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QUANTUM FIBRE BUNDLES. AN INTRODUCTION

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Abstract. An approach to construction of a quantum group gauge theory based on the quantum group generalisation of fibre bundles is reviewed.

1. Introduction and preliminaries.

1.1. *Introduction.* The algebraic approach to deformation-quantisation involves the replacing of the algebras of functions by non-commutative algebras. In recent years we have seen a rapid development of this approach to quantisation, initiated by Drinfeld's [17] realisation of Hopf algebras as deformations of Lie groups. Hopf algebras are now commonly called quantum groups. Quantum groups originated in the quantum inverse scattering method developed by the Petersburg School and applied to quantisation of completely integrable Hamiltonian systems. Nowadays, however, it is believed that quantisation-deformation and quantum groups in particular may be applied to the description of spaces at the Planck scale. Having this application in mind, it is important to develop a kind of gauge theory involving quantum groups. Such a theory was introduced by S. Majid and the author in [7] in the framework of fibre bundles with quantum structure groups. In this paper we review the main elements of the quantum group gauge theory of [7].

The article is organised as follows. In the remaining part of Section 1 we give a crash introduction to Hopf algebras and non-commutative differential geometry. The reader familiar with these topics may go directly to Section 2, where we describe elements of the theory of quantum fibre bundles. Then in Section 3 we present gauge theory of such fibre bundles. We conclude the paper with some remarks on other developments of quantum group gauge theory and open problems in Section 4.

1.2. *Hopf algebras.* A unital algebra H over a field k is called a *Hopf algebra* if there exist linear multiplicative maps: a *coproduct* $\Delta : H \to H \otimes H$ and a *counit* $\epsilon : H \to k$,

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and a linear antimultiplicative map $\mathrm{S} : H \to H$ (an *antipode*) which satisfy the following axioms [28]:

1. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$;
2. $(\mathrm{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$;
3. $m \circ (\mathrm{id} \otimes \mathrm{S}) = m \circ (\mathrm{S} \otimes \mathrm{id}) = 1\epsilon$.

Here and in what follows m denotes the multiplication map. One should think of a Hopf algebra as a non-commutative generalisation of the algebra of regular functions on a group. In this case Δ corresponds to the group multiplication and the axiom 1. states the associativity of this multiplication. Axiom 2. states the existence of the unit in a group and 3. is the existence of inverses of group elements, written in a dual form. For this reason Hopf algebras are also called *quantum groups*.

For a coproduct we use an explicit expression $\Delta(a) = a_{(1)} \otimes a_{(2)}$, where the summation is implied according to the Sweedler sigma convention [28], i.e. $a_{(1)} \otimes a_{(2)} = \sum_{i \in I} {a_{(1)}}^i \otimes {a_{(2)}}^i$ for an index set I. We also use the notation

$$a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n)} = (\Delta \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n-2}) \circ \cdots \circ (\Delta \otimes \mathrm{id}) \circ \Delta$$

which describes a multiple action of Δ on $a \in H$.

A vector space C with a coproduct $\Delta : C \to C \otimes C$ and the counit $\epsilon : C \to k$, satisfying axioms 1. and 2. is called a *coalgebra*.

A vector space V is called a *right H-comodule* if there exists a linear map $\rho_R : V \to V \otimes H$, called a *right coaction*, such that $(\rho_R \otimes \mathrm{id}) \circ \rho_R = (\mathrm{id} \otimes \Delta) \circ \rho_R$ and $(\mathrm{id} \otimes \epsilon) \circ \rho_R = \mathrm{id}$. We say that a unital algebra P over k is a *right H-comodule algebra* if P is a right H-comodule with a coaction $\Delta_R : P \to P \otimes H$, and Δ_R is a linear multiplicative or, equivalently, an algebra map. The algebra structure of $P \otimes H$ is that of a tensor product algebra. For a coaction Δ_R we use an explicit notation $\Delta_R u = u_{(0)} \otimes u_{(1)}$, where the summation is also implied. Notice that $u_{(0)} \in P$ and $u_{(1)} \in H$. If P is a right H-comodule so is $P \otimes P$ with a coaction Δ_R

$$\Delta_R(u \otimes v) = u_{(0)} \otimes v_{(0)} \otimes u_{(1)} v_{(1)}. \tag{1}$$

If P is a right H-comodule algebra then P^{coH} denotes a fixed point subalgebra of P, i.e. $P^{coH} = \{u \in P : \Delta_R u = u \otimes 1\}$. P^{coH} is a subalgebra of P with a natural inclusion $j : P^{coH} \hookrightarrow P$ which we do not write explicitly later on.

Let H be a Hopf algebra, B be a unital algebra over k, and let $f, g : H \to B$ be linear maps. A *convolution product* of f and g is a linear map $f * g : H \to B$ given by $(f * g)(a) = f(a_{(1)}) g(a_{(2)})$, for any $a \in H$. With respect to the convolution product, the set of all linear maps $H \to B$ forms an associative algebra with the unit 1ϵ. We say that a linear map $f : H \to B$ is *convolution invertible* if there is a map $f^{-1} : H \to B$ such that $f * f^{-1} = f^{-1} * f = 1\epsilon$. The set of all convolution invertible maps $H \to B$ forms a multiplicative group. Similarly if V is a right H-comodule and $f : V \to B$, $g : H \to B$ are linear maps then we define a convolution product $f * g : V \to B$ to be $(f * g)(v) = f(v_{(0)}) g(v_{(1)})$.

