independent of R. The compactness property which is also valid for the manifold $[0,1]\times Y$ with boundary produces a convergent subsequence $(\Phi^{R_i}, A^{R_i})|_{[0,1]\times Y} \to (\Phi, A)|_{[0,1]}$ for which C is constant. (Φ, A) is translation invariant in a temporal gauge and can be extended to a translation invariant solution on $\mathbb{R}\times Y$. ∎

P r o o f o f L e m m a 10.2. Consider now in our case

$$\tilde{\Sigma} \hookrightarrow X^4 = \mathbb{C}P^2 \# d^2\overline{\mathbb{C}P^2}, \quad c = c_1(L) = 3H - E.$$

There exists a sequence $R_i \to \infty$ such that

$$c_1(L) \cup [\omega_g R_i] < 0. \tag{10.2}$$

Then $\mathcal{M}_L(X, g^{R_i}) \neq \emptyset$. The assertion follows from Proposition 10.6. It remains to prove (10.2). Consider

$$\begin{aligned} Y &= S^1 \times \tilde{\Sigma}, \\ X_-^R &= X_- \cup_Y [0, R] \times Y, \\ X_-^\infty &= X_- \cup_Y [0, \infty[\times Y, \\ X_+^R &= [-R, 0] \times Y \cup_Y X_+, \\ X_+^\infty &=]-\infty, 0] \times Y \cup_Y X_+, \\ X^R &= X_- \cup_Y [-R, R] \times Y \cup_Y X_+ \end{aligned}$$

and the isometric embeddings

$$X_-^{R'} \hookrightarrow X_-^R \hookrightarrow X_-^\infty \text{ for } R' \leq R \ \text{ and } X_-^R \overset{\varphi_R}{\hookrightarrow} (X^R, g^R),$$

similarly for X_+. Set $\omega_R := \varphi_R^* \omega_{g^R}$. ω_R lies on X_-^R and for $R' \leq R$ on $X_-^{R'}$ too. Normalize ω_{g^R} so that $H \cup [\omega_{g^R}] = 1$. $([\omega_{g^R}] - H) \leq 0$ since $(\omega_{g^R} - H)$ is anti-self-dual. Then we obtain immediately $|\omega_R|^2_{L_2(X_-^{R'})} \leq 1$. The same holds for their Sobolev norms since the ω_R are closed and coclosed. This yields a subsequence ω_{R_i} converging in the first Sobolev norm on exhausting compact subsets against a harmonic self-dual form ω' on $X_-^\infty \sqcup X_+^\infty$. It follows from [16] that $[\omega'] \in \operatorname{im}(H_c^2(X_\pm^\infty; \mathbb{R}) \to H_{dR}^2(X_\pm^\infty))$, hence $[\omega'] = 0$. Finally,

$$c_1(L) \cup [\omega_{R_i}] = (PD[\tilde{\Sigma}] - (d-3)H) \cup [\omega_{R_i}] = \int\limits_{\tilde{\Sigma}} \omega_{R_i} - (d-3) \underset{i \to \infty}{\longrightarrow} -(d-3) < 0.$$

This finishes the proof of (10.2) and Lemma 10.2. ∎

P r o o f o f L e m m a 10.3. We set $Y = S^1 \times \tilde{\Sigma}$, $\tilde{\Sigma}$ endowed with a metric of constant scalar curvature $\tau(\tilde{\Sigma})$ and rescaled so that $\operatorname{area}(\tilde{\Sigma}) = 1$. Then according to Gauss-Bonnet, $\tau(\tilde{\Sigma}) = -4\pi(2g-2)$. Furthermore, since Φ is translation invariant, $|\Phi|^2 \leq \max\{0, 4\pi(2g-2)\}$. Now we use $|\omega^\Phi|^2 = 2|\Phi|^4$ and obtain

$$|\Omega_A^+| = \frac{1}{4}|\omega^\Phi| = \frac{1}{4}\sqrt{2}|\Phi|^2 \leq \max\{0, \sqrt{2}\pi(2g-2)\}.$$

It follows from translation invariance that $|\Omega_A^+| = |\Omega_A^-|$, hence

$$|\Omega_A| \leq \max\{0, 2\pi(2g-2)\}.$$

The assertion now follows from

$$\langle c_1(L), [\tilde{\Sigma}] \rangle = |\frac{i}{2\pi} \int\limits_{\tilde{\Sigma}} \Omega_A| \leq \frac{1}{2\pi} \sup |\Omega_A| \cdot \operatorname{area}(\tilde{\Sigma}) \leq \max\{0, 2g-2\}.$$

This finishes the proof of Theorem 10.1. ∎

Without proof we state the following generalization of Theorem 10.1. We call a class $K \in H^2(M^4; \mathbb{Z})$ basic if $r_* K = w_2$, $K^2 = 2\chi + 3\sigma$ and $\operatorname{SW}(M^4, K) \neq 0$.

