

A GAUGE-FIELD APPROACH TO 3- AND 4-MANIFOLD INVARIANTS

BOGUSŁAW BRODA

Department of Theoretical Physics, University of Łódź

Pomorska 149/153, PL-90-236 Łódź, Poland

E-mail: bobroda@mvi.uni.lodz.pl

Abstract. An approach to construction of topological invariants of the Reshetikhin-Turaev-Witten type of 3- and 4-dimensional manifolds in the framework of SU(2) Chern-Simons gauge theory and its hidden (quantum) gauge symmetry is presented.

1. Introduction. The issue of topological classification of low-dimensional manifolds, especially of dimensions 3 and 4 (the most difficult and interesting ones), is a challenging problem in modern mathematics. One of the most spectacular events in topology of 3-dimensional manifolds took place a few years ago, when a new (numerical) topological invariant of closed orientable 3-manifolds, parametrized by the integer k , defined via *surgery* on a framed link, was discovered. The idea is due to a physicist, Edward Witten, who proposed the invariant in his famous paper on quantum field theory and the Jones polynomial [Wit1]. The first explicit and rigorous construction is due to mathematicians, Reshetikhin and Turaev [RT]. Their approach is combinatorial, whereas non-combinatorial possibilities, very straightforward though mathematically less rigorous, are offered by *topological* quantum field theory. The 3-dimensional invariant, known as the *Reshetikhin-Turaev-Witten* (RTW) invariant, is also frequently referred to as the SU(2)-*invariant* because the *Kauffman bracket* it bases upon (denoted in mathematical literature as ' $\langle \rangle$ ') formally corresponds, in Witten's approach, to the average with respect to the connection A (defined on a trivial SU(2) bundle on the 3-manifold M) modulo gauge transformations, weighted by $\exp [ikS_{\text{CS}}(A)]$. Here $S_{\text{CS}}(A)$ is the Chern-Simons *secondary characteristic class*. Incidentally, the average is also denoted as ' $\langle \rangle$ '. The construction of the RTW invariant makes use of the *fundamental theorem of surgery* of Lickorish and Wallace on presentation of every closed connected orientable 3-manifold M via surgery

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on a framed link in \mathcal{S}^3 , and the *linear skein theory* associated with the Kauffman bracket. The construction of the RTW invariant amounts to showing its invariance with respect to the *Kirby moves* (K_1 and K_2). The Kirby moves are the allowable moves on a framed link, changing in general (isotopy class of) the link but not changing the 3-manifold \mathcal{M} obtained from the link. In fact, K_2 does not even change the 4-manifold \mathcal{M} ($M = \partial\mathcal{M}$), because it corresponds to sliding handles, whereas K_1 adds a complex projective space, which changes \mathcal{M} but still it does not change the 3-dimensional boundary M . The Kirby move K_1 means that we can add an unknotted, unlinked component with framing ± 1 , i.e. $\bigcirc_{\pm 1}$, whereas K_2 amounting to sliding a (upper) line over a (lower) trivial unknot in an annulus (a bordered exterior of “ \bullet ”), which is, in general, non-trivially immersed in \mathcal{S}^3 is schematically depicted as

$$\overline{\bigcirc} \longleftrightarrow \bigcirc \bigcirc.$$

In Section 2 we aim to propose a new, heuristic, non-combinatorial derivation of the RTW invariant in the framework of non-perturbative (topological) quantum Chern-Simons (CS) gauge theory [Bro3]. The idea is extremely simple, and in principle it applies to an arbitrary compact (semi-)simple Lie group G (not only to the $SU(2)$ one). Our invariant is essentially the partition function of CS theory on the 3-manifold M_L , defined via *surgery* on the framed link L in the 3-dimensional sphere \mathcal{S}^3 . Actually, surgery instructions are implemented in the most direct and literal way. The method of cutting and pasting back is explicitly used in the standard field-theoretical fashion. Roughly speaking, cutting corresponds to fixing, whereas pasting back to identification and summing up the boundary conditions.

