SYMPLECTIC SINGULARITIES AND GEOMETRY OF GAUGE FIELDS
BANACH CENTER PUBLICATIONS, VOLUME 39
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

ON SEIBERG–WITTEN EQUATIONS ON SYMPLECTIC 4–MANIFOLDS

KLAUS MOHNKE

Institut für Reine Mathematik, Math.–Nat. Fakultät II, Humboldt–Universität zu Berlin
Ziegelstraße 13A, 10099 Berlin, Germany
E-mail: mohnke@mathematik.hu-berlin.de

Abstract. We discuss Taubes' idea to perturb the monopole equations on symplectic manifolds to compute the Seiberg–Witten invariants in the light of Witten's symmetry trick in the Kähler case.

1. Introduction. In 1994 a new field equation on 4–manifolds came up simplifying lots of questions in low–dimensional topology, generally referred to as Donaldson theory. The main advantage of the new theory is, that it is an *Abelian* field theory coupled to the Riemannian metric. The more or less direct consequence is that the moduli space of solutions of the field equations is compact (unlike in the case of anti–selfdual $\mathbf{SU}(\mathbf{2})$–connections there is no bubbling–off phenomenon). With a few (but important) exceptions one can give a sequence of easier proofs of facts derived from Donaldson theory (some of them with even stronger statements). See [F] for a more detailed discussion. Based on the vanishing of the Seiberg–Witten invariants there are new theorems in Riemannian geometry of spaces with positive scalar curvature (see [LB]).

Most importantly, the computation of the new invariants started a development at the end of which we will probably have a much better understanding of the differential topology of symplectic 4–manifolds.

Let us recall the state of the art before the new invariants came up. A symplectic structure on an (oriented) 4–manifold is a closed 2–form ω such that $\omega \wedge \omega$ gives an orientation class. There exists a calibrating almost complex structure J or, equivalently, a Riemannian metric g such that ω is a self–dual and harmonic 2–form. So, we already have the basic two homotopy obstructions against the existence of a symplectic structure

1991 *Mathematics Subject Classification*: Primary 57R57, 57N13; Secondary 57R40.

Research of the author supported by the Sonderforschungsbereich 288 of the Deutsche Forschungsgemeinschaft.

The paper is in final form and no version of it will be published elsewhere.

on a 4–manifold: $b_2^+ \geq 1$ and the (anti)–canonical bundle $K(K^{-1})$ satisfies

$$c_1(K)^2[M] = c_1(K^{-1})^2[M] = (2\chi(M) + 3\sigma(M))$$

where the existence of such a class $c \in \mathbf{H^2}(M;\mathbf{Z})$ satisfying this equation implies the existence of an almost complex structure. With that description at hand one could prove various results like

Proposition 1 [A].

(i) *If M, N are closed, compact, almost complex 4–manifolds then there is no almost complex structure on $M \sharp N$.*

(ii) *There exists an almost complex structure on $k\mathbf{CP^2}\sharp l\overline{\mathbf{CP^2}}$ inducing the given orientation iff $k \equiv 1 \bmod 2$.*

But it was an open problem whether $k\mathbf{CP^2}\sharp l\overline{\mathbf{CP^2}}$ admits a symplectic structure or not. This was settled by Taubes with the following

Theorem 1 [T]. *Let M be a closed, compact, symplectic 4–manifold with $b_2^+ \geq 2$. Then the $\mathbf{Spin^C}$–structure associated to the calibrating almost complex structure has Seiberg–Witten invariant ± 1.*

On the other hand there exist a Riemannian metric with positive scalar curvature on $k\mathbf{CP^2}\sharp l\overline{\mathbf{CP^2}}$. The invariants vanish on such manifolds (see [KM]) so he concludes

Corollary 1. *For $k > 1$ there is no symplectic structure on $k\mathbf{CP^2}\sharp l\overline{\mathbf{CP^2}}$.*

Another circle of problems is posed by the question of rigidity of symplectic structures. So far, with the powerful techniques of Gromov's pseudo–holomorphic curves, there have been rigidity results only for noncompact 4–manifolds:

Theorem 2 [G,D]. *Suppose M is a (noncompact) symplectic 4–manifold with one end which is standard at infinity, then M is the blowing up of $\mathbf{C}^2$ in a finite collection of points.*

The problem in applying Gromov's method is usually the existence of just one such pseudo–holomorphic curve.

By showing the existence of a pseudo–holomorphic curve homologous to the hyperplane Taubes proves the following conjecture of Gromov

Theorem 3 [T2]. *Every symplectic structure on $\mathbf{CP^2}$ is diffeomorphic to the standard one.*

The paper is organized as follows. In Chapter 2 we discuss $\mathbf{Spin^C}$–structures and study the monopole equations on almost complex and symplectic 4–manifolds, in Chapter 3 we compute the Lagrangian in this situation and discuss the failure of it to catch the symmetry of the more special Kähler case. Finally, we derive the family of equations Taubes considers to circumvent this problem. In Chapter 4 we prove ellipticity of the linearization and a priori estimates for solutions of this family of equations (from which the compactness of the moduli space of monopoles and its smoothness in a nondegenerate solution follows). Chapter 5 is basically extracted from Taubes paper, explaining in less detail all necessary steps in the proof of Proposition 3. In Chapter 6 we prove the nondegeneracy of solutions having the special form proposed in Proposition 3 (which was left out by Taubes). In the odds and ends of Chapter 7 we will discuss the limits of the application of Taubes' techniques to keep the fortunate reader from applying it to enthusiastically and give some more corollaries and an overview of the important research announcements of Taubes in [T2] in Chapter 8.

2. Spinor bundles and Dirac operators on almost complex manifolds. Let (M^{2n}, g) be an oriented Riemannian manifold.

DEFINITION 1. A *spinor bundle* $S \longrightarrow M$ is a complex vector bundle which is an irreducible representation of the Clifford bundle, i.e.

$$\mathbf{Cliff}^{\mathbf{C}}(M) \cong \mathbf{End}(S)$$

as bundles of algebras.

Remark 1. Such a spinor bundle is unique up to twisting by line bundles. This can be easily deduced from the uniqueness of the irreducible representation of the Clifford algebra $\mathbf{Cliff}^{\mathbf{C}}(V)$ of a $2n$–dimensional Euclidean vector space V as algebra of endomorphisms(see [BGV]) $\mathbf{End}(S)$ of a complex vector space S.

Now we discuss the question of the existence of such spinor bundles.