THEOREM 10.7. *If $\Sigma \hookrightarrow M^4$ is a smoothly embedded surface of genus $g \geq 1$ and K is a basis class of M^4 then $2g - 2 \geq K \cdot [\Sigma] + [\Sigma] \cdot [\Sigma]$.*

We refer to [1] for the proof.

When $M^4 = \mathbb{C}P^2$ this is just Theorem 10.1.

11. Applications of Seiberg-Witten theory to symplectic manifolds. Let (M^4, ω) be a closed symplectic manifold, i.e. ω is a closed everywhere non-degenerate 2-form. M^4 has a canonical orientation given by $\omega \wedge \omega$. There arise several natural questions.

(1) Given a closed M^4, does there exist a symplectic form on M?
(2) How many symplectic forms do exist on M^4?

Concerning the first question, there are topological obstructions, M^4 must be orientable and $H^2(M^4; \mathbb{R}) \neq 0$. The second question leads to a classification theory. Locally all symplectic 4-manifolds look equally according to Darboux's theorem. One has to establish global properties of symplectic manifolds and to compare them. This has been done e.g. by Dusa McDuff and Gromov. Seiberg-Witten theory as elaborated by Cliff Taubes has brought a big progress as well for the existence as the classification problem.

To apply Seiberg-Witten theory we must have a $\mathrm{Spin}^{\mathbb{C}}(4)$ structure $c = c_1(L)$ and a Riemannian metric g on M^4. For doing this we introduce the notion of an almost complex structure J compatible with a symplectic structure $\omega : \omega(X, JX) > 0$ for $X \neq 0$, $\omega(JX, Y) + \omega(X, JY) = 0$, i.e. $g(X, Y) := \omega(X, JY)$ is a Hermitian metric.

Proposition 11.1. *The space $\mathfrak{J}(\omega)$ of all ω-compatible almost complex structures on M is non-empty and can be after choice of an J_0 parametrized by the space of sections of the vector bundle $h(TM, J_0)$,*

$$h(TM, J_0) = \{A \in \mathrm{End}(TM) \mid AJ_0 = -J_0 A, A^t = A\}.$$

As we already mentioned in Section 9, after choice of a J and corresponding g, ω is a section of Ω^2_+ and can locally be described as

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4. \tag{11.1}$$

Locally there exist an orthonormal basis $e_1 = \tilde{e}_1$, $e_2 = J\tilde{e}_1$, $e_3 = \tilde{e}_2$, $e_4 = J\tilde{e}_2$. Then an easy calculation immediately shows that $\omega = \sum_{i<j} \omega(e_i, e_j) e^i \wedge e^j$ equals to (11.1). (11.1) implies in particular that ω is self-dual with respect to g and $|\omega| = \sqrt{2}$. As in the Kähler case, ω acts on S^+ with the eigenvalues $\pm 2i$, $S^+ = S^+(2i) \oplus S^+(-2i)$, $S^+(2i) \cong \Lambda^{0,2}$, $S^+(-2i) \cong \Lambda^{0,0}$, $S^- \cong \Lambda^{0,1}$. $K = \Lambda^{2,0} = \det(T^{1,0})$ or $K^{-1} = \Lambda^{0,2}$ are called the canonical or anticanonical bundle, respectively. We have $b_2^+ \geq 1$ since ω is self-dual with respect to a compatible metric g above. If $t \mapsto \omega_t$ is a continuous 1-parameter family of symplectic forms on M^4 then the canonical bundles for (M^4, ω_c) and (M^4, ω_1) will be isomorphic. The same holds for the corresponding $\mathrm{Spin}^{\mathbb{C}}(4)$ structure according to $S = S^+ \oplus S^-$,

$$S^+ = \Lambda^{0,0} \oplus K^{-1} \equiv I \oplus K^{-1}, \quad S^- = \Lambda^{0,1},$$