Section 3 is devoted to a 4-dimensional generalization of the RTW invariant, and is mathematically more rigorous [Bro1]. In dimension 4, there is a celebrated theorem of Freedman on classification of closed orientable *simply-connected* 4-manifolds, provided by the *intersection form* $Q(\mathcal{M})$ (and the Kirby-Siebenman invariant $\alpha(\mathcal{M})$). The intersection form $Q(\mathcal{M})$ corresponds to, and for 4-manifolds with boundary defined via surgery on a link in \mathcal{S}^3 is equal to, the *linking matrix* ℓ_k . The elements of the symmetric matrix ℓ_k , the *linking numbers* (with framing numbers on the diagonal), are the simplest numerical invariants of a link. Therefore, one can ask the following questions. Can one use ‘non-abelian’ invariants of links, for example the Kauffman bracket polynomial, to obtain ‘non-abelian’ invariants of 4-dimensional manifolds? Can one extend the idea of RTW to the 4-dimensional case? Can one treat simply-connected and non-simply-connected manifolds uniquely? The answer to these questions seems to be affirmative. Namely, we propose an invariant of closed connected orientable 4-manifolds, defined via surgery on a special link in \mathcal{S}^3 . Thus, we have found a quantity invariant with respect to the 4-dimensional version of the ‘Kirby moves’. The idea as well as the construction resembles the original one, proposed by RTW in the 3-dimensional case, whereas the 4-dimensional version of the *Kirby calculus* we need has been developed by César de Sá [CdS].

In Section 4, we mention some other known invariants of 3- and 4-dimensional manifolds.

2. Gauge-field approach to 3-manifold invariants. Our principal goal is to compute the partition function $Z(M_L)$ of CS theory on the manifold M_L , defined via honest/integer surgery on the framed link $L = \bigcup_{i=1}^N K_i$ in \mathcal{S}^3 (i.e. attaching 2-handles to a 4-ball along a framed link—a particular case of rational surgery manipulating only tori), for the $SU(2)$ (gauge) Lie group. Obviously, the starting point is the partition function [Wit1] of CS theory $Z(\mathcal{S}^3)$ on the sphere \mathcal{S}^3

$$Z(\mathcal{S}^3) = \int e^{ikS_{CS}(A)} \mathcal{D}A,$$

where the functional integration is performed with respect to the connections A modulo gauge transformations, defined on a trivial $SU(2)$ bundle on \mathcal{S}^3 . The classical action is the *CS secondary characteristic class*

$$S_{CS}(A) = \frac{1}{4\pi} \int_{\mathcal{S}^3} \text{Tr} \left(AdA + \frac{2}{3} A^3 \right),$$

and the expectation value of an observable \mathcal{O} is defined as

$$\langle \mathcal{O} \rangle = \int \mathcal{O} e^{ikS_{CS}(A)} \mathcal{D}A.$$

According to the surgery prescription [Rol], we should cut out a closed tubular neighborhood N_i of K_i (a solid torus), and paste back a copy of a solid torus T , matching the meridian of T to the (twisted by framing number) longitude on the boundary torus ∂N_i in \mathcal{S}^3 . To this end, in the first step, we should fix boundary conditions for the field A on the twisted longitude represented by K_i . Since the only gauge-invariant (modulo conjugation) quantity defined on a closed curve is holonomy, we associate the *holonomy* operator $\text{Hol}_{K_i}(A)$ to each knot K_i . Thus the symbol

$$Z(\mathcal{S}^3, L; g_1, g_2, \dots, g_N)$$

should be understood as the *constrained* partition function of CS theory, i.e. the values of holonomies along K_i are fixed

$$\text{Hol}_{K_i}(A) = g_i, \quad i = 1, 2, \dots, N.$$

Now, we can put

$$(1) \quad Z(\mathcal{S}^3, L; g_1, g_2, \dots, g_N) = \left\langle \prod_{i=1}^N \delta(g_i, \text{Hol}_{K_i}(A)) \right\rangle,$$