PROPOSITION 2. *An oriented $2n$–dimensional Riemannian manifold (M, g) admits a spinor bundle iff it admits a $\mathbf{Spin}^{\mathbf{C}}$–structure.*

Proof. The statement is rather elementary and probably well–known. In [BGV] the spinor module S is constructed locally using a maximal isotropic subspace of the complexified Euclidean vector space $P \subset T^{\mathbf{C}}M$. Having chosen an orienting orthonormal frame $\{e_j\}$ of TM, P may be set $P = \mathcal{L}_{\mathbf{C}}(\{e_{2k-1} + ie_{2k}\})$. S is then defined to be the exterior algebra $S = \Lambda P$ and the Clifford action is given by exterior multiplication and contraction. (Equivalently, given locally an almost complex structure J, e.g. via $J(e_{2k-1}) = e_{2k}$, then P is the eigenspace to the eigenvalue i of J.) The algebra of matrices is simple, so given any other spinor module S' there is up to scalar multiplication a unique isomorphism of S and S' as Clifford modules. Now assume we have chosen a covering $\{U_\alpha\}_\alpha$ of M such that the open sets and pairwise intersections are contractible together with trivializations of (TM, g) and S. Trivialising the tangent bundle and the spinor bundle compatibly, we obtain transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{SO(2n)}$ and $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Gl}(2^n; \mathbf{C})$ with

$$\phi_{\alpha\beta}(cs) = g_{\alpha\beta}(c)\phi_{\alpha\beta}(s)$$

for all $c \in \mathbf{Cliff(2n)}$ and $s \in S$. On the other hand take liftings $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{Spin(2n)}$. These do not define a cocycle in general. But via Clifford multiplication they do define transition functions for S commuting with the Clifford action. Denote these by $\tilde{g}_{\alpha\beta}$, too. By irreducibility of the representation of the Clifford module the difference between $\tilde{g}_{\alpha\beta}$ and $\phi_{\alpha\beta}$ is a scalar $\lambda_{\alpha\beta}$, i.e.

$$\tilde{g}_{\alpha\beta}\lambda_{\alpha\beta} = \phi_{\alpha\beta}$$

is a cocycle and so $\tilde{g}_{\alpha\beta} \times_{\mathbf{Z_2}} \lambda_{\alpha\beta}$ defines a $\mathbf{Spin}^{\mathbf{C}}$–structure. ∎

To begin with consider an almost complex 4–manifold (X, J) equipped with a Hermitian metric g. Denote by ω the corresponding skew–symmetric 2–form. Clifford multiplication by ω splits any $\mathbf{Spin^C}$–bundle S into its eigenspaces. Assume a scaling of ω (by a constant) such that the corresponding eigenvalues on S^+ are $\pm\mathrm{i}$, i.e.

$$\omega = \frac{\mathrm{i}}{4}(v^1 \wedge \bar{v^1} + v^2 \wedge \bar{v^2}),$$

for $v^1 = e^1 + \mathrm{i}e^2$ and $v^2 = e^3 + \mathrm{i}e^4$ with a compatible orthonormal frame $\{e^i\}$. Note that ω is self dual. J defines a maximal isotropic subspace of $T^{\mathbf{C}}M$ globally. The line bundle is $L = K^{-1}$ for the corresponding $\mathbf{Spin^C}$–bundle

$$\begin{aligned} S &\cong S^+ \oplus S^- \\ S^+ &\cong \mathbf{\Lambda^0} \oplus \mathbf{\Lambda^{0,2}} \\ S^- &\cong \mathbf{\Lambda^{0,1}}, \end{aligned}$$

(see [BGV] p. 110) where ω acts as $-\mathrm{i}$ on the first and as $+\mathrm{i}$ on the second summand and trivially on S^-. At first let us compute the unique spinor connection $\nabla^S = \nabla^+ \oplus \nabla^-$ associated to the metric and the connection A_0 on $K^{-1} \cong \mathbf{\Lambda^{0,2}}$ induced by the Levi–Civita connection ∇. Restricted and projected to the eigenbundles of the ω–action ∇^S agrees with ∇. That is equivalent to the assumption that $\det(\nabla^+)$ coincides with the covariant derivative ∇_{A_0} of the connection A_0 on $K^{-1} \cong \mathbf{\Lambda^{0,2}}$. Unfortunately, ∇ is not compatible with the Clifford multiplication. One computes

$$\nabla^+(\omega \cdot (f, \phi)) = \nabla^+(-\mathrm{i}f, \mathrm{i}\phi) = \nabla\omega \cdot (f, \phi) + \omega \cdot \nabla^+(f, \phi),$$

from which one deduces

$$\begin{aligned} \nabla^+ &= \nabla + b - b^* \\ b &\in \mathbf{\Omega^1}(\mathbf{Hom}(\mathbf{\Lambda^0}; \mathbf{\Lambda^{0,2}})) \\ &\cong \mathbf{\Omega^1}(K^{-1}) \\ b &= \mathrm{i}(\nabla\omega)^{0,2}, \end{aligned}$$

where $(\nabla\omega)^{0,2}$ is the projection of the covariant derivative on the $(0, 2)$–part. One easily checks that this is indeed the desired spinor connection.

Now we write down a formula for the twisted Dirac operator. Note first that in general for a spinor $\psi = (f, \phi)$

$$D_A\psi = \sqrt{2}(\bar\partial_A f + \bar\partial_A^* \phi + \frac{\mathrm{i}}{2}(((\partial + \partial^*)\omega)f + ((\bar\partial + \bar\partial^*)\omega)\phi)) \in S^- \cong \mathbf{\Lambda^{0,1}}.$$

But if ω is closed then $\bar\partial\omega = \partial\omega = 0$ ($d = \partial + \bar\partial$ and $d^* = \partial^* + \bar\partial^*$ on $\mathbf{\Lambda^{1,1}}$ in dimension 4). So, if ω is closed the Dirac operator takes the form

$$D_A\psi = \sqrt{2}(\bar\partial_A f + \bar\partial_A^* \phi)$$

Here A is a connection on K^{-1} which induces a spinor connection and D_A is the corresponding Dirac operator. Note that $\bar\partial_{A_0} = \bar\partial$. In general, for the Dirac operator we have the Weitzenböck formula

$$D_A^2\psi = (\nabla_A^S)^* \nabla_A^S \psi + \frac{R}{4}\psi + \frac{1}{2}F_A \cdot \psi.$$

Weitzenböck's formula applied to $D^2_{A_0}\mathbf{1} = 0$ with $\mathbf{1} = (1,0)$ and evaluated on the $(0,2)$–form part gives

$$\nabla^* b = -F^{0,2}_{A_0}.$$

Evaluating on the function component yields

$$2\langle(\nabla^*(\nabla\omega)^{0,2}),\omega\rangle - |(\nabla\omega)^{0,2}|^2 = -\frac{F^\omega_{A_0}}{2} - \frac{R}{4}$$

here $F^+_{A_0} = \mathrm{i}F^\omega_{A_0}\omega + F^{0,2}_{A_0} + F^{2,0}_{A_0}$. Note that $\langle(\nabla^*(\nabla\omega)^{0,2}),\omega\rangle = |(\nabla\omega)^{0,2}|^2$ pointwise (just pair both sides with a cut–off and use $\langle(\nabla\omega)^{0,2},\omega\rangle = 0 \in \mathbf{\Omega^1}(M)$). It follows that

$$|b|^2 = -\frac{F^\omega_{A_0}}{2} - \frac{R}{4}.$$

So, the right–hand side is a nonnegative function which vanishes exactly in the Kähler case.