where I is the trivial complex line bundle. Consider the line bundle $L = \det S^+ \cong K^{-1}$. A connection A on L and the Levi-Civita connection determine the $\mathrm{Spin}^{\mathbb{C}}(4)$ connection ∇^A on S^+. The projections onto I or K^{-1} are given by $\frac{1}{2}(1 + \frac{i}{2}\omega)$ or $\frac{1}{2}(1 - \frac{i}{2}\omega)$, respectively. There is a (up to gauge transformations) uniquely determined connection A_0 on K^{-1} such that the induced $\mathrm{Spin}^{\mathbb{C}}(4)$ connection ∇^{A_0} induces on I the flat connection, $\frac{1}{2}(1 + \frac{i}{2}\omega)\nabla^{A_0}|_I = \text{flat}$. There is a nontrivial section $u_0 \in \Omega(I)$ of constant length, which should be normalized to have length equal to 1. u_0 is only annihilated by $\frac{1}{2}(1+\frac{i}{2}\omega)\nabla^{A_0}$, but

$$\nabla^{A_0}u_0 = b, \quad b = \frac{1}{2}\nabla^{0,2}\omega \operatorname{im}\Omega^1(K^{-1}). \tag{11.2}$$

LEMMA 11.2. *u_0 solves the Dirac equation*

$$D_{A_0}u_0 = 0.$$

This follows from an easy calculation using $*\omega = \omega$, $\mathrm{d}\omega = 0$ and (11.2). ■

The $\mathrm{Spin}^{\mathbb{C}}(4)$ structure above corresponds to the first Chern class of the complex vector bundle. After fixing this canonical $\mathrm{Spin}^{\mathbb{C}}(4)$ structure the set of equivalence classes of $\mathrm{Spin}^{\mathbb{C}}(4)$ structures can be identified with the set of equivalence classes of complex line bundles over $M^4 \cong H^2(M^4;\mathbb{Z})$. Let E be such a line bundle. Then the corresponding $S^+ = S^+(E)$ is given by

$$S^+ = E \oplus (K^{-1}\otimes E) \tag{11.3}$$

and $L = L(E) = \det S^+(E)$ is given by

$$L = K^{-1}\otimes E^2. \tag{11.4}$$

A connection A on L is determined by A_0 on K^{-1} and a connection a on E. They are related as follows. Let ∇^A be the corresponding $\mathrm{Spin}^{\mathbb{C}}(4)$ connection and $\alpha\cdot u_0 \in \Omega(E) \subset \Omega(E\oplus K^{-1}\otimes E) = \Omega(S^+)$. Then

$$\nabla^A(\alpha\cdot u_0) = (\nabla^a\alpha)\cdot u_0 + \alpha\cdot\nabla^{A_0}u_0. \tag{11.5}$$

Moreover, define for $\beta\in\Omega(K^{-1}\otimes E)$

$$\nabla'^A\beta := \frac{1}{2}(1-\frac{i}{2}\omega)\nabla^A\beta.$$

For $\Psi = \alpha\cdot u_0 + \beta \in \Omega(S^+(E))$ the Seiberg-Witten equations

$$D_A\Psi = 0, \qquad \Omega_A^+ = \frac{1}{4}\omega^\Psi \tag{SW}$$

can be rewritten as

$$u_0\cdot\nabla^a\alpha + D_A\beta = 0 \tag{11.6}$$

$$\Omega_A^+ = \frac{i}{8}(|\alpha|^2 - |\beta|^2)\omega + \frac{i}{4}(\alpha\beta^* + \alpha^*\beta). \tag{11.7}$$

Taubes and other authors do not discuss (11.6) and (11.7) but certain perturbations of (SW) depending on a parameter r which give the same Seiberg-Witten invariant. Let $\mu\in\Omega^2_+$ be a self-dual 2-form and consider

$$D_A\Phi = 0, \qquad \Omega_A^+ = \frac{1}{4}\omega^\Phi + i\mu. \tag{SW_μ}$$

Quite analogously as above one defines moduli spaces $\mathcal{M}_{L,\mu}$. Let $\mathcal{M}^*_{L,\mu}$ be the irreducible part.

