SYMPLECTIC CAPACITIES IN MANIFOLDS

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Abstract. Symplectic capacities coinciding on convex sets in the standard symplectic vector space are extended to any subsets of symplectic manifolds. It is shown that, using embeddings of non-smooth convex sets and a product formula, calculations of some capacities become very simple. Moreover, it is proved that there exist such capacities which are distinct and that there are star-shaped domains diffeomorphic to the ball but not symplectomorphic to any convex set.

1. Preliminaries. For an introduction to symplectic capacities, non-smooth Hamiltonian systems and characteristic differential inclusions we refer to a previous talk given at the Banach Center in October 93 [K93].

The aim of this note is to show that some calculations of symplectic capacities can be simplified through embeddings of non-smooth convex sets. No approximations by families of Hamiltonian functions are needed. We show that definitions of capacities of convex sets in the symplectic model space \((\mathbb{R}^{2n}, \omega)\) suffice to define and to calculate in some cases symplectic capacities for subsets in any symplectic manifolds. Moreover, some applications of the product formula for convex sets derived in [K90] are given.

To define the setting, let us consider the standard symplectic linear space \(V := (\mathbb{R}^{2n}, \omega)\). The non-degenerate closed 2-form \(\omega\) is expressed by the almost complex structure \(J_0 : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}\), which is described in standard coordinates by an \(n\)-fold tensor product of the matrix \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cong i\). We write in these coordinates \(x.y = \sum_{i=1}^{2n} x_i y_i\) for the scalar product and \(\omega(x,y) = J_0 x.y\) for the symplectic form. A differentiable map \(\varphi : V \to V\) is called symplectic if \(\varphi^* \omega = \omega\), i.e. \(d\varphi(x)^T J_0 d\varphi(x) = J_0\). We denote the set of symplectic embeddings of open subsets of \(\mathbb{R}^{2n}\) into \(\mathbb{R}^{2n}\) by \(E_\omega(\mathbb{R}^{2n})\) and the symplectic diffeomorphisms of \(\mathbb{R}^{2n}\) by \(D_\omega(\mathbb{R}^{2n})\).

Let \(B(r) = B^{2n}(r) = \{ x \in \mathbb{R}^{2n} \mid |x| < r \}\) be the ball and \(Z(r) = B^2(r) \times \mathbb{R}^{2n-2} = \{ x \in \mathbb{R}^{2n} \mid q_1^2 + p_1^2 < r^2 \}\) be a cylinder with a symplectic base disc, where \(p_1, q_1\) are the

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first two coordinates.

Let $\mathcal{K}$ be the set of possibly unbounded convex sets with perhaps empty interior. Given such a convex set $K$, let $n_K(x)$ be the section of elements of length 1 in the normal cone (see e.g. [A84]) at a point $x$. We study the periodic characteristic differential inclusion of a non-smooth convex set $K$ which depends in fact only of the boundary of $K$:

\[
\begin{align*}
(i) & \quad \dot{\gamma}(t) \in Jn_K(\gamma(t)) \quad \text{a.e.} \\
(ii) & \quad \gamma(t) \in \partial K \quad \forall t \in [0, T_\gamma] \\
(iii) & \quad \gamma(t + T_\gamma) = \gamma(t) \quad \forall t \in [0, T_\gamma]
\end{align*}
\]

and $T_\gamma > 0$ is the minimal period of $\gamma$

whose moduli space of solutions is called $\Gamma(K)$, which is in a well defined way equivalent to the periodic solutions of a non-smooth Hamiltonian system (see [K93]). The set of symplectic actions $A(\gamma) = \frac{1}{2} \int \dot{\gamma}, J_0 \gamma \, dt$ of elements of $\Gamma(K)$ is called the action spectrum of $K$.

**Definition 1.** Let $c$ be the map

\[c : \mathcal{K} \longrightarrow [0, \infty] \]

\[K \mapsto c(K) = \inf\{A(\gamma) \mid \gamma \in \Gamma(K)\}\]

assigning to $K$ the minimal characteristic action of $\partial K$, using the convention that $\inf = \infty$ if $\Gamma(K)$ is empty.