Another identity which could be derived exploiting $D(\alpha\cdot\mathbf{1})$ for a one form α is, the following: the $(0,2)$–part of its exterior derivative is expressed by

$$(d\alpha)^{0,2} = \bar\partial(\alpha^{0,1}) + b\circ(\alpha^{1,0})$$

where "$\circ$" denotes the $\mathbf{C}$–linear contraction of the common T^*M–component. Accordingly one obtains on functions

$$\bar\partial^2_A + b\circ\partial_A = (F_A - F_{A_0})^{0,2}.$$

We conclude this collection of useful formulas with an identity resembling a similar one in Kähler geometry but which holds in the more general symplectic context on $(0,1)$–forms only: Denote by Λ the operator on forms dual to the exterior multiplication by ω. Let $\alpha\in\mathbf{\Omega}^{0,1}$. Then

$$2\Lambda\partial\alpha = \mathrm{i}\bar\partial^*\alpha.$$

3. The monopole equations on symplectic 4–manifolds. Using the Hermitian product on S^+ (see [KM]) we write

$$\sigma(\psi) = \mathrm{i}(\psi\bar\psi^T)_0 \in \underline{su}(S^+)$$

For a spinor $\psi = (f,\phi)\in\Gamma(S^+)\cong\Omega^0(M)\oplus\Omega^{0,2}(M)$. The action of the matrix entries is, of course, also defined with the help of the Hermitian product. This yields

$$\sigma(\psi) = \mathrm{i}\begin{pmatrix} \frac{|f|^2-|\phi|^2}{2} & f\bar\phi \\ \bar f\phi & -\frac{|f|^2-|\phi|^2}{2}\end{pmatrix}.$$

Via Clifford multiplication one obtains

$$\underline{su}(S^+)\cong\Omega^+(M).$$

Then $\sigma(\psi)$ corresponds to

$$\sigma(\psi) = -\frac{1}{2}(|f|^2-|\phi|^2)\omega + \frac{\mathrm{i}}{2}(\bar f\phi - f\bar\phi).$$

Note that for the $(2,0)$–form $f\bar\phi$ the Clifford action is $f\bar\phi\cdot\Psi = -2\langle\Psi,\bar f\phi\rangle$. The second monopole equation is

$$F^+_A = -2\mathrm{i}\sigma(\psi) = \mathrm{i}(|f|^2-|\phi|^2)\omega + (\bar f\phi - f\bar\phi).$$

For the Lagrangian we compute

$$\begin{aligned}
&\int_M (2|D_A\psi|^2 + |F_A^+ + 2\mathrm{i}\sigma(\psi)|^2)dM \\
&\quad = \int_M (2|\nabla_A^+\psi|^2 + \frac{R}{2}|\psi|^2 + \langle F_A^+\cdot\psi,\psi\rangle + |F_A^+|^2 + 2\langle F_A^+,\sigma(\psi)\rangle + 4|\sigma(\psi)|^2)dM \\
&\quad = \int_M (2|\nabla_A^+\psi|^2 + \frac{R}{2}|\psi|^2 + F_A^\omega(|f|^2-|\phi|^2) + 2\langle F_A^{0,2}f,\phi\rangle - 2\langle\phi,\overline{F_A^{2,0}}f\rangle + |F_A^+|^2 \\
&\qquad - 2\langle \mathrm{i}F_A^\omega\omega + F_A^{0,2} + F_A^{2,0}, \mathrm{i}(|f|^2-|\phi|^2)\omega + (\bar f\phi - f\bar\phi)\rangle + 4|\sigma(\psi)|^2)dM \\
&\quad = \int_M (2|\nabla_A^+\psi|^2 + \frac{R}{2}|\psi|^2 - F_A^\omega(|f|^2-|\phi|^2) + 4\,\mathrm{Re}(f\langle F_A^{0,2},\phi\rangle) + |F_A^+|^2 \\
&\qquad + F_A^\omega(|f|^2-|\phi|^2) - 4\,\mathrm{Re}(f\langle F_A^{0,2},\phi\rangle) + \frac{1}{2}(|f|^2+|\psi|^2)^2)dM \\
&\qquad\quad = \int_M (2|\nabla_A^+\psi|^2 + \frac{R}{2}(|f|^2+|\psi|^2) + \frac{1}{2}(|f|^2+|\psi|^2)^2)dM.
\end{aligned}$$

Now

$$|\nabla_A^+\psi|^2 = |\nabla_A f|^2 + |\nabla_A\phi|^2 + 2\,\mathrm{Re}(\langle bf,\nabla_A\phi\rangle - \langle\langle\phi,b\rangle,\nabla_A f\rangle) + (|bf|^2 + |b^*\phi|^2),$$

so in the non–Kähler case (where $b = (\nabla\omega)^{0,2} \neq 0$) we have terms in the integrand of the Lagrangian mixing f and ϕ. In the Kähler case ($b \equiv 0$) there are no such terms and we obtain an additional symmetry

$$\begin{aligned}
A &\mapsto A \\
f &\mapsto -f \\
\phi &\mapsto \phi.
\end{aligned}$$

(see [W] for a discussion). We would like to adopt this idea for the general symplectic 4–manifold. Observe that

$$d^*(\mathrm{Re}\langle bf,\phi\rangle) = \mathrm{Re}(-\langle bf,\nabla_A\phi\rangle - \langle\langle\phi,b\rangle,\nabla_A f\rangle + \langle\nabla^* b,\bar f\phi\rangle).$$

which vanishes after integration. We change the second equation to

$$F_A^+ = \mathrm{i}(|f|^2 - |\phi|^2 + \delta)\omega + (\bar f\phi - f\bar\phi) - (\nabla^* b - \overline{\nabla^* b}),$$

where $\delta \in \mathbf{C}^\infty(M)$. Then the Lagrangian changes to

$$\begin{aligned}
&\int_M (2(|\nabla_A f|^2 + |\nabla_A\phi|^2) + \frac{R}{2}(|f|^2+|\phi|^2) + (|bf|^2 + |b^*\phi|^2) - 8\,\mathrm{Re}\langle\langle\phi,b\rangle,\nabla_A f\rangle \\
&\qquad + \langle\delta,(|f|^2-|\phi|^2)\rangle + |F_A^+ + (\nabla^* b - \overline{\nabla^* b}) - \mathrm{i}\delta\omega|^2 + \frac{1}{2}(|f|^2+|\phi|^2)^2)dM.
\end{aligned}$$