PROPOSITION 11.3. $\bigcup_{\mu\in\Omega^2_+}\mathcal{M}^*_{L,\mu}\times\{\mu\}$ *is a manifold and the projection π onto Ω^2_+ is Fredholm of index $\frac{1}{4}(c_1(L)^2 - (2\chi+3\sigma))$. For regular values μ_1, μ_2*

$$\mathrm{SW}(L,g,\mu_1) = \mathrm{SW}(L,g,\mu_2).$$

We refer to [12] for a proof which is strongly adapted to the corresponding proof in gauge theory as presented e.g. in [6]. ∎

In his absolutely fundamental contributions to Seiberg-Witten theory Taubes intensively studied (SW_μ) for (11.6), (11.7):

THEOREM 11.4 (Taubes [17]). *Assume $(M^4\omega)$ closed, oriented with orientation given by $\omega \wedge \omega$, $b_2^+ \geq 2$. Then the first Chern class of the associated almost complex structure has Seiberg-Witten invariant equal to ± 1.*

Proof. Consider for $E =$ trivial line bundle the following perturbation of (11.6), (11.7)

$$D_A \Psi = 0 \tag{11.8}$$

$$\Omega_A^+ = \Omega_{A_0}^+ + \frac{i}{8}(|\alpha|^2 - |\beta|^2)\omega + \frac{i}{4}(\alpha\beta^* + \alpha^*\beta) \tag{11.9}$$

The advantage of (11.8), (11.9) over (11.6), (11.7) is that the pair (A_0, u_0) is a solution of (11.8), (11.9). (11.8), (11.9) is the $r = 0$ version of a 1-parameter family of perturbations

$$D_A \Psi = 0 \tag{11.8}$$

$$\begin{aligned} \Omega_A^+ = \Omega_{A_0}^+ &+ \frac{i}{8}(|\alpha|^2 - |\beta|^2 - 1)\omega + \frac{i}{4}(\alpha\beta^* + \alpha^*\beta) - \\ &- i4r(1 + r|\alpha|^2)^{-1}(\alpha^*(b, \nabla_a \alpha) - \alpha(b, \nabla_a \alpha)^*). \end{aligned} \tag{11.10}$$

Taubes now shows that for any $r \geq 0$ (11.8), (11.10) can be used to compute the Seiberg-Witten invariant for (M^4, ω) and the canonical $\mathrm{Spin}^{\mathbb{C}}(4)$ structure. In the second step he shows that there exists an increasing unbounded sequence $(r_m)_m$ of parameter values for which (11.8), (11.10) has a solution $(A_m, \Psi_m = (\alpha_m, \beta_m))$. In the third step he shows that for large m, (A_m, Ψ_m) is gauge equivalent to (A_0, u_0). As usual, the proofs consist essentially of careful estimates. ∎

$\underset{n}{\#}\, \mathbb{C}P^2 \,\#\, (\underset{m}{\#}\, \overline{\mathbb{C}P^2})$ admits a metric with positive scalar curvature and hence the Seiberg-Witten invariant vanishes.

COROLLARY 11.5. *Assume M_i^4 closed, oriented with $b_2^+(M_i) > 0$, $i = 1, 2$. Then $M_1^4 \# M_2^4$ does not admit a symplectic form which defines the given orientation. For example, when $n > 1$, $m \geq 0$ then $\underset{n}{\#}\, \mathbb{C}P^2 \,\#\, (\underset{m}{\#}\, \overline{\mathbb{C}P^2})$ has no symplectic form which defines the given orientation.*

THEOREM 11.6. *Assume (M^4, ω) closed, oriented with orientation given by ω, $b_2^+ \geq 2$. Let $c \in H^2(M^4; \mathbb{Z})$ have non-zero Seiberg-Witten invariant. Then $|c \cdot [\omega]| \leq c_1(K) \cdot [\omega]$ and if the equality holds then either $\pm c = c_1(K)$. In particular, if a closed 4-manifold admits a symplectic form then $c_1(K) \cdot [\omega] \geq 0$.*

We refer to [18] for a proof which essentially uses details of the proof of Theorem 11.4 and (11.8), (11.10) for arbitrary E. ∎

We mention that for the standard Kähler structure on $\mathbb{C}P^2$ $c_1(K) \cdot [\omega] < 0$.

THEOREM 11.7. *$\mathbb{C}P^2$ has no symplectic form ω for which $c_1(K) \cdot [\omega] > 0$.*