1.3. *Differential structures.* Let P be a unital algebra over k. Denote by $\Omega^1 P$ the P-bimodule $\ker m$, where $m : P \otimes P \to P$ is a multiplication map. Let $\mathrm{d_U} : P \to \Omega^1 P$ be a linear map

$$\mathrm{d_U} u = 1 \otimes u - u \otimes 1. \tag{2}$$

It can be easily checked that $\mathrm{d_U}$ is a differential, known as the Karoubi differential. We call the pair $(\Omega^1 P, \mathrm{d_U})$ the *universal differential structure on* P [21, 22]. $\Omega^1 P$ should be understood as a bimodule of 1-forms. We say that $(\Omega^1(P), \mathrm{d})$ is a *first order differential calculus* on P if there exists a subbimodule $\mathcal{N} \subset \Omega^1 P$ such that $\Omega^1(P) = \Omega^1 P/\mathcal{N}$ and $\mathrm{d} = \pi \circ \mathrm{d_U}$, where $\pi : \Omega^1 P \to \Omega^1(P)$ is a canonical projection. It is then said that $(\Omega^1(P), \mathrm{d})$ is generated by $\mathcal{N}$. Let a differential structure $(\Omega^1(H), \mathrm{d})$ on a Hopf algebra H be generated by $\mathcal{N} \subset \Omega^1 H$. We say that $(\Omega^1(H), \mathrm{d})$ is a *bicovariant differential calculus* [31] if there exists a unique right ideal $\mathcal{Q} \subset \ker \epsilon$ such that $H \otimes \mathcal{Q} = \kappa(\mathcal{N})$, where $\kappa : H \otimes H \to H \otimes H$, $\kappa : a \otimes b \mapsto ab_{(1)} \otimes b_{(2)}$, and $\mathrm{Ad_R}(\mathcal{Q}) \subset \mathcal{Q} \otimes H$, where $\mathrm{Ad_R} : H \to H \otimes H$ is a right adjoint coaction

$$\mathrm{Ad_R} : a \mapsto a_{(2)} \otimes (\mathrm{S}a_{(1)}) a_{(3)}. \tag{3}$$

The universal differential envelope is the unique differential algebra $(\Omega P, \mathrm{d})$ containing $(\Omega^1 P, \mathrm{d_U})$ as its first order part.

2. Fibre bundles. In this section we report the basic elements of the theory of quantum fibre bundles of S. Majid and the author [7]. The detailed analysis of quantum group gauge theory on classical spaces may be found in [8]. All the algebras are over a field k of complex or real numbers. Except for Section 2.4 and Example 3.1.4 we work with the universal differential structure.

2.1. *Quantum principal bundles.* Let H be a Hopf algebra, P a right H-comodule algebra with a coaction $\Delta_R : P \to P \otimes H$. We define a canonical map $\chi : P \otimes P \to P \otimes H$,

$$\chi = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R). \tag{4}$$

Explicitly, $\chi(u \otimes v) = uv_{(0)} \otimes v_{(1)}$, for any $u, v \in P$. We say that the coaction Δ_R is *free* if χ is a surjection and it is *exact* if $\ker \chi = P(\mathrm{d}P^{coH})P$, where d denotes the universal differential (2) and P^{coH} is a fixed point subalgebra of P. We denote $P(\mathrm{d}P^{coH})P$ by $\Omega^1 P_{\mathrm{hor}}$ and call its elements *horizontal forms.* Although the freeness and exactness conditions are algebraic in this formulation one should notice that in fact the latter one is a condition on differential structures on P and P^{coH}. This becomes clear in Section 2.4. The map $\chi\mid_{\Omega^1 P}$ has a natural geometric interpretation as a dual to the map $\mathcal{G} \to T_u X$, which to each element of the Lie algebra $\mathcal{G}$ of a group G associates a fundamental vector field on a manifold X on which G acts.

DEFINITION 2.1.1. Let H be a Hopf algebra, (P, Δ_R) be a right H-comodule algebra and let $B = P^{coH}$. We say that $P(B, H)$ is a *quantum principal bundle* within the differential envelope, with a structure quantum group H and a base B if the coaction Δ_R is free and exact.

This definition reproduces the classical situation (but in a dual language) in which a group G acts freely on a total space X from right, and a base manifold M is defined as $M = X/G$. The freeness of the action of G on X means that a map $X \times G \to X \times X$, $(u, g) \mapsto (u, ug)$ is an inclusion. In the classical situation and the commutative differential structure the exactness follows from the freeness. This is no longer true in a non-commutative extension.

The notion of a quantum principal bundle is strictly related to the theory of algebraic extensions [27,2] since $P(B,H)$ is a Hopf-Galois extension of B to P by a Hopf algebra H. Yet another way of defining of a quantum principal bundle makes use of the notion of a *translation map*, which proves very useful in analysis of the structure of quantum bundles [5].

PROPOSITION 2.1.2. *Let H be a Hopf algebra, P a right H-comodule algebra and $B = P^{coH}$. Assume that the coaction Δ_R is free. Then $P(B,H)$ is a quantum principal bundle iff there exists a linear map $\tau : H \to P \otimes_B P$, given by $\tau(a) = \sum_{i\in I} u_i \otimes_B v_i$, where $\sum_{i\in I} u_i \otimes v_i \in \chi^{-1}(1 \otimes a)$. The map τ is called a translation map.*

A translation map is a well-known object in the classical bundle theory [20, Definition 4.2.1]. Classically, if X is a manifold on which a Lie group G acts freely then the translation map $\hat{\tau} : X \times_M X \to G$, where $M = X/G$, is defined by $u\hat{\tau}(u,v) = v$. Dualising this construction we arrive immediately at the map τ above.