Now, if one adds, hypothetically, the unfortunately singular term

$$\frac{2}{|f|^2}(\bar f b\circ\nabla_A f - f\overline{b\circ\nabla_A f})$$

to the second equation the last mixed term vanishes in the corresponding Lagrangian. So, we have a good heuristic to change the second equation to

$$F_A^+ = \mathrm{i}(|f|^2 - |\phi|^2 + \delta)\omega + (\bar f\phi - f\bar\phi) - (\nabla^* b - \overline{\nabla^* b}) + \frac{2r}{1+r|f|^2}(\bar f b\circ\nabla_A f - f\overline{b\circ\nabla_A f}),$$

where we obtain the singular term in the process $r \to \infty$. Now we choose the parameters $\delta = F^{\omega}_{A_0} - 1$ and use the above computed identity $\nabla^* b = -F^{0,2}_{A_0}$ and obtain the final version of the equations

$$\bar\partial_A f + \bar\partial^*_A \phi = 0$$
$$F^+_A - F^+_{A_0} = \mathrm{i}(|f|^2 - |\phi|^2 - 1)\omega + (\bar f\phi - f\bar\phi) + \frac{2r}{1 + r|f|^2}(\bar f b \circ \nabla_A f - f\overline{b \circ \nabla_A f}).$$

For each r the tuple $(A_0, \mathbf{1})$ is a solution, where $\mathbf{1} \equiv (1, 0)$. The main result of Taubes' paper is to prove the following

PROPOSITION 3. *For r sufficiently large this is up to gauge transformation the only solution of the above monopole equations. Moreover, this solution is nondegenerate for r sufficiently large.*

4. Ellipticity and compactness. First one should note that the monopole equations are still a (nonlinear) elliptic problem, i.e. the linearization of the equations modulo gauge equivalence is an elliptic operator (of first order). We just compute it to be

$$\begin{pmatrix} D_A & 0 \\ \frac{2r}{1+r|f|^2}(\bar f b \circ \nabla_A|_{\mathbf{\Lambda^0}} - f\overline{b \circ \nabla_A|_{\mathbf{\Lambda^0}}}) & d^* \oplus d^+ \end{pmatrix} + \text{terms of zeroth order.}$$

The diagonal part is known to be elliptic and an off–diagonal smooth first order differential operator cannot destroy that property.

The second important issue is the a priori estimate of solutions: if we loose the compactness of the moduli space we did not gain much, introducing the r–dependent term. But this is as easily established as the former:

PROPOSITION 4. *For arbitrary r a solution (A, ψ) of the monopole equations satisfies*

$$|\psi| \le \max(0, -R + 2\|F^+_{A_0}\|_\infty + 2\|b\|^2_\infty).$$

Consequently, the moduli space $\mathcal{M}_r$ of solutions is compact for all r. Moreover, the right–hand side does not depend on r. In addition, we have an r–independent bound

$$\int_M |\nabla_A\psi|^2 \le \max(0, -\frac{R}{2} + \|b\|^2_\infty + \|F_{A_0}\|_\infty)\int_M |\psi|^2.$$

Proof. The proof goes along the line of [KM]. Suppress the issue of smoothness (just suppose the solution is smooth). We compute

$$\begin{aligned} d^*d|\psi|^2 &= 2\langle(\nabla^+_A)^*\nabla^+_A\psi, \psi\rangle - 2\langle\nabla^+_A\psi, \nabla^+_A\psi\rangle \\ &= -\frac{1}{2}\big\langle\big(\mathrm{i}(|f|^2 - |\phi|^2 - 1)\omega + (\bar f\phi - f\bar\phi) + \frac{4r}{1 + r|f|^2}\,\mathrm{Im}(\bar f b \circ \nabla_A f) + F^+_{A_0}\big)\cdot\psi, \psi\big\rangle \\ &\qquad - \frac{R}{2}|\psi|^2 - 2|\nabla^+_A\psi|^2, \end{aligned}$$

using Weitzenböck's formula. We end up with

$$d^*d|\psi|^2 + 2|\nabla^+_A\psi|^2 + \frac{1}{2}|\psi|^4 = -\frac{R}{2}|\psi|^2 + \langle F^+_{A_0}\cdot\psi, \psi\rangle + \frac{4r|f|^2}{1 + 2r|f|^2}\,\mathrm{Re}\langle b \circ \nabla_A f, \phi\rangle$$

for solutions of the monopole equations. The proposition is now derived as in the before mentioned paper: considering the equality at a maximal point of $|\psi|^2$, where $d^*d|\psi|^2 \ge 0$

for the first, and integrating over M for the second statement. We only have to take care of the r–dependent part:

$$\left|\frac{4r|f|^2}{1+2r|f|^2}\operatorname{Re}\langle b\circ\nabla_A f,\phi\rangle\right|\le\epsilon\|b\|_\infty^2|\nabla_A^S\psi|^2+\frac{1}{\epsilon}|\psi|^2.$$

Choose $\epsilon\|b\|_\infty^2=1$ to obtain the result. ■

5. The behaviour for large r. We now turn to the proof of Taubes' main result. The idea goes as follows: Instead of having the symmetry of the Kähler case the Lagrangian is "nearly symmetric" where the failure of having the symmetry decreases with growing r. To be more precise: Denote by $\mathcal{L}(A,\psi)$ the Lagrangian described above. Then

$$\mathcal{L}(A,(f,\phi))-\mathcal{L}(A,(f,-\phi))=\int_M\frac{-16}{1+r|f|^2}\operatorname{Re}\langle b\circ\nabla_A f,\phi\rangle. \tag{1}$$

Remember that we have r–independent a priori estimates on the solutions for a parameter r. So the only serious contribution on the right–hand side comes from the set where f is small compared to r. The left–hand side will be expressed differently to get $\mathbf{L_1^2}$ estimates on ψ.