It has been shown in [K90] that $c(K)$ (for a convex set $K$ with non-empty interior) can be expressed with a simple formula through the minimum of the classical dual Hamiltonian functional introduced by Clarke and Ekeland [CE80] and that it satisfies the axioms of a capacity of convex sets in the standard symplectic vector space. This means that $c$ coincides on smooth convex sets with the Ekeland-Hofer [EH89] and the Hofer-Zehnder capacity [HZ90] which are defined with the classical non-definite Hamiltonian functional and approximation by well chosen families of Hamiltonian functions. Moreover, $c$ satisfies a useful formula for symplectic products [K90] which we will use later: $c(K_1 \times K_2) = \min\{c(K_1), c(K_2)\}$.

In this paper, we study the symplectic capacities extending $c$:

**Definition 2.** Let $\mathcal{M}^{2n}$ be the family of symplectic manifolds of given dimension $2n$ and $\mathcal{S}$ a family of symplectic embeddings defined on open domains of such manifolds. Let further $\mathcal{F}$ be an $\mathcal{S}$-invariant family of subsets of these manifolds containing $\mathcal{K}$. We denote by $(D, \omega)$ the set $D$ with the symplectic form of the ambient manifold restricted to $D$ (which may be degenerate on $D$). A symplectic capacity for $\mathcal{F}$ and $\mathcal{S}$ extending $c$ is a map $C$ of $\mathcal{F}$ to $\mathbb{R}_+$ satisfying

(a) $D, D' \in \mathcal{F}, \ D \subset D' \Rightarrow C(D) \leq C(D')$,  
(b) $D \in \mathcal{F}, \ \varphi \in \mathcal{S} \Rightarrow C(\varphi(D)) = C(D)$,  
(c) if $K \in \mathcal{K}$, then $C(K) = c(K)$.

Capacities in $V$ are therefore obtained by taking $\mathcal{M}^{2n} := \{\mathbb{R}^{2n}, \omega\}, \ \mathcal{F} \subset \mathcal{P}(\mathbb{R}^{2n})$, where $\mathcal{P}(\mathbb{R}^{2n})$ is the set of all subsets of $\mathbb{R}^{2n}$, and we distinguish two cases: If $\mathcal{S} := D_\omega$, we call $C$ diffeomorphism capacity and if $\mathcal{S} := E_\omega$ we call it embedding capacity.
The axioms are designed in the way that the existence of a symplectic capacity for $V$ implies Gromov’s squeezing theorem: The existence of a symplectic embedding of the ball of radius $r$ into $Z(R)$ implies that $r \leq R$. However, to give a new proof of this theorem is not the aim of the present article.

2. Extensions in $\mathbb{R}^{2n}$. In order to control all extensions of $c$ to any subset of $\mathbb{R}^{2n}$ at the same time, the idea is to consider the smallest and biggest functions satisfying monotonicity and $D_\omega$-invariance for $D \in \mathcal{P}(\mathbb{R}^{2n})$:

**Definition 3.**

$$\ell(D) = \sup \{ c(K) \mid K \in \mathcal{K} \text{ such that } \exists \varphi \in D_\omega \text{ with } \varphi(K) \subset D \}$$

$$u(D) = \inf \{ c(K) \mid K \in \mathcal{K} \text{ such that } \exists \varphi \in D_\omega \text{ with } D \subset \varphi(K) \}.$$  

Let analogously $\ell_e$ and $u_e$ be defined with symplectic embeddings $\varphi \in \mathcal{E}_\omega$ with open domain of definition $\text{dom } \varphi \supset \bar{K}$ instead of diffeomorphisms $D_\omega$. As usual, we set $0$ the supremum and $\infty$ the infimum on the empty set.

We call $u$ and $\ell$ upper and lower symplectic capacity in $\mathbb{R}^{2n}$ respectively because any capacity extending $c$ is estimated above and below by $u$ and $\ell$:

**Theorem 1.**

(i) All symplectic capacities $C : \mathcal{F} \to [0, \infty]$ coinciding on $\mathcal{K}$ with $c$ are estimated by $u$ and $\ell$: $\ell(D) \leq C(D) \leq u(D)$ for every $D \in \mathcal{F}$. If $C_e$ is moreover $\mathcal{E}_\omega$-invariant (an embedding capacity), it satisfies $\ell \leq \ell_e \leq C_e \leq u_e \leq u$.