2.2. *Examples of quantum principal bundles.*

EXAMPLE 2.2.1. *A trivial quantum principal bundle.* Let H be a Hopf algebra, P a right H-comodule algebra and $B = P^{coH}$. Assume there is a convolution invertible map $\Phi : H \to P$ such that $\Delta_R\Phi = (\Phi \otimes \mathrm{id})\Delta,\quad \Phi(1) = 1$, i.e. Φ is an intertwiner. Then $P(B,H)$ is a quantum principal bundle called a *trivial quantum principal bundle* and denoted by $P(B,H,\Phi)$. The word *trivial* refers to the fact that $P \cong B \otimes H$ as vector spaces with an isomorphism $\Theta_\Phi : P \to B \otimes H$, $\Theta_\Phi : u \mapsto u_{(0)}\Phi^{-1}(u_{(1)}) \otimes u_{(2)}$. Moreover, as algebras $P \cong B_\Phi\#H$, where ${}_\Phi\#$ denotes a crossed product [1], with the isomorphism Θ_Φ above. Explicitly, the product in $B_\Phi\#H$ is given by

$$(b_1 \otimes a^1)(b_2 \otimes a^2) = b^1\Phi({a^1}_{(1)})b_2\Phi({a^2}_{(1)})\Phi^{-1}({a^1}_{(2)}{a^2}_{(2)}) \otimes a^1_{(3)}a^2_{(3)}.$$

Such an algebra P is also known as a *cleft extension* of B [29, 16].

The map $\tau = (\Phi^{-1} \otimes_B \Phi) \circ \Delta$ is a translation map in $P(B,H,\Phi)$.

For a trivial quantum principal bundle $P(B,H,\Phi)$ we define a *gauge transformation* as a convolution invertible map $\gamma : H \to B$ such that $\gamma(1) = 1$. The set of all gauge transformations of $P(B,H,\Phi)$ forms a group with respect to the convolution product. This group is denoted by $\mathcal{H}(B)$. Gauge transformations relate different trivialisations of $P(B,H,\Phi)$: $\Psi : H \to P$ is a trivialisation of $P(B,H,\Phi)$ iff there exists $\gamma \in \mathcal{H}(B)$ such that $\Psi = \gamma * \Phi$. They also have a clear meaning in the theory of crossed products. The following proposition is a special case of the result of Doi [15] (see also [23, Proposition 4.2]).

PROPOSITION 2.2.2. *Let $P(B,H,\Phi)$ be a trivial quantum principal bundle. Let for any trivialisation Ψ of $P(B,H,\Phi)$, $\Theta_\Psi : B_\Psi\#H \to B_\Phi\#H$ be a crossed product algebra isomorphism such that $\Theta_\Psi\mid_B = \mathrm{id}$ and $\Delta_R\Theta_\Psi = (\Theta_\Psi \otimes \mathrm{id})\Delta_R$. Then there is a bijective correspondence between all isomorphisms Θ_Ψ corresponding to all trivialisations Ψ and the gauge transformations of $P(B,H,\Phi)$.*

EXAMPLE 2.2.3. *Quantum principal bundle on a quantum homogeneous space.* Let H and P be Hopf algebras. Assume, there is a Hopf algebra projection $\pi : P \to H$. Define a right coaction of H on P by $\Delta_R = (\mathrm{id} \otimes \pi)\Delta : P \to P \otimes H$. Then $B = P^{coH}$ is a *quantum quotient space*, a special case of a *quantum homogeneous space*. Assume that $\ker \pi \subset m \circ (\ker \pi \mid_B \otimes P)$. Then $P(B, H)$ is a quantum principal bundle within the differential envelope. This bundle is denoted by $P(B, H, \pi)$.

The translation map $\tau : H \to P \otimes_B P$ in $P(B, H, \pi)$ is given by $\tau(a) = \mathrm{S}u_{(1)} \otimes_B u_{(2)}$, where $u \in \pi^{-1}(a)$.

A large number of examples of quantum bundles on quantum homogeneous spaces has been found in [24]. The simplest and probably the most fundamental one is

EXAMPLE 2.2.4. *The quantum Hopf fibration* [7, Section 5.2]. The total space of this bundle is the quantum group $SU_q(2)$, as an algebra generated by the identity and a matrix $T = (t_{ij}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, subject to the homogeneous relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\delta = \delta\alpha + (q - q^{-1})\beta\gamma, \quad \beta\gamma = \gamma\beta, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma,$$

and a determinant relation $\alpha\delta - q\beta\gamma = 1$, $q \in k^*$. $SU_q(2)$ has a matrix quantum group structure,

$$\Delta t_{ij} = \sum_{k=1}^{2} t_{ik} \otimes t_{kj}, \quad \epsilon(t_{ij}) = \delta_{ij}, \quad \mathrm{S}T = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}.$$

The structure quantum group of the quantum Hopf bundle is an algebra of functions on $U(1)$, i.e. the algebra $k[Z, Z^{-1}]$ of formal power series in Z and Z^{-1}, where Z^{-1} is an inverse of Z. It has a standard Hopf algebra structure

$$\Delta Z^{\pm 1} = Z^{\pm 1} \otimes Z^{\pm 1}, \quad \epsilon(Z^{\pm 1}) = 1, \quad \mathrm{S}Z^{\pm 1} = Z^{\mp 1}.$$