Let us first turn to this latter issue. Having a solution (A,ψ) to the monopole equations of parameter r $(\psi=(f,\phi))$ we compute

$$\begin{aligned}(2)\qquad &\mathcal{L}(A,(f,\phi))-\mathcal{L}(A,(f,-\phi))\\ &=\int_M\Big(16\operatorname{Re}\langle\bar\partial_A f,\bar\partial_A^*\phi\rangle-4\operatorname{Re}\Big\langle\bar f\phi-f\bar\phi,F_A^+-F_{A_0}^+-\frac{2r(\bar f b\circ\nabla_A f-f\overline{b\circ\nabla_A f})}{1+r|f|^2}\Big\rangle\Big)dM\\ &=-\int_M(16|\bar\partial_A f|^2+8|f|^2|\phi|^2)dM=-\int_M(16|\bar\partial_A^*\phi|^2+8|f|^2|\phi|^2)dM.\end{aligned}$$

We need another expression for $2|\bar\partial_A f|^2=|D_A(f,0)|^2$, namely

$$\int_M|D_A(f,0)|^2dM=\int_M(|\nabla_A^+(f,0)|^2+\frac{R}{4}|f|^2+\frac{F_A^\omega}{2}|f|^2)dM,$$

via Weitzenböck which is with one of the identities of the second chapter easily seen to be

$$\int_M(|\nabla_A f|^2+|(\nabla\omega)^{0,2}|^2|f|^2+\frac{R}{4}|f|^2+\frac{F_A^\omega}{2}|f|^2)dM=\int_M(|\nabla_A f|^2+\frac{(F_A^\omega-F_{A_0}^\omega)}{2}|f|^2)dM.$$

Proof of Proposition 3. Assume we have an unbounded increasing sequence $\{r_i\}$ and solutions $\{(A_i,\psi_i)\}$ such that $\phi_i\neq 0$ for all i and $\psi_i=(f_i,\phi_i)$. By Proposition 4 and some standard arguments we have a subsequence such that $|\psi_i|$ converges strongly in $\mathbf{L^p}$ for any $p\in[2,\infty)$ and $|\nabla_{A_i}^+\psi_i|$ converges strongly in $\mathbf{L^2}$. Denote by V_i the set $\{x\in M\mid|f_i|<\frac12\}$. Taubes proves successively

1. For the above subsequence the measure
$$\lim_{i\to\infty}\mu(V_i)=0.$$
2. For the same subsequence
$$\int_M(|\nabla_{A_i}\phi_i|^2+|\phi_i|^2)dM\le\text{const.}\int_{V_i}|\phi_i|^2dM$$

independently of the parameter r_i.

3.
$$\begin{aligned}\int_{V_i} |\phi_i|^2 dM &\leq \mathrm{vol}(V_i)^{\frac{1}{2}}\big(\int_{V_i} |\phi_i|^4 dM\big)^{\frac{1}{2}} \\ &\leq \text{const. } \mathrm{vol}(V_i)^{\frac{1}{2}}\big(\int_M (|d|\phi_i||^2 + |\phi|^2)dM\big) \\ &\leq \text{const. } \mathrm{vol}(V_i)^{\frac{1}{2}} \int_M (|\nabla_{A_i}\phi_i|^2 + |\phi_i|^2)dM \\ &\leq \text{const. } \mathrm{vol}(V_i)^{\frac{1}{2}} \int_{V_i} |\phi_i|^2 dM\end{aligned}$$

by Kato's inequality $|d|\phi|| \leq |\nabla_{A_i}\phi_i|$. This contradicts the assumption of nonvanishing ϕ_i if $\mathrm{vol}(V_i)$ tends to zero and proves together with Section 6 the main theorem.

It remains to explain the first two steps in the proof. Note that the parameter r in (1) is in the denominator. So, wherever $|f_i|$ is big the contribution to the integral will be small (if you like only $r_i^\epsilon |f_i|^2$ has to be big for $\epsilon < 1$, because we have $\mathbf{L^2}$–bounds on $|\nabla_{A_i} f_i|$ and $\mathbf{L}^\infty$–bounds on ϕ_i). So we will just divide the domain of integration in the part where $|f_i|$ is big and where it is small. First we integrate the right–hand side of (1) by parts to obtain

$$\int_M (2|\bar\partial_A^*\phi_i|^2 + |f_i|^2|\phi_i|^2)dM \leq \int_M \frac{2}{1+r_i|f_i|^2}(|f_i|\,|\nabla_A\phi_i| + |f_i|\,|\phi_i| + |\phi_i|\,|d|f_i||)dM.$$

Now using the convergence assumption on the subsequence, observing $\frac{s}{1+rs^2} \leq \frac{\sqrt{2}}{3\sqrt{r}}$ and dividing the domain in $V^i = \{x \in M \mid r_i^{\frac{1}{4}}|f_i| \leq 1\}$ and its complement we have

$$\int_M (2|\bar\partial^*\phi_i|^2 + |f_i|^2|\phi_i|^2)dM \leq \text{const.}\big(\frac{1}{\sqrt{r_i}} + \int_{V^i} |\phi_i|\frac{|d|f_i||}{1+r_i|f_i|^2}dM\big)$$

The key observation at this point is that the right–hand side of the inequality has to converge to zero under the assumption made on the subsequence. This is proved by standard techniques. Remember that the $|f_i|$ converge to some positive function f in $\mathbf{L_1^2}$. So it is sufficient to prove that

$$\int_M \frac{|df|}{1+r_i|f_i|^2}dM$$

converges to zero. We first show that for

$$\int_M \frac{|df|}{1+r_i f^2}dM$$

and then for the difference. We divide the domain in the part where $f^2 > \frac{1}{n}$ and its complement. Over the first domain the integral is bounded by

$$\text{const.}\frac{n}{r_i}\|df\|_{\mathbf{L^2}}$$

which goes to zero for fixed n as r tends to infinity. For the integral over the latter region Taubes uses

Lemma 1. $0 \le f \in \mathbf{L^2_1}(M)$. *Then*

$$\lim_{\epsilon\to 0}\int_{\{x\in M|f<\epsilon\}} |df| = 0.$$

For the difference compute the integrand

$$|df| r_i^2 \frac{|f_i|^2 - |f|^2}{(1+r_i|f|^2)(1+r_i|f_i|^2)}.$$

Integrate over $\{f^2 > \frac{1}{n}\}$ and its complement. By appeal to Lemma 1 the latter is bounded by a sequence which r–independently tends to zero as n tends to infinity where the former tends to zero for fixed n as r tends to infinity because the measure of the set $\{\big||f_i|^2 - |f|^2\big| > \frac{1}{2n}\}$ tends to zero. Now go back to (2). We obtain

$$\int_M (|\nabla_{A_i} f_i|^2 + \frac{(|f_i|^2 - |\phi_i|^2 - 1)}{2}(|f_i|^2 - 1) + |f_i|^2|\phi_i|^2) dM$$
$$= \int_M \frac{2}{1+r_i|f_i|^2}\langle b\circ\nabla_{A_i} f_i, \phi_i\rangle dM$$

using

$$\int_M (|f_i|^2 - |\phi_i|^2 - 1) dM = \int_M (F_A - F_{A_0})\wedge\omega = 0.$$

With $w_i = (1 - |f_i|^2)$ we end up with

$$\int_M (|\nabla_{A_i} f_i|^2 + \frac{w_i^2}{2} + \frac{|f_i|^2|\phi_i|^2}{2} + |\phi_i|^2) dM = \int_M \frac{1}{1+r_i|f_i|^2}\langle b\circ\nabla_{A_i} f_i, \phi_i\rangle dM. \tag{3}$$

But we have seen that the right–hand side of this equation tends to zero forcing the $\mathbf{L^2}$–norm of w_i to vanish at infinity, so the measure for *any* $\delta > 0$ of $\{x\in M \mid |f_i|^2 < \delta\}$ converges to zero, completing the first step.