(ii) $u$ and $\ell$ (and also $u_e$ and $\ell_e$) are symplectic diffeomorphism capacities for $\mathcal{P}(\mathbb{R}^{2n})$. $\ell_e$ is moreover an embedding capacity, whereas $u_e$ is not $\mathcal{E}_\omega$-invariant.

(iii) They all coincide on $\mathcal{K}$ with $c$;

(iv) $u$ and $\ell$ are distinct,

(v) and $u(D) = \inf_{\varphi \in \mathcal{E}_\omega} c(\text{conv } \varphi(D))$, where $\text{conv } D$ is the closed convex hull of $D$.

**Notation.** We denote inward and outward approximation sets by

$$\mathcal{I}(D) = \{ K \in \mathcal{K} \mid \exists \varphi \in D_\omega \text{ with } \varphi(K) \subset D \}$$

$$\mathcal{O}(D) = \{ K \in \mathcal{K} \mid \exists \varphi \in D_\omega \text{ with } D \subset \varphi(K) \}.$$

then the proofs for $u$ and $\ell$ can simply be deduced from the properties of these sets.

**Proof.**

(i) We show only $\ell \leq C$. If $\ell = 0$, there is nothing to prove since any capacity $C$ is non-negative. We may therefore suppose that there is $K \in \mathcal{K}$ and $\varphi \in D_\omega$ with $\varphi(K) \subset D$; then

$$C(D) \overset{(a)}{\geq} C(\varphi(K)) \overset{(b)}{=} C(K) \overset{(c)}{=} \ell(K),$$

therefore $C(D) \geq \sup c(K) = \ell(D)$. An analogous argument yields $u(D) \geq C(D)$. The other inequalities can be proved in a similar way.
(ii) Monotonicity: \( D_1 \subseteq D_2 \implies I(D_1) \subseteq I(D_2), O(D_1) \supset O(D_2), \) therefore
\[
\ell(D_1) = \sup_{I(D_1)} c \leq \sup_{I(D_2)} c = \ell(D_2)
\]
\[
u(D_1) = \inf_{O(D_1)} c \leq \inf_{O(D_2)} c = \nu(D_2).
\]

Symplectic invariance: Let \( \psi \in \mathcal{D}_\omega. \) For \( K \in I(\psi(D)) \), there is \( \varphi(K) \subseteq \psi(D) \implies \psi^{-1} \circ \varphi(K) \subseteq D \implies K \in I(D) \), by the group property of \( \mathcal{D}_\omega \), thus \( I(\psi(D)) = I(D) \).
Analogously, \( O(\psi(D)) = O(D) \), from where
\[
\ell(\psi(D)) = \ell(D)
\]
\[
u(\psi(D)) = \nu(D).
\]
The function \( u_e(D) := \inf\{ c(K) \mid K \in K \text{ such that } \exists \varphi \in \mathcal{E}_\omega \text{ with } D \subseteq \varphi(K) \} \) satisfies immediately \( u_e(D) \leq \nu(D) \). But \( u_e \) is not \( \mathcal{E}_\omega \)-invariant (only \( \mathcal{D}_\omega \)-invariant):
\[
\psi(D) \subseteq \varphi(K) \quad \psi, \varphi \in \mathcal{E}_\omega \neq D \subseteq \psi^{-1} \circ \varphi(K)
\]
as \( \psi^{-1} \) may not be defined on \( \varphi(K) \). But \( \ell_e \) is \( \mathcal{E}_\omega \)-invariant:
\[
\varphi(K) \subseteq \psi(D) \quad \psi, \varphi \in \mathcal{E}_\omega \implies \psi^{-1} \varphi(K) \subseteq D
\]
since \( \psi^{-1} \) is defined on the (smaller) set \( \varphi(K) \).

(iii) To show \( \ell(K) = c(K) = u(K) \) for all \( K \in K \), first note that
\[
\ell(K) \geq c(K) \geq u(K)
\]
because we can take \( \varphi = id \) in the definition of \( \ell \) and \( u \). For the reverse inequality, we need the monotonicity of a symplectic capacity on smooth convex domains such as \( c_{EH} \): For all \( \varphi(K_1) \subseteq K \subseteq \varphi(K_2) \) one gets \( c(K_1) \leq c(K) \leq c(K_2) \) and therefore the claim by taking the infimum respectively the supremum on \( K \).