Proof. Let ω a symplectic form on $\mathbb{C}P^2$, J a compatible almost complex structure. As we have already seen, the perturbed Seiberg-Witten equation

$$D_A\Psi = 0 \tag{11.8}$$

$$\begin{aligned}\Omega_A^+ = \Omega_{A_0}^+ + \frac{i}{8}(|\alpha|^2 - |\beta|^2 - 1) + \frac{ir}{4}(\alpha\beta^* + \alpha^*\beta) - \\ -i4r(1+r|\alpha|^2)^{-1}(\alpha^*(b,\nabla_a\alpha) - \alpha(b,\nabla_a\alpha)^*)\end{aligned} \tag{11.11}$$

has for r sufficiently large a unique solution which is gauge equivalent to (A_0, u_0). For no $r \geq 0$ (11.11) does have a solution with $\Psi = 0$: If this would be the case then $\Omega_A^+ = \Omega_{A_0}^+ - i\omega$, i.e. Ω_A and Ω_{A_0} would not be cohomologous which is impossible. Hence the $r = 0$ version of (11.11) computes a Seiberg-Witten invariant of ± 1 for K^{-1}. On the other hand, $\mathbb{C}P^2$ has a metric with positive scalar curvature which implies at least $\Psi \equiv 0$ for K^{-1} when using the original Seiberg-Witten equation

$$\begin{aligned} D_A\Psi &= 0 \\ \Omega_A^+ &= \frac{i}{8}(|\alpha|^2 - |\beta|^2 - 1)\cdot\omega + \frac{i}{4}(\alpha\beta^* + \alpha^*\beta).\end{aligned} \tag{11.12}$$

Consider the interpolation

$$\begin{aligned} D_A\Psi &= 0 \\ \Omega_A^+ &= s\cdot\Omega_{A_0}^+ + \frac{i}{8}(|\alpha|^2 - |\beta|^2 - s)\cdot\omega + \frac{i}{4}(\alpha\beta^* + \alpha^*\beta).\end{aligned} \tag{11.13}$$

If $s = 1$ in (11.13) then we obtain the $r = 0$ version of (11.11) and if $s = 0$ in (11.13) then we obtain (11.12). Hence there should exist an $s \in]0,1[$ with solution $\Psi \equiv 0$ of (11.13). Take the wedge product of (11.13) with ω and integrate over $\mathbb{C}P^2$. This yields

$$c_1(K^{-1})\cdot[\omega] = s\cdot c_1(K^{-1})\cdot[\omega] + s\cdot[\omega]\cdot[\omega], \tag{11.14}$$

which is impossible if $c_1(K^{-1})\cdot[\omega] < 0$. ■

Remark. For the standard Kähler structure $c_1(K)\cdot[\omega] < 0$, $c_1(K^{-1})\cdot[\omega] < 0$.

12. The Seiberg-Witten equation and pseudo-holomorphic curves. The coincidence of Gromov and Seiberg-Witten invariants is up to now one of the greatest achievements of Seiberg-Witten theory. It belongs essentially to the deep work of Cliff Taubes.

The main intention of Gromov was as follows. All symplectic manifolds of a fixed dimension are locally isomorphic. To distinguish one from the other one should look at some global object attached to them. This is Gromov's space of pseudo-holomorphic curves. Let Σ be a compact Riemann surface and (M, J) an almost complex manifold. The complex structure of Σ gives a canonical almost complex structure on it.

Definition. Let $f : \Sigma \to M$ be a smooth map. It is called a *pseudo-holomorphic curve* if its differential $\mathrm{d}f : T\Sigma \to TM$ is (i, J)-linear, i.e. $\mathrm{d}f \circ i = J \circ \mathrm{d}f$.

The general strategy is to show that the set of holomorphic curves can be provided with the structure of a compact manifold. This has been done by Gromov. If one additionally requires that Σ contains a finite set $\Omega \subset M$ of points then the set of all Σ becomes zero-dimensional.

Here we start with a closed symplectic manifold (M^4, ω). g and J will be assumed to be compatible with ω. The notions then can be reformulated. A 2-dimensional submanifold $\Sigma \subset (M^4, \omega)$ is called symplectic if the restriction of the symplectic form to $T\Sigma$ is nondegenerate. Then Σ is symplectic if and only if there is an ω-compatible almost complex structure which makes Σ pseudo-holomorphic.

Denote for (M^4, ω), J ω-compatible, $e \in H^2(M; \mathbb{Z})$

$$\mathcal{M}_{\mathrm{Gr}}(M, \omega, J, e) = \{\Sigma \mid \Sigma \text{ pseudo-holomorphic and } PD[\Sigma] = e\}.$$

THEOREM 12.1. *For a generic choice of a compatible almost complex structure J, the space*

$$\mathcal{M}_{\mathrm{Gr}}(M, \omega, J, e)$$

is a smooth $(-c_1(K) \cdot e + e \cdot e)$-dimensional manifold.

Remark. Since Σ is symplectic it is endowed with a canonical orientation and fundamental class $[\Sigma]$.