On the other hand taking absolute values and integrating the right–hand side over $V_i = \{|f_i|^2 < \frac{1}{2}\}$ and its complement gives

$$\int_M (|\nabla_{A_i} f_i|^2 + w_i^2 + |\phi_i|^2 + \frac{|f_i|^2|\phi_i|^2}{2}) dM$$
$$\le \frac{2\|b\|_\infty}{r_i}\int_M |\nabla_{A_i} f_i|\,|\phi_i| dM + \|b\|_\infty \int_{V_i} |\nabla_{A_i} f_i|\,|\phi_i| dM.$$

Use inequalities like $2xy \le \frac{1}{\epsilon}x^2 + \epsilon y^2$ and suppose r_i to be large enough to get

$$\int_M (2|\bar\partial_{A_i} f_i|^2 + |f_i|^2|\phi_i|^2) dM$$
$$= \int_M (|\nabla_{A_i} f_i|^2 + w_i^2 + |\phi_i|^2 + \frac{|f_i|^2|\phi_i|^2}{2}) dM \le \frac{\|b\|_\infty^2}{2\epsilon(1-\epsilon-\frac{\|b\|_\infty}{r_i})}\int_{V_i} |\phi_i|^2 dM.$$

Then with (1) and (2) we have an estimate

$$\int_M (2|\bar\partial^*_{A_i}\phi_i|^2 + |f_i|^2|\phi_i|^2) dM \le \text{const.}\int_{V_i} |\phi_i|^2 dM$$

for i sufficiently large. Again Weitzenböck, the second monopole equation and the formula for $|(\nabla\omega)^{0,2}|^2$ give

$$\int_M |\bar\partial^*_{A_i}\phi_i|^2 dM = \int_M (|\nabla_{A_i}\phi_i|^2 - \frac{|f_i|^2 - |\phi_i|^2 - 1}{2}|\phi_i|^2 - F^\omega_{A_0}|\phi_i|^2)dM$$

or

$$\int_M |\nabla_{A_i}\phi_i|^2 dM \le |\int_M (|\bar\partial^*_{A_i}\phi_i|^2 + \frac{|f_i|^2|\phi_i|^2}{2} + (F^\omega_{A_0} - 1)|\phi_i|^2)dM| \le \text{const.}\int_{V_i}|\phi_i|^2 dM$$

proving the second statement and completing the proof of the theorem. ∎

6. Uniqueness and nondegeneracy. The uniqueness follows easily from the computations of the last section. We proved that for r sufficiently large each solution has the form $(A,(f,0))$. But then on the other hand equation (3) shows that $|f| \equiv 1$. Now we gauge the solution with $g = \bar f \in \mathcal{G}$ and obtain a solution $(A,\mathbf{1})$. If we remember that

$$0 = \bar\partial_A \mathbf{1} = \bar\partial\mathbf{1} + (A - A_0)^{(0,1)}\mathbf{1} = ((A - A_0)^{(0,1)}, 0) \in (\mathbf{\Omega^0} \oplus \mathbf{\Omega^{0,2}})(M),$$

$A = A_0$ follows easily.

It remains to verify the nondegeneracy of the functional at $(A_0,\mathbf{1})$. That means to verify that the second cohomology of the deformation complex vanishes which in the case the virtual dimension of the moduli space is zero is equivalent to the vanishing of the first cohomology at irreducible solutions. We use the observation of LeBrun (see [LB]) which can be carried out for the irreducible solution $(A_0,\mathbf{1})$.

Assume $(a,g,\varphi) \in \mathbf{\Omega^1}(\mathrm{i}\mathbf{R}) \times (\mathbf{\Lambda^0} \oplus \mathbf{\Lambda^{0,2}})$ is a 1–cocycle. Write $a = \alpha - \bar\alpha$ with $\alpha \in \mathbf{\Omega^{0,1}}$. This is equivalent to lying in the kernel of the Hessian $\mathcal{H}$ of the corresponding Lagrangian (which has semi–definite Hessian at solutions of the equations). With Weitzenböck the Lagrangian can be transformed to

$$\begin{aligned}\int_M \big(&4(|\bar\partial_A f|^2 + |\bar\partial^*_A\phi|^2) + \frac{1}{2}\big|F^\omega_A - F^\omega_{A_0} + 1 - (|f|^2 - |\phi|^2)\big|^2 \\ &+ 2\big|F^{0,2}_A - F^{0,2}_{A_0} - \bar f\phi - \frac{2r}{1 + r|f|^2}\bar f b \circ \nabla_A f\big|^2\big)dM \\ = \int_M \big(&4(|\bar\partial_A f|^2 + |\bar\partial^*_A\phi|^2) - \frac{8}{1 + r|f|^2}\operatorname{Re}\langle b \circ \partial_A f, \phi\rangle + \frac{1}{2}\big|F^\omega_A - F^\omega_{A_0} + 1 - (|f|^2 - |\phi|^2)\big|^2 \\ &+ 2\big|F^{0,2}_A - F^{0,2}_{A_0} - \frac{2r}{1 + r|f|^2}\bar f b \circ \nabla_A f\big|^2 + 2|f|^2|\phi|^2\big)dM.\end{aligned}$$

Computing the Hessian at $(A_0,\mathbf{1})$ we obtain

$$\begin{aligned}\int_M \big(&8(|\bar\partial g + \frac{\alpha}{2}|^2 + |\bar\partial^*\varphi|^2) + |\Lambda(da) - 2\mathrm{i}\operatorname{Re} g|^2 + 4\big|(da)^{0,2} - \frac{2r}{1+r} b \circ (\partial g - \frac{\bar\alpha}{2})\big|^2 \\ &+ 4|\varphi|^2 - \frac{16}{1+r}\operatorname{Re}\langle \varphi, b \circ \nabla g + b \circ \frac{\bar\alpha}{2}\rangle\big)dM.\end{aligned}$$

Using (1) we derive for the difference

$$\mathcal{H}(\alpha,(g,\varphi)) - \mathcal{H}(\alpha,(g,-\varphi)) = \int_M \big(- \frac{32}{1+r}\operatorname{Re}\langle\varphi, b \circ \nabla g + b \circ \frac{\bar\alpha}{2}\rangle\big)dM.$$