where δ is a (group-theoretic) Dirac delta-function. Its explicit form following from the (group-theoretic) Fourier expansion is

$$(2) \quad \delta(g, h) = \sum_n \overline{\chi_n(g)} \chi_n(h),$$

where n labels inequivalent irreducible representations (irrep's) of $SU(2)$, and χ_n is a character. Physical observables being used in CS theory are typically Wilson loops, defined as

$$(3) \quad W_n^K(A) = \text{Tr}_n(\text{Hol}_K(A)) \equiv \chi_n(\text{Hol}_K(A)).$$

By virtue of (2–3)

$$(4) \quad \delta(g_i, \text{Hol}_{K_i}(A)) = \sum_n \overline{\chi_n(g_i)} W_n^{K_i}(A).$$

Inserting (4) into (1) yields, as a basic building block, the following representation of the constrained partition function

$$Z(\mathcal{S}^3, L; g_1, g_2, \dots, g_N) = \left\langle \prod_{i=1}^N \sum_{n_i} \overline{\chi_{n_i}(g_i)} W_{n_i}^{K_i}(A) \right\rangle.$$

In the second step of our construction, we should paste back the tori matching the pairs of “longitudes” (the twisted longitudes and the meridians), i. e. we should identify and sum up the boundary conditions. Since the interior of a solid torus is homeomorphic to \mathcal{S}^3 with a removed solid torus, actually the meridians play the role of longitudes in analogous cutting procedures for an unknot $\{\circ\}$ (with reversed orientation). Thus the partition function of CS theory on M_L is

$$\begin{aligned} Z(M_L) &= \int \prod_{i=1}^N dg_i Z(\mathcal{S}^3, \circ; g_i^{-1}) Z(\mathcal{S}^3, L; g_1, g_2, \dots, g_N) \\ &= \int \prod_{i=1}^N dg_i \sum_{m_i} \sum_{n_i} \overline{\chi_{m_i}(g_i^{-1})} \overline{\chi_{n_i}(g_i)} \langle W_{m_i}^\circ(A) \rangle \left\langle \prod_{j=1}^N W_{n_j}^{K_j}(A) \right\rangle, \end{aligned}$$

where the reversed orientation of the unknots $\{\circ\}$ (corresponding to the meridians of the pasted back tori) accounts for the power -1 of the group elements g_i . From the orthogonality relations for characters and unitarity of irrep’s, it follows that the 3-manifold invariant is of the form

$$(5) \quad Z(M_L) = \left\langle \prod_{i=1}^N \omega_{K_i}(A) \right\rangle,$$

where

$$\omega_{K_i}(A) \equiv \sum_{n_i} \langle W_{n_i}^\circ(A) \rangle W_{n_i}^{K_i}(A)$$

is an element of the linear skein of an annulus, immersed in the plane as a regular neighborhood of K_i . $\langle W_n^\circ(A) \rangle$ are some computable coefficients depending on n and k . Eq. (5) can be easily generalized to accommodate an ordinary link $\mathcal{L} = \bigcup_{i=1}^M \mathcal{K}_i$ embedded in M_L

$$\left\langle \prod_{i=1}^M W_{n_i}^{\mathcal{K}_i}(A) \right\rangle_{M_L} = \left\langle \prod_{i=1}^M W_{n_i}^{\mathcal{K}_i}(A) \prod_{j=1}^N \omega_{K_j}(A) \right\rangle.$$