(iv) We prove this by exhibiting an example: Consider the shell \( A^{2n} = B(R) \setminus B(r) \), \( r < R \). To calculate \( u(A^{2n}) \), observe that all images of convex sets by diffeomorphisms containing \( A^{2n} \) contain \( B(R) \), which is itself convex; therefore \( u(A) = c(B(R)) = \pi R^2 \).
For \( \ell_c \), look first at an area-preserving embedding \( \varphi_0 \in \mathcal{E}_\omega \) in 2 dimensions \( \varphi_0 : K := (0, 2\pi) \times (0, \frac{R^2 - r^2}{\pi}) \longrightarrow A^2 \). Its image \( \tilde{A}^2 \setminus \{(p, q) \mid p = 0, q > 0\} \) has the same area as \( K \):
\[
c(K) = \pi (R^2 - r^2) = \ell_c(\varphi_0(K))
\]
and fills out \( B(R) \setminus B(r) \) with respect to the area measure. Therefore, the lower embedding capacity \( \ell_e(A^2) := \sup \{ c(K) \mid K \in \mathcal{K} \text{ such that } \exists \varphi \in \mathcal{E}_\omega \text{ with } \varphi(K) \subseteq A^2 \} \) equals \( \pi (R^2 - r^2) \). But \( \ell(A^2) \) is less than \( \ell_e(A^2) \) because \( \mathcal{D}_\omega \subseteq \mathcal{E}_\omega \), from where we get the claim for dimension 2:
\[
\ell(A^2) \leq \ell_e(A^2) = \pi (R^2 - r^2) < \pi R^2 = u(A^2).
\]
The product formula for the symplectic product \( P = A^2 \times \cdots \times A^2 \) yields finally \( \ell(P) < u(P) \) for arbitrary dimensions.
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Fig. 1. Existence of distinct symplectic capacities.

(v) \( \inf_{\varphi \in \mathcal{D}} c(\text{conv} \varphi(D)) = \inf_{\varphi} \inf_{K \in K} \{ c(K) \mid \varphi(D) \subset K \} \) by the definition of the convex hull and monotonicity for convex sets. This is equal to \( \inf_{\varphi} \inf_{K \in K} \{ c(K) \mid D \subset \varphi^{-1}(K) \} = u(D) \).

Remark. To complete the calculation for the example in (iv), consider an area-preserving diffeomorphism \( \psi \in \mathcal{E}_0 \):

\[
\psi : A^2 = B(R) \setminus B(r) \rightarrow \tilde{B}(\sqrt{R^2 - r^2 + \varepsilon^2}) \setminus B(\varepsilon)
\]

which yields, together with the above result \( t_\varepsilon(A^2) = \pi(R^2 - r^2) \) that all \( \mathcal{E}_0 \)-invariant capacities of \( A^2 \) are \( \pi(R^2 - r^2) \).

This example shows that \( \mathcal{E}_0 \)-invariant capacities \( C_\varepsilon \) do not distinguish between annuli of the same area whereas \( u \) does. On the other hand, \( u \) does not distinguish between discs and annuli of the same (outer) radius, whereas \( C_\varepsilon \) might.

3. Applications to closed characteristics and action inequalities.

As \( C_{HZ}(D) \leq u(D) \) for all \( D \), one can draw a consequence of Theorem 4 in [HZ90]: If \( u(D) \) is finite and \( \partial D \) admits a foliation \( S_\varepsilon \in [0, 1] \) by hypersurfaces such that \( S_0 = \partial D \), then there exists a periodic solution on \( S_\varepsilon \) for almost every \( \varepsilon \) in \( [0, 1] \). This contains the almost existence theorem of Hofer and Zehnder in [HZ87] which generalized Viterbo’s proof [V87] that every hypersurface of contact type carries at least one periodic orbit.

On the other hand, for a given \( D \), the characterization of \( c \) as a minimum of the dual Hamiltonian action functional together with Theorem 1(v) may be useful to show that \( u(D) \) is finite.