It follows from $c_1(K) \equiv w_2(M) \bmod 2$ that $d = -c_1(K) \cdot e + e \cdot e \equiv 0 \bmod 2$. Pick a set Ω of $\frac{d}{2}$ points in M generically.

THEOREM 12.2. *Then*

$$H_\Omega = \{\Sigma \in \mathcal{M}_{\mathrm{Gr}}(M, \omega, J, e) \mid \Omega \subset \Sigma\} = H_\Omega(e)$$

is a finite set of signed points.

DEFINITION. Let (M^4, ω) be a closed symplectic manifold. Then $\mathrm{Gr} : H^2(M^4; \mathbb{Z}) \to \mathbb{Z}$, $e \mapsto \mathrm{Gr}(e)$ is defined as follows.

a) When $d < 0$, set $\mathrm{Gr}(e) := 0$.

b) When $d = 0$, then for generic J, $\mathcal{M}_{\mathrm{Gr}}(M, \omega, J, e)$ is a finite set of signed points $(x_i, (-1)^{\varepsilon_i})$, $\varepsilon_i = 0$ or 1. Define $\mathrm{Gr}(e) := \sum\limits_i (-1)^{\varepsilon_i}$.

c) When $d > 0$, then for generic J and Ω, H_Ω is a finite set of signed points $(x_i, (-1)^{\varepsilon_i})$. Define $\mathrm{Gr}(e) := \sum\limits_i (-1)^{\varepsilon_i}$.

$\mathrm{Gr}(\cdot)$ is generically well defined.

MAIN THEOREM 12.3 (Taubes). *Let (M^4, ω) be a closed, oriented symplectic 4-manifold, E a nontrivial complex line bundle defining a $\mathrm{Spin}^{\mathbb{C}}(4)$ structure $L = \det S^+$, $S^+ = E \oplus K^{-1} \otimes E$. Then*

$$\mathrm{SW}(L) = \mathrm{Gr}(c_1(E)). \tag{12.1}$$

The general conclusion coming from Theorem 12.3 is the existence of certain pseudo-holomorphic curves for topological reasons. This shall be indicated by the following theorems.

THEOREM 12.4. *Assume (M^4, ω), E as in Theorem 12.3 with non-zero Seiberg-Witten invariant. Then $PD(c_1(E))$ can be represented by the fundamental class of an embedded symplectic curve which consists of some number N of components. Let Σ be any such component of genus $g = g(\Sigma)$ and let $e = PD[\Sigma] \in H^2(M^4; \mathbb{Z})$. Then $g = 1 + e \cdot e$. In* *particular, the Poincaré dual to* $c_1(K)$ *is represented by a symplectic curve. If* M^4 *has no embedded spheres of self-intersection* -1*, i.e.* M^4 *is minimal, then* $c_1(K) \cdot c_1(K) \geq 0$.

The first assertion follows immediately from Theorem 12.3. The other use other ingredients, in particular the general adjunction formula and the generalized Thom conjecture which has been formulated in Section 10. ∎

DEFINITION. Let M^4 be a closed oriented manifold, $b_2^+ \geq 1$, $c = c_1(L)$ a $\mathrm{Spin}^{\mathbb{C}}(4)$ structure. c is called a *basic class* if $\mathrm{SW}(M, c) \neq 0$. M^4 is said to *have simple type* if all basic classes have $d = \text{v-dim}\, \mathcal{M}_L = 0$.

THEOREM 12.5. *Let* (M^4, ω) *be as in Theorem* 12.3. *Then* (M^4, ω) *is simple.*

THEOREM 12.6. *Let* (M^4, ω) *be as in Theorem* 12.3 *and suppose that* M^4 *admits no symplectically embedded* 2*-spheres with self-intersection number* -1*, i.e.* (M^4, ω) *is minimal. Then*

$$-\frac{4}{3}(1 - b_1) - \frac{2}{3} b_2 \leq \sigma(M).$$

Proof. We obtain from (M^4, ω) minimal

$$0 = 4d = c_1^2(K) - (2\chi(M) + 3\sigma(M)) \geq -(2\chi(M) + 3\sigma(M))$$

which implies the assertion immediately. ∎

There are many other implications coming from Theorem 12.3. We refer to [19], [20].