It appears that a very convenient way of organization of irrep’s of $SU(2)$ group is provided by the polynomials $S_n(x)$, closely related to the Chebyshev polynomials. $S_n(x)$ are defined recursively by the formula

$$(6a) \quad S_{n+2}(x) = xS_{n+1} - S_n(x), \quad n = 0, 1, \dots,$$

together with the initial conditions

$$(6b) \quad S_0(x) = 1, \quad S_1(x) = x.$$

By virtue of (6), $S_n(x)$ expresses n -th irrep of $SU(2)$ in terms of powers of the fundamental representation x , denoted as $\mathbf{1}$ henceforth. The explicit solution of (6) is

$$S_n(2 \cos \alpha) = \frac{\sin((n + 1)\alpha)}{\sin \alpha}.$$

The skein relations for the fundamental representation ($n = 1$) of $SU(2)$ are given by the expression

$$(7a) \quad q^{\frac{1}{4}} \left\langle \cdots \begin{array}{c} \diagup \\ \diagdown \end{array} \cdots \right\rangle - q^{-\frac{1}{4}} \left\langle \cdots \begin{array}{c} \diagdown \\ \diagup \end{array} \cdots \right\rangle = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \left\langle \cdots \parallel \cdots \right\rangle,$$

$$(7b) \quad \left\langle \cdots \left| \pm 1 \cdots \right. \right\rangle = -q^{\pm \frac{3}{4}} \left\langle \cdots \left| 0 \cdots \right. \right\rangle,$$

where the integers in (7b) mean framings, and $q = \exp \frac{2\pi i}{k}$. Closing the left legs of all the (three) diagrams in (7a) with arcs, as well as the right ones, next applying (7b), and using the property of locality, we obtain

$$-(q - q^{-1}) \left\langle W_1^\circ(A) \right\rangle = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \left\langle W_1^{\circ\circ}(A) \right\rangle = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \left\langle W_1^\circ(A) \right\rangle^2.$$

Hence

$$\left\langle W_1^\circ(A) \right\rangle = - \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right) = -2 \cos \frac{\pi}{k},$$

and by virtue of the so-called satellite formula

$$(8) \quad \left\langle W_n^\circ(A) \right\rangle = S_n \left(-2 \cos \frac{\pi}{k} \right) = (-)^n \frac{\sin \frac{(n+1)\pi}{k}}{\sin \frac{\pi}{k}} = (-)^n \frac{q^{\frac{n+1}{2}} - q^{-\frac{n+1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

We can observe a remarkable property of (8) for $n = k - 1$, namely

$$(9) \quad \left\langle W_{k-1}^\circ(A) \right\rangle = 0.$$

It appears that for any \mathcal{K}

$$(10) \quad \left\langle \cdots W_{k-1}^\mathcal{K}(A) \cdots \right\rangle = 0.$$

In particular, Eq. (10) immediately follows from (9) for any \mathcal{K} that can be unknotted with corresponding skein relations. Thus we can truncate representations of $SU(2)$ above the value $k - 2$, and assume

$$0 \leq n \leq k - 2, \quad k = 2, 3, \dots$$

The final explicit form of ω_K for the group $SU(2)$ is then

$$(11) \quad \omega_K(A) = \sum_{n=0}^{k-2} (-)^n \frac{q^{\frac{n+1}{2}} - q^{-\frac{n+1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} S_n \left(W_1^K(A) \right).$$

Strictly speaking, $Z(M_L)$ is invariant with respect to the second Kirby move K_2 . It means that it is insensitive to the operation of sliding one of its handles over another one. But up to now we have not considered the issue of the determination of normalization. It appears that proper normalization of the partition function $Z(M_L)$ universally follows from the requirement of its invariance with respect to the first Kirby move K_1 .

The approach proposed above differs from Witten’s one [Wit1] in that we explicitly construct the invariant via the cutting and pasting procedure using standard field-theoretic tools.

3. Generalization to 4 dimensions. An arbitrary closed connected orientable 4-manifold \mathcal{M} can be obtained via surgery in \mathcal{S}^3 on a *special framed link* (L, f) [CdS].