For convex sets with \( B(r) \subset K \subset B(R) \), a theorem by Croke–Weinstein and a theorem by Ekeland (see [E90] for both) state

(a) \( \forall \gamma \in \Gamma(K) \quad A(\gamma) \geq \pi r^2 \) (Croke–Weinstein)

(b) \( \exists \gamma \in \Gamma(K) \quad A(\gamma) \leq \pi R^2 \) (Ekeland)

These estimates can now be understood naturally in terms of capacities and are readily generalized:

Proposition 1. Consider \( K \in K \). If \( D_1 \subset K \subset D_2 \) for two sets \( D_i \in \mathcal{P}(\mathbb{R}^{2n}) \), then for any extensions \( C_1, C_2 \) of \( c \) one gets

(a) \( \forall \gamma \in \Gamma(K) \quad A(\gamma) \geq C_1(D_1) \),

(b) \( \exists \gamma \in \Gamma(K) \quad A(\gamma) \leq C_2(D_2) \).

Proof. Monotonicity and \( C_1(K) = C_2(K) = c(K) \) imply

\[
C_1(D_1) \leq c(K) = \min_{\gamma \in \Gamma(K)} A(\gamma) \leq C_2(D_2).
\]
4. Star-shaped domains need not be symplectomorphic to any convex set.

Theorem 1 together with the definition of $c$ by closed characteristics on any set has an immediate

**Corollary.** Consider a subset $D_0$ of $\mathbb{R}^{2n}$ with non-empty interior. Let $C(D_0)$ be its value for any symplectic capacity extending $c$. Then all sets $D \supset D_0$ carrying a characteristic loop on their boundary $\partial D$ with action strictly less than $C(D_0)$ cannot be symplectomorphic to a convex set. Consequently there are star-shaped domains which are not symplectomorphic to any convex set.

**Proof.** Assume $D = \varphi(K)$ for $K \in \mathcal{K}$, $\varphi \in \mathcal{D}_\omega$, and show that this leads to a contradiction. On the one hand

$$C(D_0) \leq C(D) = C(\varphi(K)) = c(K) = \inf \{A(\gamma) \mid \gamma \in \Gamma(K)\};$$

but on the other, $\varphi$ induces a bijection between characteristic curves leaving the actions invariant, because $K$ and $\varphi(K)$ are simply connected, implying that for all characteristic loops on $\partial \varphi(K)$, $A(\gamma) \geq c(K) = C(D)$, contradiction. For $C(D_0) = \infty$ the theorem means: If $\partial D$ carries a characteristic loop with finite action, then $D$ cannot be symplectically diffeomorphic to a convex set.

As examples, consider $D_0 = B(r)$; then all sets $D \supset B(r)$ with a “neck loop” $\gamma$ as in the theorem are not symplectomorphic to a convex set. In particular, there are star-shaped domains which are not symplectomorphic to any convex set.

![Fig. 2. A star-shaped domain which is not symplectomorphic to any convex set.](image)

5. Further examples.

**Proposition 2.**

(i) If $D \subset \mathbb{R}^{2n-1} \subset \mathbb{R}^{2n}$ is bounded, then $C(D) = 0$ for all symplectic capacities $C$. For example $u(S^{2n-2}) = 0$, whereas $u(S^{2n-1}) = u(B(1)) = \pi$.

(ii) A Lagrangian plane $L$ satisfies $u(L) = \infty$.

(iii) Let $D_1 \supset D_2$, then $u(D_1 \setminus D_2) = u(D_1)$.

(iv) Let $T^d = \partial B_1 \times \cdots \times \partial B_d$ be a standard isotropic torus, where $B_i$ are simply connected 2-dimensional domains in the standard symplectic 2-space. Put $B_i = 0$, $i = d + 1, \ldots, n$. Then $u(T^d) = \min_{i=1,\ldots,n} \text{Area}(B_i) < \infty$ for all $d \leq n$ which is 0 for all $d < n$. Moreover, $C_e(\Lambda) = 0$ for all $E_\omega$-invariant capacities $C_e$ and for all Lagrangian tori $\Lambda$. 
(v) Let \( \{ D_i \mid i \in I \} \) be a collection of open bounded subsets with \( \bar{D}_i \cap \bar{D}_j = \emptyset \) for \( i \neq j \) and let \( D = \bigcup_{i \in I} D_i \). Then \( u(D) \geq \sup \{ u(D_i) \} \geq \ell(D) \).

(vi) \( u(D) = u(\bar{D}) \), but \( \ell(D) \neq \ell(\bar{D}) \) in general.