The proof of Theorem 12.3 occupies more or less 100 pages. We can only indicate the main idea. One has to derive from a solution of (SW) a pseudo-holomorphic curve and, conversely, from such a curve the existence of a solution of (SW) and finally that equivalence classes are in a $1-1$-relation. For the first step, Taubes studies the perturbed equation with $\Psi = r^{\frac{1}{2}}(\alpha u_0 + \beta)$,

$$\begin{aligned} &u_0 \cdot \nabla_a \alpha + D_A \beta = 0 \\ &\Omega_A^+ = -\frac{i}{8} r(1 - |\alpha|^2 + |\beta|^2) \cdot \omega + \frac{ir}{4}(\alpha\beta^* + \alpha^*\beta). \end{aligned} \tag{12.2}$$

By a long series of estimates he can show that for a sequence $r_I \to \infty$ one obtains a reasonable limit for the sequence of zero sets $\alpha_{r_i}^{-1}(0)$. Finally he shows that this limit is pseudo-holomorphic. In the second step, Taubes constructs from a pseudo-holomorphic curve Σ in the class $c_1(L)$ a solution of (12.2) as follows. He grafts a rescaled by a factor $r^{\frac{1}{2}}$ solution of the vortex equation for $\mathbb{R}^2 = \mathbb{C}$,

$$\bar{\partial}_A \alpha = 0, \quad *iF_a = \frac{1}{8}(1 - |\alpha|^2),$$

into the normal bundle of Σ. The resulting (A, Ψ) is not yet a solution of (12.2). But by a correcting procedure similar to that in [21] an honest solution of (12.2) can be constructed. We refer to [20] for all details.

13. Further results. One of the achievements of Donaldson's theory was the smooth irreducibility of algebraic surfaces with respect to 4-manifolds with $b_2^+ > 0$. This result here comes out as a simple corollary. The main point is a formula for the Seiberg-Witten moduli space under gluing of 4-manifolds. We consider here the simplest case of gluing, namely the connected sum.

PROPOSITION 13.1. *Assume M^4, X_1^4, X_2^4 closed, oriented, with* $\mathrm{Spin}^{\mathbb{C}}(4)$ *structures* L, L_1, L_2, $M^4 = X_1 \# X_2$, $L = L_1 + L_2$. *If* $b_2^+(X_i) > 0$, $i = 1, 2$, *then* $\mathrm{SW}(M^4, L) = 0$.

Proof. Write $M^4 = X_1 \# X_2 \cong (X_1 \setminus \mathring{D}^4) \cup S^3 \times [-R, R] \cup (X_2 \setminus \mathring{D}^4)$ and endow $S^3 \times [-R, R]$ with the product metric. $\tau(S^3) > 0$ implies that for the limit $R \to \infty$ on the stretched neck there is only the solution ($\Phi = 0, A =$ flat). Consider now the based moduli spaces $\mathcal{M}_L^0(M^4)$, $\mathcal{M}_L^0(X_i^4)$, $i = 1, 2$. Then we obtain, if $\mathcal{M}_L(M^4) \neq 0$,

$$\mathcal{M}_L^0(M) \cong \mathcal{M}_{L_1}^0(X_1) \times \mathcal{M}_{L_2}^0(X_2)$$

since ($\Phi = 0, A =$ flat) is the only solution on the (sufficiently long) neck. This gives an $(S^1 \times S^1)$-action on $\mathcal{M}_L(M^4)$ given by the (independent) S^1-action on each side. Since $b_2^+(X_i) > 0$, neither based moduli space contains reducibles if generic metrics are used on both sides. Hence each S^1-action is free. Restrict now the $S^1 \times S^1$-action to the diagonal action and divide out $\mathcal{M}_L^0(M)$ by this section. This gives $\mathcal{M}_L(M)$. Then $\mathcal{M}_L(M)$ is a circle bundle over $\mathcal{M}_{L_1}(X_1) \times \mathcal{M}_{L_2}(X_2)$. For $d = 0$ this is impossible. Assume $d > 0$. Then one has to calculate $\int\limits_{\mathcal{M}_L(M)} (\varphi^* c_1^u)^{\max}$, where c_1^u is the universal Chern class and $\varphi : \mathcal{M}_L(M) \to B_{\mathrm{U}(1)} = \mathbb{C}P^\infty$ the classifying map of the bundle $\mathcal{M}_L^0(M) \to \mathcal{M}_L(M)$. Factorizing φ via $\mathcal{M}_{L_1}(X_1) \times \mathcal{M}_{L_2}(X_2)$, we obtain immediately

$$\int\limits_{\mathcal{M}_L(M)} (\varphi^* c_1^u)^{\max} = 0.$$

If we work with the bordism version of (SW) then it is clear that the total space of a smooth compact S^1-bundle is zero bordant. ■