DEFINITION. The special framed link L is a sum of two sorts of knots

$$L = \bigcup_{i=1}^N K_i \cup \bigsqcup_{i=1}^{\dot{N}} \dot{K}_i,$$

where $\{K_i\}_{i=1}^N$ are ordinary knots (corresponding to 2-handles), and $\{\dot{K}_i\}_{i=1}^{\dot{N}}$ are special knots (corresponding to 1-handles). The special knots, denoted with dots, are trivial (with zero framing), and mutually unlinked unknots, and the whole link, when regarded as a description of a 3-manifold, represents a connected sum of copies of $\mathcal{S}^1 \times \mathcal{S}^2$. The symbol “ \bigsqcup ” means the “distant sum”—the components are mutually unlinked.

Now, we introduce the following decomposition (*gradation*) of ω , defined in (11), into an even (+) and odd (−) parts (integer and half-integer ‘spins’, respectively)

$$\omega = \omega^+ + \omega^-.$$

Let us denote as \tilde{K} the result of pushing a knot K off itself (missing the rest of the link L) using the framing f of K , whereas as $K_1 \#_b K_2$ a (band) connected sum of the two knots K_1, K_2 , where b is any band missing the rest of L .

PROPOSITION. *For arbitrary complex numbers, a^+, a^- , we have the following ‘Kirby calculus’*

$$\begin{aligned} \langle \cdots \alpha_{K_1} \cdots \omega_{K_2} \cdots \rangle &= \langle \cdots \alpha_{K_1 \#_b \tilde{K}_2} \cdots \omega_{K_2} \cdots \rangle, \\ \langle \cdots \alpha_{K_1}^2 \cdots (a^+ \omega^+ + a^- \omega^-)_{K_2} \cdots \rangle &= \langle \cdots \alpha_{K_1 \#_b \tilde{K}_2}^2 \cdots (a^+ \omega^+ + a^- \omega^-)_{K_2} \cdots \rangle, \end{aligned}$$

where α is a linear combination of S_n with arbitrary coefficients (compare with the definition of ω (11)).

Remark. The first equality expresses a standard property of ω , whereas the second one follows from the observation that the even element α^2 respects the gradation in all cablings.

COROLLARY. *From the Proposition we can derive the following ‘Kirby equalities’*

$$\begin{aligned} \langle \cdots \omega_{K_1} \cdots \omega_{K_2} \cdots \rangle &= \langle \cdots \omega_{K_1 \#_b \tilde{K}_2} \cdots \omega_{K_2} \cdots \rangle, \\ \langle \cdots \omega_{K_1}^+ \cdots \omega_{K_2}^+ \cdots \rangle &= \langle \cdots \omega_{K_1 \#_b \tilde{K}_2}^+ \cdots \omega_{K_2}^+ \cdots \rangle, \\ \langle \cdots \omega_{K_1}^+ \cdots \omega_{K_2} \cdots \rangle &= \langle \cdots \omega_{K_1 \#_b \tilde{K}_2}^+ \cdots \omega_{K_2} \cdots \rangle. \end{aligned}$$

The fourth equality,

$$\langle \cdots \omega_{K_1} \cdots \omega_{K_2}^+ \cdots \rangle = \langle \cdots \omega_{K_1 \#_b \tilde{K}_2} \cdots \omega_{K_2}^+ \cdots \rangle,$$

is, in general, not true.

Henceforth, H and \dot{H} are two components of the *special Hopf link* \mathcal{H} , ordinary and special one respectively, $\mathcal{H} = H \cup \dot{H}$.

THEOREM. Let ν be the nullity of the (extended) linking matrix ℓk . Then

$$I_k(\mathcal{M}) = \frac{\langle \prod_{i=1}^N \omega_{K_i}^+ \prod_{i=1}^{\dot{N}} \omega_{\dot{K}_i} \rangle}{\langle \omega_{\circ}^+ \rangle^\nu \langle \omega_H^+ \omega_{\dot{H}} \rangle^{(N+\dot{N}-\nu)/2}}$$

is an invariant of the closed, connected, orientable 4-manifold $\mathcal{M} = \mathcal{M}_L$, a complex number parametrized by the integer k , independent of the choice of the representative (L, f) .