There is a Hopf algebra projection $\pi : SU_q(2) \to k[Z, Z^{-1}]$,

$$\pi : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} Z & 0 \\ 0 & Z^{-1} \end{pmatrix},$$

which defines a right coaction $\Delta_R : SU_q(2) \to SU_q(2) \otimes k[Z, Z^{-1}]$ by $\Delta_R = (\mathrm{id} \otimes \pi) \circ \Delta$. Finally $S_q^2 \subset SU_q(2)$ is a quantum two-sphere [26], defined as a fixed point subalgebra, $S_q^2 = SU_q(2)^{cok[Z,Z^{-1}]}$. S_q^2 is generated by $\{1, b_- = \alpha\beta, b_+ = \gamma\delta, b_3 = \alpha\delta\}$ and the algebraic relations in S_q^2 may be deduced from those in $SU_q(2)$.

It was shown in [7] that $SU_q(2)(S_q^2, k[Z, Z^{-1}], \pi)$ is a non-trivial quantum principal bundle over the homogeneous space.

The other examples of quantum principal bundles constructed in [24] include:

$$U_q(n)(S_q^{2n-1}, U_q(n-1), \pi),$$
$$SU_q(n)(S_q^{2n-1}, SU_q(n-1), \pi),$$
$$SU_q(n)(\mathbf{CP}_q^{n-1}, U_q(n-1), \pi),$$
$$U_q(n)(G_k(\mathbf{C}_q^n), U_q(k) \otimes U_q(n-k), \pi),$$

where $G_k(\mathbf{C}_q^n)$ is a quantum Grassmannian.

R e m a r k 2.2.5. The quantum sphere S_q^2 considered in Example 2.2.4. is the special case of the most general quantum sphere $S_q^2(\mu,\nu)$, where $\mu \neq \nu$ are real parameters such that $\mu\nu \geq 0$ (see [26] for details). Precisely $S_q^2 = S_q^2(1,0)$. It can be shown that S_q^2 is the only quantum sphere which can be interpreted as a quotient space of $SU_q(2)$ by $k[Z, Z^{-1}]$ in the sense of Example 2.2.3. It turns out, however, that $S_q^2(\mu,\nu)$ may be viewed as a quotient space of $SU_q(2)$ by a *coalgebra* $C = SU_q(2)/J$, where J is a right ideal in $SU_q(2)$ generated by

$$p(q\alpha^2 - \beta^2) + \alpha\beta - pq, \quad p(q\gamma^2 - \delta^2) + \gamma\delta + p, \quad p(q\alpha\gamma - \beta\delta) + q\beta\gamma,$$

where $p = \sqrt{\mu\nu}/(\mu - \nu)$ [6]. Precisely

$$S_q^2(\mu,\nu) = \{u \in SU_q(2);\ \ u_{(1)} \otimes \pi(u_{(2)}) = u \otimes \pi(1)\},$$

where $\pi : SU_q(2) \to C$ is the canonical surjection. It can be shown that the vector space C is spanned by $1 = \pi(1)$, $x_n = \pi(\alpha^n)$ and $y_n = \pi(\delta^n)$ (cf. definition of π in Example 2.2.4).

One would like to view $SU_q(2)$ as a total space of a quantum principal bundle over $S_q^2(\mu,\nu)$ similarly as in Example 2.2.4. Since C is not a Hopf algebra one needs to generalise the notion of a bundle. In [6] we proposed the following generalisation of Definition 2.1.1 (this generalisation of quantum group gauge theory is further developed in [9]). Let C be a coalgebra and let P be an algebra and a right C-comodule. Assume that there is an action $\rho : P \otimes C \otimes P \to P \otimes C$ of P on $P \otimes C$ and an element $1 \in C$ such that $\Delta_R \circ m = \rho \circ (\Delta_R \otimes \mathrm{id})$ and for any $u, v \in P$, $\rho(u \otimes 1 \otimes v) = \chi(u \otimes v)$. Then $B = \{u \in P;\ \ \Delta_R u = u \otimes 1\}$ is a subalgebra of P, and we say that $P(B, C, \rho)$ is a *quantum ρ-principal bundle* over B if the coaction Δ_R is free and exact.

In the above example of the quantum sphere $S_q^2(\mu,\nu)$ the action ρ is given by $\rho(u \otimes c, v) = uv_{(1)} \otimes \rho_0(c, v_{(2)})$, where ρ_0 is a natural right action of $SU_q(2)$ on C.

2.3. *Quantum associated bundles.*

Definition 2.3.1. Let $P(B,H)$ be a quantum principal bundle and let V be a right H^{op}-comodule algebra, where H^{op} denotes the algebra which is isomorphic to H as a vector space but has an opposite product, with coaction $\rho_R : V \to V \otimes H$. The space $P \otimes V$ is naturally endowed with a right H-comodule structure $\Delta_E : P \otimes V \to P \otimes V \otimes H$ given by $\Delta_E(u \otimes v) = u_{(0)} \otimes v_{(0)} \otimes u_{(1)} v_{(1)}$ for any $u \in P$ and $v \in V$. We say that the fixed point subalgebra E of $P \otimes H$ with respect to Δ_E is a *quantum fibre bundle associated to* $P(B,H)$ over B with structure quantum group H and standard fibre V. We denote it by $E = E(B,V,H)$.