On the other hand, because $\mathbf{1} = (1, 0)$, (2) leads to

$$= -\int_M (32|\bar\partial g + \frac{\alpha}{2}|^2 + 16|\varphi|^2) dM.$$

Now via gauge fixing we assume without loss of generality that g is purely real. Then we end up with

$$\begin{aligned}\int_M (32|\bar\partial g + \frac{\alpha}{2}|^2 + 16|\varphi|^2) dM &= \Big|\int_M \big(-\frac{32}{1+r}\operatorname{Re}\langle\varphi, b\circ\nabla g + b\circ\frac{\bar\alpha}{2}\rangle\big) dM\Big| \\ &\le \int_M \big|\frac{32}{1+r}\operatorname{Re}\langle\varphi, b\circ\nabla g + b\circ\frac{\bar\alpha}{2}\rangle\big| dM \\ &\le \frac{\text{const.}}{1+r}\int_M (32|\partial g - \frac{\bar\alpha}{2}|^2 + 16|\varphi|^2) dM = \frac{\text{const.}}{1+r}\int_M (32|\bar\partial g - \frac{\alpha}{2}|^2 + 16|\varphi|^2) dM,\end{aligned}$$

on account of $\overline{\partial g} = \bar\partial g$ in the gauge fixing, from which we conclude for r sufficiently large the vanishing of $(\alpha, (g, \varphi))$: $\varphi = 0$ is obvious,

$$\bar\partial g = -\frac{\alpha}{2}$$

and

$$\Lambda(da) = 2\mathrm{i}\operatorname{Re} g = 2\mathrm{i} g$$

imply

$$\bar\partial^* \bar\partial g + \partial^* \partial g + g = 0,$$

so $g = 0$ follows and consequently the vanishing of α. ∎

7. Further consequences and limits. From the computations of Section 5 one obtains some more immediate obstructions against symplectic structures and vanishing results for Seiberg–Witten invariants summarized in [T1]. Proving the first step in the verification of Proposition 3 we made use of the fact that

$$\int_M (|f_i|^2 - |\phi_i|^2 - 1) dM = \int_M -\mathrm{i}(F_A - F_{A_0}) \wedge \omega = 0.$$

Note that if we have a general $\mathbf{Spin}^{\mathbf{C}}$–structure with $\det S^+(L) = K^{-1} \otimes L^2$, or more precisely

$$S^+(L) = L \oplus K^{-1} \otimes L$$

then everything will go as in the case where L was the trivial bundle, but

$$\int_M -\mathrm{i}(F_A - F_{A_0}) \wedge \omega = -4\pi c_1(L) \cup [\omega][M]$$

(note that A and A_0 live on different line bundles in general). So, carrying out the remaining estimates one concludes the vanishing of the Seiberg–Witten invariant for $S^+(L)$ if $c_1(L) \cup [\omega][M] < 0$. The case when this number vanishes is more subtle: One still obtains the vanishing of the form–part of the spinor, i.e. $0 = \phi_i \in \mathbf{\Omega^{0,2}}(L)$. But equation (3) forces the function component of the spinor to satisfy $|f| \cong 1$. That means that L has to be trivial. One should also note that the monopole equations have a symmetry in the two components of the spinor. If we play with the parameter δ and consider the monopole equation

$$\bar\partial_A f + \bar\partial_A^* \phi = 0$$
$$F_A^+ + F_{A_0}^+ = \mathrm{i}(|f|^2 - |\phi|^2 + 1)\omega + (\bar f\phi - f\bar\phi) - \frac{2r}{1+r|\phi|^2}(\bar\phi b^* \circ \nabla_A\phi - \phi\overline{b^* \circ \nabla_A\phi})$$

instead, exchanging in all the estimates the f's and ϕ's we obtain a similar result for the Chern class $c_1(K \otimes L^{-1})$. We summarize

THEOREM 4 [T1]. *Suppose M is a compact, closed, symplectic 4–manifold with $b_2^+ > 1$. Then the Seiberg–Witten invariants for the $\mathbf{Spin}^{\mathbf{C}}$–structure with*

$$S^+(L) = L \oplus K^{-1} \otimes L$$

vanishes if

(i) $c_1(L) \cup [\omega][M] < 0$

(ii) $c_1(K \otimes L^{-1}) \cup [\omega] < 0$ *or*

(iii) $c_1(L) \cup [\omega][M] = 0$ *and L is a nontrivial line bundle or*

(iv) $c_1(K \otimes L^{-1}) \cup [\omega][M] = 0$ *and $K \otimes L^{-1}$ is a nontrivial line bundle.*

As an immediate consequence we obtain that $c_1(K) \cup [\omega][M]$ has to be positive or K has to be the trivial line bundle. A somewhat different observation can be made in the case of $\mathbf{CP^2}$:

THEOREM 5 [T1]. *There is no symplectic structure on $\mathbf{CP^2}$ with*

$$c_1(K) \cup [\omega][M] > 0.$$

Proof. One observes that the original monopole equation

$$D_A\psi = 0$$
$$F_A^+ = \mathrm{i}(|f|^2 - |\phi|^2)\omega + (\bar f\phi - f\bar\phi)$$

gives zero Seiberg–Witten invariant on account of the existence of a metric with positive scalar curvature, while the perturbed version

$$D_A\psi = 0$$
$$F_A^+ - F_{A_0}^+ = \mathrm{i}(|f|^2 - |\phi|^2 - 1)\omega + (\bar f\phi - f\bar\phi)$$

gives nonzero invariant. The only reason for that can be a "wall crossing" for the family of equations

$$D_A\psi = 0$$
$$F_A^+ - sF_{A_0}^+ = \mathrm{i}(|f|^2 - |\phi|^2 - s)\omega + (\bar f\phi - f\bar\phi)$$

at some parameter $s \in (0,1)$. At this parameter exists a reducible solution ($\psi \equiv 0$) and we end up with

$$(1-s)c_1(K^{-1}) \cup [\omega][M] = s \cdot \mathrm{vol}_\omega(M) > 0. \ \blacksquare$$

The considerations above also show to what extend these techniques of Taubes can be used: If $c_1(L) \cup [\omega][M] > 0$ and $c_1(K \otimes L^{-1}) \cup [\omega][M] > 0$ the line of arguments stops at the point described in this section. One has to use different methods which Taubes proposes to do in [T2].

8. Monopoles and pseudo–holomorphic curves. Being so powerful in detecting holomorphic sections of line bundles in the Kähler case there was strong hope that the Seiberg–Witten invariants could compute the Gromov invariants defined in [G] counting pseudo–holomorphic curves homologous to a certain homology class. In his big research announcement [T2] Taubes proposes to settle this issue. We shortly discuss the results.