Remark. Assume $b_2^+(X_2) = 0$. Irreducible solutions on X_2 give no contribution to $\mathcal{M}_L(M)$ as above, but reducible solutions can contribute. If $d = \text{v-dim}\, \mathcal{M}_{L_2}(X_2) = 0$ and $\mathcal{M}_{L_2}(X_2)$ consists only of reducible solutions then

$$\mathcal{M}_L(M^4) = \mathcal{M}_{L_1}(X_1) \times \mathcal{M}_{L_2}(X_2).$$

COROLLARY 13.2. *L is a basic class on M^4 if and only if $L \pm E$ is basic on $M^4 \# \overline{\mathbb{C}P^2}$* (*blow-up formula*).

For a detailed proof of Corollary 13.2 we refer to [1], p. 43.

We mention below two other important theorems concerning connected sums.

THEOREM 13.3. *Let Y and N be closed oriented* 4-*manifolds. If* $\mathrm{SW}(Y) \neq 0$, $b_1(N) = b_2^+(N) = 0$ *then* $\mathrm{SW}(Y \# N) \neq 0$.

THEOREM 13.4. *Assume* $\mathrm{SW}(Y^4) \neq 0$, *e.g.* Y^4 *symplectic with* $b_2^+(Y) > 1$, N^4 *with* $b_1(N) = b_2^+(N) = 0$ *and* $\pi_1(N)$ *with a nontrivial finite quotient. Then* $X = Y \# N$ *has nontrivial Seiberg-Witten invariant but does not admit any symplectic structure.*

The proof of Theorem 13.4 follows immediately from Theorem 13.3 and

Proposition 13.5. *Let (X^4, ω) be a closed symplectic manifold which decomposes as a smooth connected sum. Then one of the summands has negative definite intersection form and its fundamental group has nontrivial finite quotients.*

We refer to [11] for the proofs of Theorem 13.3 and Proposition 13.5. ■

Finally we conclude with several new results presented by Ono and Le Hong at the Aarhus conference on geometry and physics.

Let (M^{2n}, g, J) be Kählerian, H an ample line bundle, $c_1(M) = \lambda \cdot c_1(H)$, $\lambda > 0$. Then $\lambda \le n+1$. If $\lambda = n+1$, then $M^{2n} \cong \mathbb{C}P^n$. Another version of this consideration has been formulated as the following

Conjecture. Let M be a projective variety, $H^*(M;\mathbb{Z}) \cong H^*(\mathbb{C}P^n;\mathbb{Z})$ as rings, $c_1(M)$ positive. Does there hold

$$M \cong \mathbb{C}P^n?$$

For $n \le 5$ the answer is: yes.

One can consider a symplectic analogue. Let (M^n, ω) be symplectic. (M, ω) is called monotone (or positive) if $c_1(M) = \lambda \cdot [\omega]$, $\lambda > 0$.

Theorem 13.6. *Let (M^4, ω) be monotone. Then*
(1) $c_1^2(M) \le 9$, $b_1(M) = 0$, $b_2^+(M) = 1$, $\mathrm{td}(M) = 1$.
(2) $\pi_1(M)$ *has no proper subgroup of finite index.*

The proof essentially uses $\mathrm{SW}(K_M^{-1}) = \pm 1$ and reduces to a simple but carefully done of several cases coming from the assumption. ■

Concerning the classification of closed 4-dimensional symplectic manifolds, Dusa McDuff proved the following

Theorem 13.7. *Let (M^4, ω) be a closed symplectic 4-manifold which contains a symplectically embedded S^2 with non-negative self-intersection. Then M^4 is rational or a ruled surface up to blow-up or blow-down.*

Using Taubes' results, one presents several classes of (M^4, ω) admitting symplectically embedded S^2 with $S \cdot S \ge 0$.

Consider the following question. Let be given a homotopy class $[J]$ of almost complex structures on a closed 4-manifold M^4. Does there exist a compatible symplectic structure? A similar question can be formulated for the existence of Kähler structures. Then Ono and Le Hong proved that at least 50% of the possible $[J]$ do not admit a compatible symplectic structure if M^4 is minimal rational or a ruled surface.

For the proofs we refer to their paper in preparation.

Finally we mention some other developments. In [22] Zhang, Wang and Carey introduce a topological quantum field theory which reproduces the Seiberg-Witten invariants for 4-manifolds. Labastida and Marino develop in [13] a non-Abelian generalization of Witten's monopole equation and analyze the associated moduli problem.

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