It can be easily shown that B is a subalgebra of E with the inclusion $j_E = b \otimes 1$. The inclusion j_E provides E with the structure of a left B-module.

Example 2.3.2. Let $P(B,H,\Phi)$ be a trivial quantum principal bundle and let V be as in Definition 2.3.1. Assume also that H has a bijective antipode. The associated bundle $E(B,V,H)$ is called a *trivial quantum fibre bundle*. Trivialisation $\Phi : H \to P$ induces a map $\Phi_E : V \to E$, $\Phi_E(v) = \sum \Phi(\mathrm{S}^{-1} v_{(1)}) \otimes v_{(0)}$ which allows one to identify E with $B \otimes V$ as vector spaces via the linear isomorphism $b \otimes v \mapsto b\Phi_E(v)$. As an algebra, E is isomorphic to a certain crossed product algebra $B\#V$ [3].

The following proposition shows that a quantum principal bundle is a fibre bundle associated to itself.

PROPOSITION 2.3.3. *A quantum principal bundle $P(B, H)$ is a fibre bundle associated to $P(B, H)$ with the fibre which is isomorphic to H as an algebra and with the coaction $\rho_R = (\mathrm{id} \otimes \mathrm{S}) \circ \Delta'$, where Δ' denotes the opposite coproduct, $\Delta'(a) = a_{(2)} \otimes a_{(1)}$, for any $a \in H$.*

From the point of view of a gauge theory it is important to consider cross-sections of a vector bundle. In this algebraic setting a cross-section is defined as follows

DEFINITION 2.3.4. Let $E(B, V, H)$ be a quantum fibre bundle associated to a quantum principal bundle $P(B, H)$. A left B-module map $s : E \to B$ such that $s(1) = 1$ is called a *cross section* of $E(B, V, H)$. The set of cross sections of $E(B, V, H)$ is denoted by $\Gamma(E)$.

LEMMA 2.3.5. *If $s : E \to B$ is a cross section of a quantum fibre bundle $E(B, V, A)$ then $s \circ j_E = \mathrm{id}$.*

The result of trivial Lemma 2.3.5 justifies the term *cross section* used in Definition 2.3.4. We remark that the definition of a cross section of a quantum fibre bundle analogous to the one we use here was first proposed in [19]. We analyse cross-sections more closely in Section 3.3.

2.4. *Quantum principal bundles with general differential structures.* The detailed analysis of quantum principal bundles with general differential structures goes far beyond the scope of this paper. Here we give only a definition of a quantum principal bundle with general differential structure. We refer the interested reader to the fundamental paper [7]. More explicit exposition may be also found in [3].

Let $(\Omega^1(P), \mathrm{d})$ be a first order differential calculus on a right H-comodule algebra P generated by $\mathcal{N} \subset \Omega^1 P$ and let $(\Omega^1(H), \mathrm{d})$ be a bicovariant differential structure on H generated by the right ideal $\mathcal{Q} \subset \ker \epsilon$. We say that differential structures $(\Omega^1(P), \mathrm{d})$ and $(\Omega^1(H), \mathrm{d})$ *agree* with each other if $\Delta_R(\mathcal{N}) \subset \mathcal{N} \otimes H$, where Δ_R is given by (1), and $\chi(\mathcal{N}) \subset P \otimes \mathcal{Q}$. If differential structures on P and H agree we can define a map $\chi_{\mathcal{N}} : \Omega^1(P) \to P \otimes \ker \epsilon/\mathcal{Q}$ as follows. Let $\pi_{\mathcal{N}} : \Omega^1 P \to \Omega^1(P)$ and $\pi_{\mathcal{Q}} : \ker \epsilon \to \ker \epsilon/\mathcal{Q}$ be canonical projections. Then for any $\rho \in \Omega^1(P)$ take any $\rho_U \in \pi_{\mathcal{N}}^{-1}(\rho)$ and define $\chi_{\mathcal{N}}(\rho) = (\mathrm{id} \otimes \pi_{\mathcal{Q}}) \circ \chi(\rho_U)$, where χ is a canonical map (4). We say that the coaction $\Delta_R : P \to P \otimes H$ is *exact* with respect to differential structures generated by $\mathcal{N}$ and $\mathcal{Q}$ if $\ker \chi_{\mathcal{N}} = P\Omega^1(P^{coH})P$. Finally we define a quantum principal bundle with $P(B, H)$ with differential structure generated by $\mathcal{N}$ and $\mathcal{Q}$ if the coaction Δ_R is free and exact with respect to this structure.

3. Gauge Theory. In this section we analyse more closely the structure of quantum bundles. We introduce the formalism of connections and take a closer look at cross sections and gauge transformations in general (non-trivial) quantum bundles.

3.1. *Connections = gauge fields.* From the point of view of gauge theories connections in principal bundles are the gauge fields. In the definition of a connection an important rôle is played by a right adjoint coaction of H on itself (3). Since $\mathrm{Ad_R}(\ker \epsilon) \subset \ker \epsilon \otimes H$, we can define a coaction $\Delta_R : P\otimes\ker\epsilon \to P\otimes\ker\epsilon\otimes H$ by $\Delta_R(u\otimes a) = u_{(0)}\otimes a_{(2)}\otimes u_{(1)}(\mathrm{S}a_{(1)})a_{(3)}$.The canonical map $\chi:\Omega^1P\to P\otimes\ker\epsilon$ is equivariant, i.e. $\Delta_R\chi = (\chi\otimes\mathrm{id})\Delta_R$, where Δ_R on Ω^1P is given by (1). From the definition of a quantum principal bundle we deduce that the following sequence