Denote by $\mathcal{M}_c(M,\omega)$ the space of smooth pseudo–holomorphic curves in a closed, compact symplectic 4–manifold (M,ω) equipped with a compatible almost complex structure J homologous to $c \in \mathbf{H_2}(M;\mathbf{Z})$. Note that only in the case of a torus there are topologically different parametrizations with smooth immersions, producing smooth pseudo–holomorphic tori with multiplicities. For a generic almost complex structure this is a smooth, oriented manifold of even dimension

$$c \cdot c - c \cdot c_1(K) \equiv 0 \bmod 2.$$

Pick a set of $(c \cdot c - c \cdot c_1(K))/2$ points $\{x_i\}$ in M. For a generic choice of the latter

$$\mathcal{M}_{c,\{x_i\}} = \{C \in \mathcal{M}_c | \{x_i\} \subset C\}$$

is a discrete set of points with orientation. The Gromov invariant as an invariant of the symplectic manifold (M,ω) is defined as the oriented number of points

$$G_c(M,\omega) = \hat{\sharp}\mathcal{M}_{c,\{x_i\}}$$

Now the main theorem Taubes announces to prove is

Theorem 6. *Let (M,ω) be a compact, closed, symplectic 4–manifold with $b_+^2 > 1$. Then the Seiberg–Witten invariant for the $\mathbf{Spin}^{\mathbf{C}}$–structure with $S^+(L) = L \oplus K^{-1} \otimes L$ is $\pm G_{c_1(L)}$. In other words the Gromov invariants are diffeomorphism invariants of the 4–manifold.*

From the theorem and this observation one concludes a series of remarkable statements: Suppose $b_2^+ > 1$ and J is a generic, compatible almost complex structure.

1. $c_1(K)$ is represented by a smooth pseudo–holomorphic curve consisting of several components C of genus $g(C) = C \cdot C + 1$ (possibly with multiplicities only if the component is a torus). If M is minimal (i.e. admits no embedded pseudo–holomorphic 2–spheres of self intersection -1) then $c_1(K) \cdot c_1(K) \geq 0$.

Assume the Seiberg–Witten invariant is nonzero for $S^+(L) = L \oplus K^{-1} \otimes L$. Then $c_1(L)$ and $c_1(K \otimes L^{-1})$ are represented by smooth pseudo–holomorphic curves. Their components are components of the representative of $c_1(K)$ with smaller multiplicities. The converse statement is true.

An immediate consequence is the vanishing of the Seiberg–Witten invariants in the case of a positive dimensional moduli space of monopoles if $b_2^+ > 1$.

2. If M is minimal, then

$$-\frac{4}{3}(1 - b_1) - \frac{2}{3}b_2 \leq \sigma(M).$$

3. Suppose M admits an embedded sphere S with $S \cdot S = -1$ and $c_1(K) \cdot S \neq 0$. Then M admits a pseudo–holomorphic 2–sphere in the same homology class.

4. Suppose M is minimal and $c_1(K) \cdot c_1(K) = 0$. Then $c_1(K)$ is Poincaré dual to a union of disjoint, embedded, pseudo–holomorphic tori (possibly with multiplicities) with, of course, zero self intersection (by adjunction formula). Moreover, for any $S^+(L)$ with nontrivial Seiberg–Witten invariant, $c_1(L)$ is Poincaré dual to a subset of these tori (with smaller multiplicities).

Remark 2. The statements are (although reminiscent) slightly weaker than those obtained in Kähler geometry. There, in the case of $c_1(K) \cdot c_1(K) > 0$, one obtains that the holomorphic curve representing $c_1(K)$ is connected obtained by purely algebraic means using the Hodge index theorem (see [FM]). It would be interesting to know whether there are symplectic manifolds with nonconnected pseudo–holomorphic representatives of $c_1(K)$ allowing more $\mathbf{Spin}^{\mathbf{C}}$–structures $S^+(L)$ with nontrivial Seiberg–Witten invariant then $L = \underline{\mathbf{C}}$ or $L = K$ as in the Kähler case under this assumption.

Proof. The first statement is a direct consequence of the two main Theorems. Recall that the dimension of the moduli space of monopoles for $S^+(L)$ is

$$d(L) = c_1(L) \cdot c_1(L) - c_1(K) \cdot c_1(L)$$

but $P.D.(c_1(L))$ consists of components of $P.D.(c_1(K))$ with smaller multiplicities by the second part of the statement. So, the dimension obstruction follows. The second statement plays with the index formula for the virtual dimension of the moduli space using $c_1(K) \cdot c_1(K) \geq 0$. The third statement observes that there is a diffeomorphism acting on $\mathbf{H_2}(M; \mathbf{Z})$ as reflection on the hyperplane orthogonal to the class $[S]$. But then $c_1(K)$ changes to $c_1(K) - (2c_1(K) \cdot S)(P.D.([S]))$ which has to have nontrivial Seiberg–Witten invariant. That means that a multiple of $[S]$ has to be represented by a multiple of pseudo–holomorphic embedded spheres. But the only way to achieve this is to represent $[S]$ by a pseudo–holomorphic sphere of self intersection -1. The last statement is the special case of the first. ∎

References

[A] M. Audin, *Exemples de variétés presque complexes*, Enseign. Math. (2) 37 (1991), 175–190.

[BGV] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and the Dirac Operators*, Grundlehren der Mathematischen Wissenschaften, Springer 1991.

[FM] R. Friedman, J. W. Morgan, *Algebraic surfaces and Seiberg–Witten invariants*, alg-geom/9502026, 1995.

[F] Th. Friedrich, *Neue Invarianten der 4–dimensionalen Mannigfaltigkeiten*, SFB 288–Preprint 156, 1995.

[G] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307–347.

[KM] P. B. Kronheimer, T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. 1 (1994), 797–808.

[LB] C. LeBrun, *Einstein Metrics and Mostow Rigidity*, Math. Res. Lett. 2 (1995), 1–8.

[D] D. McDuff, *Rational and ruled symplectic 4–manifolds and Erratum*, J. Amer. Math. Soc. 3 (1992), 988–997.

[T] C. H. Taubes, *The Seiberg–Witten Invariants and Symplectic Forms*, Math. Res. Lett. 1 (1994), 809–822.

[T1] C. H. Taubes, *More Constraints on Symplectic Forms from Seiberg–Witten Invariants*, Math. Res. Lett. 2 (1995), 9–14.

[T2] C. H. Taubes, *The Seiberg–Witten and the Gromov invariants*, Math. Res. Lett. 2 (1995), 221–238.

[W] E. Witten, *Monopoles and 4–Manifolds*, Math. Res. Lett. 1 (1994), 769–796.