Below, we give a list of all the allowable ‘4-dimensional Kirby moves’, so-called Γ -moves [CdS]:

- (a) sliding one of the special knots over another special one;
- (b) sliding one of the ordinary knots over one of the special ones;
- (c) sliding one of the ordinary knots over another ordinary one;
- (d) introducing or deleting a special Hopf link;
- (e) introducing or deleting a trivial unknot;
- (f) isotoping the link picture in \mathcal{S}^3 .

Proof. We should show that $I_k(\mathcal{M})$ is invariant with respect to all the Γ -moves. a -, b - and c -invariance of $I_k(\mathcal{M})$ immediately follows from the Corollary as well as from the invariance of N , \dot{N} and ν . d -invariance is a consequence of the following transformation rule of the linking matrix ℓk , accompanying the introduction of a special Hopf link \mathcal{H} ,

$$\ell k \longrightarrow \begin{pmatrix} \ell k & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence the corresponding shift of the dimension and nullity of ℓk

$$\begin{aligned} N &\longrightarrow N + 1 \\ \dot{N} &\longrightarrow \dot{N} + 1 \\ \nu &\longrightarrow \nu, \end{aligned}$$

compensates the (factorized out) Kauffman bracket in the numerator. Similarly, e -invariance corresponds to the transformation rule

$$\ell k \longrightarrow \begin{pmatrix} \ell k & 0 \\ 0 & 0 \end{pmatrix},$$

and consequently the shift

$$\begin{aligned} N &\longrightarrow N + 1 \\ \dot{N} &\longrightarrow \dot{N} \\ \nu &\longrightarrow \nu + 1, \end{aligned}$$

also compensates the numerator. f -invariance directly follows from fundamental properties of the Kauffman bracket and the linking matrix ℓk . ■

Remark 1. The invariant $I_k(\mathcal{M})$ possesses the following obvious properties:

1) *multiplicativity*,

$$I_k(\mathcal{M}\#\mathcal{N}) = I_k(\mathcal{M}) \cdot I_k(\mathcal{N}),$$

2) *orientation sensitivity*,

$$I_k(\overline{\mathcal{M}}) = \overline{I_k(\mathcal{M})},$$

3) *normalization*,

$$I_k(\mathcal{S}^4) = 1,$$

where $\mathcal{M}\#\mathcal{N}$ denotes a connected sum of \mathcal{M} and \mathcal{N} .

Remark 2. Crane and Yetter [CY] have found a 4-dimensional topological invariant, defined via triangulation, which is basically equivalent to ours.

Remark 3. It has been proved that $I_k(\mathcal{M})$ is expressible by classical invariants (signature and Euler character) [CKY2], but there is a gap in the proof announced in [CKY1].

4. Final remarks. In the 3-dimensional case, besides the original RTW invariant, we have some other invariants of the same CS origin, and therefore more or less mutually related. The so-called *Turaev-Viro* invariant [TV] (the Crane-Yetter invariant, mentioned in Remark 2, is its 4-dimensional counterpart) can be calculated from triangulation of the manifold, and corresponds to the square of the modulus of the RTW invariant. Whereas the Kohno invariant can be calculated from the Heegaard decomposition of the manifold, and is also basically equivalent to the RTW invariant [Bro2]. Perturbative expansion provides us with some further family of invariants, defined for homology spheres, so-called *Ohtsuki-Garoufalidis* invariants [Oht]. The first perturbative term of this family is the famous *Casson(-Walker)* invariant, which was originally defined non-perturbatively, and has a surgical description.

In the 4-dimensional situation, we have also a distinct world of ‘differentiable invariants’, the famous *Donaldson* invariants and their building block, the *Seiberg-Witten* invariant [Wit2], mathematically described in this volume. But up to now, it is not clear whether the ‘differentiable invariants’ are related to the combinatorial ideas presented here.

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