$$0\to\Gamma_{\mathrm{hor}}\xrightarrow{j}\Omega^1P\xrightarrow{\chi}P\otimes\ker\epsilon\to 0$$

is an exact sequence of equivariant maps. A connection in $P(B,H)$ is a right-invariant splitting of this sequence. In other words, if there is a map $\sigma : P\otimes\ker\epsilon\to\Omega^1P$ such that $\Delta_R\sigma = (\sigma\otimes\mathrm{id})\Delta_R$ and $\chi\circ\sigma = \mathrm{id}$, then a connection in $P(B,H)$ is identified with a linear projection $\Pi:\Omega^1P\to\Omega^1P$, $\Pi = \sigma\circ\chi\,|_{\Omega^1P}$. Obviously, $\Delta_R\Pi = (\Pi\otimes\mathrm{id})\Delta_R$. The connection Π is *strong* if and only if $(\mathrm{id}-\Pi)\mathrm{d}P\subset\Omega^1BP$, [19].

We denote $\Omega^1P_{\mathrm{ver}} = \mathrm{Im}\,\Pi$. Every $\alpha\in\Omega^1P_{\mathrm{ver}}$ is said to be a *vertical* 1*-form*. If there is a connection in $P(B,H)$, then $\Omega^1P = \Omega^1P_{\mathrm{hor}}\oplus\Omega^1P_{\mathrm{ver}}$.

Next we define a map $\omega : H\to\Omega^1P$, by

$$\omega(a) = \sigma(1\otimes(a-\epsilon(a))).$$

The map ω is called a *connection* 1*-form* of the connection Π.

THEOREM 3.1.1. *Let $P(B,H)$ be a quantum principal bundle and let Π be a connection in $P(B,H)$. A connection form ω has the following properties*:

1. $\omega(1) = 0$;
2. $\forall\, a\in H,\quad \chi\omega(a) = 1\otimes(a-\epsilon(a))$;
3. $\Delta_R\circ\omega = (\omega\otimes\mathrm{id})\circ\mathrm{Ad_R}$.

Conversely, *if $\omega : H\to\Omega^1P$ is a linear map obeying* $1-3$, *then* $\Pi = m\circ(\mathrm{id}\otimes\omega)\chi\,|_{\Omega^1P}$ *is a connection with a connection* 1*-form* ω.

Having a connection Π in a quantum principal bundle $P(B,H)$ one can define the horizontal projection as a complementary part of Π, and a covariant derivative as a horizontal part of d (for details see [7]). As a result one defines a curvature of a strong connection ω as $F = \mathrm{d}\omega+\omega*\omega$ [19].

EXAMPLE 3.1.2. *Strong connection in a trivial bundle.* Let $P(B,H,\Phi)$ be a trivial quantum principal bundle as before, and let $\beta : H\to\Omega^1B$ be any linear map such that $\beta(1) = 0$. Then the map $\omega = \Phi^{-1}*\beta*\Phi+\Phi^{-1}*\mathrm{d}\Phi$ is a connection 1-form in $P(B,H,\Phi)$. Its curvature is easily computed to be $F = \Phi^{-1}*(\mathrm{d}\beta+\beta*\beta)*\Phi$.

EXAMPLE 3.1.3. *Canonical connection.* Let $P(B,H,\pi)$ be a quantum principal bundle over the homogeneous space B as described in Example 2.2.3. Assume, there is an algebra inclusion $i : H\hookrightarrow P$ such that $\pi\circ i = \mathrm{id}$, $\epsilon_P(i(a)) = \epsilon_H(a)$, for any $a\in H$ and such that $(\mathrm{id}\otimes\pi)\mathrm{Ad_R}i = (i\otimes\mathrm{id})\mathrm{Ad_R}$. Then the map $\omega(a) = \mathrm{S}i(a)_{(1)}\mathrm{d}i(a)_{(2)}$ is a connection 1-form in $P(B,H,\pi)$. This connection is strong if i is an intertwiner for the right coaction [3, Lemma 5.5.5].

EXAMPLE 3.1.4. *The Dirac q-monopole.* Consider the quantum Hopf fibration of Example 2.2.4. Let a differential structure $(\Omega^1(SU_q(2)),\mathrm{d})$ be given by the 3D calculus of Woronowicz [30]. $\Omega^1(SU_q(2))$ is generated by the forms $\omega^0 = \delta\mathrm{d}\beta-q^{-1}\beta\mathrm{d}\delta$, $\omega^1 =$

$\delta \mathrm{d}\alpha - q^{-1}\beta \mathrm{d}\gamma$, $\omega^2 = \gamma \mathrm{d}\alpha - q^{-1}\alpha \mathrm{d}\gamma$ and the relations

$$\omega^0\alpha = q^{-1}\alpha\omega^0, \quad \omega^0\beta = q\beta\omega^0, \quad \omega^1\alpha = q^{-2}\alpha\omega^1,$$
$$\omega^1\beta = q^2\beta\omega^1, \quad \omega^2\alpha = q^{-1}\alpha\omega^2, \quad \omega^2\beta = q\beta\omega^2.$$

The remaining relations can be obtained by the replacement $\alpha \to \gamma$, $\beta \to \delta$. One can show that $SU_q(2)(S^2_q, k[Z, Z^{-1}], \pi)$ is a quantum principal bundle with this differential structure. We define the connection one form $\omega : k[Z, Z^{-1}] \to \Omega^1(SU_q(2))$ by

$$\omega(Z^n) = \frac{q^{-2n} - 1}{q^{-2} - 1}\omega^1.$$

In [7] it has been shown that ω is a canonical connection in $SU_q(2)(S^2_q, k[Z, Z^{-1}], \pi)$ which reduces to the Dirac monopole of charge 1 [18] when $q \to 1$. The curvature of ω is $F(Z^n) = \frac{q^{-2n}-1}{q^{-2}-1}\omega^0 \wedge \omega^2$. The q-deformed Dirac monopole of any charge is discussed in [11].

3.2. *Cross sections = matter fields.* In this section we use the notion of a translation map in a quantum principal bundle $P(B, H)$ to identify cross sections of a quantum fibre bundle $E(B, V, H)$ with equivariant maps $V \to P$. In gauge theories such maps play a rôle of matter fields. Recall that a linear map $\phi : V \to P$ is said to be equivariant if $\Delta_R\phi = (\phi \otimes \mathrm{id})\rho_R$, where ρ_R is a right coaction of H on V. In particular, our identification implies that a quantum principal bundle is trivial if it admits a cross section which is an algebra map.

THEOREM 3.2.1. *Let H be a Hopf algebra with a bijective antipode. Cross sections of a quantum fibre bundle $E(B, V, H)$ associated to a quantum principal bundle $P(B, H)$ are in bijective correspondence with equivariant maps $\phi : V \to P$ such that $\phi(1) = 1$.*

Proof. A map $\phi : V \to P$ induces a cross section s of $E(B, V, H)$, by $s = m \circ (\mathrm{id} \otimes \phi)$. Conversely, for any $s \in \Gamma(E)$ we define a map $\phi : V \to P$ by

$$\phi : v \mapsto \tau^{(1)}(\mathrm{S}^{-1}v_{(1)})s(\tau^{(2)}(\mathrm{S}^{-1}v_{(1)}) \otimes v_{(0)}), \tag{5}$$

where $\tau(a) = \tau^{(1)}(a) \otimes_B \tau^{(2)}(a)$ is a translation map in $P(B, H)$, and then use properties of a translation map to prove that ϕ has the required properties and that the correspondence $\theta : \phi \mapsto s$ is bijective. ∎

EXAMPLE 3.2.2. Let $E(B, V, H)$ be a quantum fibre bundle associated to a trivial quantum principal bundle $P(B, H, \Phi)$ as described in Example 2.3.2. In this case every element of E has the from $\sum_{i\in I} b_i \Phi_E(v_i)$ for some $b_i \in B$ and $v_i \in V$, and the bijection θ of the proof of Theorem 3.2.1 reads

$$\theta(\phi)(\sum_{i\in I} b_i \Phi_E(v_i)) = \sum_{i\in I} b_i \Phi(\mathrm{S}^{-1}v_{i(1)})\phi(v_{i(0)}),$$

for any equivariant $\phi : V \to P$. The inverse of θ associates an equivariant map $\theta^{-1}(s) : V \to P$,

$$\theta^{-1}(s)(v) = \Phi^{-1}(\mathrm{S}^{-1}v_{(1)})s(\Phi_E(v_{(0)}))$$

to any $s \in \Gamma(E)$. Notice that the map $\theta^{-1}(s)$ obtained in this way is different from the equivariant map ϕ discussed in [7, Proposition A6].

COROLLARY 3.2.3. *Cross sections $s : P \to B$ of a quantum principal bundle $P(B, H)$ are in bijective correspondence with the maps $\phi : H \to P$ such that $\Delta_R\phi = (\phi \otimes \mathrm{S})\Delta'$ and $\phi(1) = 1$.*

Note that in Corollary 3.2.3 we do not need the invertibility of S, but if H has a bijective antipode S, the sections of a quantum principal bundle $P(B, H)$ are in one-to-one correspondence with the maps $\psi : H \to P$ such that $\psi(1) = 1$ and $\Delta_R \circ \psi = (\mathrm{id} \otimes \psi) \circ \Delta$. We simply need to define $\psi = \phi \circ \mathrm{S}^{-1}$, where ϕ is given by Corollary 3.2.3.

PROPOSITION 3.2.4. *Any trivial quantum principal bundle $P(B, H, \Phi)$ admits a section. Conversely, if a bundle $P(B, H)$ admits a section which is an algebra map then $P(B, H)$ is trivial with the total space P isomorphic to $B \otimes H$ as an algebra.*

Proof. A convolution inverse of a trivialisation Φ of a trivial quantum principal bundle $P(B, H, \Phi)$ satisfies the assumptions of Corollary 3.2.3, hence $s = \mathrm{id} * \Phi^{-1}$ is a section of $P(B, H, \Phi)$. Conversely, assume that an algebra map $s : P \to B$ is a section of $P(B, H)$. Clearly, s is a B-bimodule map, hence we can define a linear map $\Phi : H \to P$, $\Phi = m \circ (s \otimes_B \mathrm{id}) \circ \tau$. One then shows that Φ is a trivialisation and $\tilde{\theta}(s)$ constructed in Corollary 3.2.3 is its convolution inverse. ■

Remark 3.2.5. We would like to emphasise that the existence of a cross section of a quantum principal bundle does not necessarily imply that the bundle is trivial. As an example of a non-trivial quantum principal bundle admitting a cross section we consider the quantum Hopf fibration of Example 2.2.4. We consider a linear map $\phi : k[Z, Z^{-1}] \to SU_q(2)$, given by

$$\phi(1) = 1, \qquad \phi(Z^n) = \delta^n, \qquad \phi(Z^{-n}) = \alpha^n,$$

for any positive integer n. The map ϕ satisfies the hypothesis of Corollary 3.2.3, hence it induces a cross section $s : SU_q(2) \to S_q^2$, $s : u \mapsto u_{(1)}\phi(\pi(u_{(2)}))$ but s is not an algebra map since, for example, $s(\alpha\beta) = b_- \neq q^{-1}b_3b_- = s(\alpha)s(\beta)$.

3.3. *Vertical automorphisms = gauge transformations.*

DEFINITION 3.3.1. Let $P(B, H)$ be a quantum principal bundle. Any left B-module automorphism $\mathcal{F} : P \to P$ such that $\mathcal{F}(1) = 1$ and $\Delta_R\mathcal{F} = (\mathcal{F} \otimes \mathrm{id})\Delta_R$ is called a *vertical automorphism* of the bundle $P(B, H)$. The set of all vertical automorphisms of $P(B, H)$ is denoted by $Aut_B(P)$.

Elements of $Aut_B(P)$ preserve both the base space B and the action of the structure quantum group H of a quantum principal bundle $P(B, H)$. $Aut_B(P)$ can be equipped with a multiplicative group structure $\cdot : (\mathcal{F}_1, \mathcal{F}_2) \mapsto \mathcal{F}_2 \circ \mathcal{F}_1$. Vertical automorphisms are often called gauge transformations and $Aut_B(P)$ is termed a gauge group.

PROPOSITION 3.3.2. *Vertical automorphisms of a quantum principal bundle $P(B, H)$ are in bijective correspondence with convolution invertible maps $f : H \to P$ such that $f(1) = 1$ and $\Delta_R f = (f \otimes \mathrm{id})\mathrm{Ad_R}$.*

Proof. If f is a map satisfying the hypothesis of the proposition. then $\mathcal{F} = \mathrm{id} * f$. Conversely, for any $\mathcal{F} \in Aut_B(P)$ a map $f : H \to P$, $f = m \circ (\mathrm{id} \otimes_B \mathcal{F}) \circ \tau$, where τ is a translation map has all the required properties. ■

Maps $f : H \to P$ form a group with respect to the convolution product. This group is denoted by $\mathcal{H}(P)$. There is an action of $\mathcal{H}(P)$ on the space of connection one-forms in $P(B, A)$ given by $(\omega, f) \mapsto \omega^f = f^{-1} * \omega * f + f^{-1} * \mathrm{d}f$. The connection one-form ω^f is called a gauge transformation of ω. If ω is strong so is its gauge transformation. Gauge transformation of such ω induces the gauge transformation of its curvature $F \mapsto f^{-1} * F * f$. Similarly there is an action of $\mathcal{H}(P)$ on $\Gamma(E)$ viewed as equivariant maps $\phi : V \to P$ by Theorem 3.2.1, given by $(\phi, f) \mapsto \phi^f = \phi * f$. These are the transformation properties of the fields in quantum group gauge theories.

Proposition 3.3.2 implies the following:

COROLLARY 3.3.3. *For a quantum principal bundle $P(B, H)$, $Aut_B(P) \cong \mathcal{H}(P)$ as multiplicative groups.*

THEOREM 3.3.4. *Let $P(B, H, \Phi)$ be a trivial quantum principal bundle. Then the groups $Aut_B(P)$, $\mathcal{H}(P)$, and the gauge group $\mathcal{H}(B)$ are isomorphic to each other.*

Therefore Theorem 3.3.4 allows one to interpret a vertical automorphism of a (locally) trivial quantum principal bundle as a change of local variables and truly as a gauge transformation of a trivial quantum principal bundle.

4. Conclusions and open problems. In this paper we reviewed basic properties of quantum fibre bundles introduced in [7]. There is a number of constructions, already present in the literature, that we have not described here. For example, locally trivial quantum principal bundles, defined in [7] were developed by M. Pflaum in [25], using the methods of the sheaf theory. A very interesting example of the Yang-Mills theory in quantum bundles was constructed by P. Hajac in [19]. The example considered in [19] belongs to the interface of the theory described here and the Connes-Rieffel Yang-Mills theory [14], and points to the very important problem of finding the relationship between the quantum group gauge theory and Connes' non-commutative geometry [12].

There is also a number of challenging problems that need to be solved in order to obtain a full understanding of quantum group gauge theories. For example, in this article we restricted our discussion only to gauge transformations of bundles with the universal differential structure. The theory of gauge transformations of bundles with general differential structures is not yet known. In particular, we would like to define gauge transformations in such a way that a gauge transformation of a connection one-form is still a connection one-form. A couple of remarks on this problem may be found in [4]. Also, it would be interesting to equip our algebraic constructions with a some kind of topology, like C^* or Fréchet topology. Some topological aspects of quantum fibre bundles are discussed in [10]. Furthermore, the theory of quantum fibre bundles reviewed in this article is strictly related to the theory of algebraic extensions. We think that the analysis of quantum bundles from the point of view of Hopf-Galois extensions may lead to a deeper insight into the both subjects. Finally, we think it is desirable to develop generalised fibre bundles defined in Remark 2.2.5. in order to construct a gauge theory on general homogeneous spaces. The development of such a theory becomes even more important and challenging now that the appearance of the $SU_q(2)$ homogeneous spaces in the Connes description of Standard Model was announced [13].

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