(vii) \( u \) is Hausdorff-continuous on bounded domains, but \( \ell \) is not.

This illustrates how much differently from measures capacities behave.

**Proof.**

(i) Consider a vector \( e \in \mathbb{R}^{2n} \) orthogonal to \( D \) and \( \epsilon' = J e \) and the convex rectangle \( K_\epsilon := [-R, R] \epsilon' \times [-\epsilon, \epsilon] e \subset \text{span} \{ \epsilon', e \} =: E^1 \). \( D \) is contained in the symplectic product of convex sets \( K_\epsilon \times E \). By the product formula for \( \epsilon \), one gets \( C(D) \leq c(K_\epsilon \times E) = c(K_\epsilon) = 2R \cdot 2\epsilon \to 0 \) for \( \epsilon \to 0 \).

This is true for any capacity, not only for extensions of \( c \), because \( K_\epsilon \) is area-preserving diffeomorphic to a disc with area \( 2\epsilon = \pi r^2 \), i.e. \( K_\epsilon \times E \sim B(r) \times \mathbb{R}^{2n-2} \).

In conclusion, all bounded subsets of \( \mathbb{R}^{2n-1} \) have vanishing value for any capacity function \( C \).

(ii) If \( L \) is an \( n \)-dimensional plane in \( \mathbb{R}^{2n} \), its normal cone is an \( n \)-dimensional quadrant, whose image by \( J_0 \) is a quadrant in \( L \). The differential inclusion \((*)\) has therefore no closed orbit, which means that \( c(L) = \infty \).

(iii) \( \varphi(K) \supset D_1 \) if and only if \( \varphi(K) \supset D_1 \setminus D_2 \) for \( D_1 \supset D_2 \), because \( \varphi(K) \) is contractible. This implies \( \varphi(D_1 \setminus D_2) = \varphi(D_1) \) and therefore \( u(D_1 \setminus D_2) = u(D_1) \).

**Remark:** A special case is the shell \( B(R) \setminus B(r) \) we treated earlier.

(iv) \( T^d \subset \partial \left( \bigotimes_{i=1}^n B_i \right) =: \partial P \) where \( P \) is the symplectic product of \( B_i \) whose capacities can be estimated by the product formula for convex sets (with \( B_i \) area-preserving diffeomorphic to convex discs):

\[
u(P) = \min \{ u(B_1) \} = u(B_k),\]

for some \( k \). As \( u(B_k) \) is the area of the bounded set \( B_k \), \( u(T^d) \) is bounded. If \( d < n \), it is even 0.

Now we can apply Moser’s homotopy argument to show that all Lagrangian tori are symplectically equivalent, i.e. for all Lagrangian tori \( \Lambda \), there is a \( \varphi \in \mathcal{E}_\omega \) such that \( \varphi(\Lambda) = T^n \) is a standard torus. Consequently

\[
C_\epsilon(\Lambda) = C_\epsilon(\varphi(\Lambda)) = C_\epsilon(T^n) \leq u(B_k).
\]

In particular, for all \( \epsilon > 0 \), there is a standard torus \( T^n \) with \( u(T^n) = \epsilon \), i.e. \( C_\epsilon(\Lambda) = 0 \) for all \( \Lambda \) and \( C_\epsilon \).

(v) \( \varphi(K) \supset D \Rightarrow \varphi(K) \supset D_i : \mathcal{O}(D) \subset \mathcal{O}(D_i) \), implying \( u(D_i) \geq \sup_{i \in I} \{ u(D_i) \} \). If \( \varphi(K) \subset D \), then it must be contained in one of the \( D_i \), and conversely: \( \mathcal{I}(D) = \bigcup_{i \in I} \mathcal{I}(D_i) \), yielding \( \ell(\bigcup_{i \in I} D_i) = \sup_{i \in I} \{ \ell(D_i) \} \).

(vi) For any symplectic diffeomorphism \( \varphi \) defined on \( \mathbb{R}^{2n} \), one infers

\[
\mathcal{D} \subset \varphi(K) \iff \mathcal{D} \subset \varphi(K) \iff \mathcal{D} \subset \varphi(K),
\]

from where \( u(\bar{D}) = u(\bar{D}) \).

(vii) Consider \( D_\epsilon = \{ x \in \mathbb{R}^{2n} \mid \text{dist}(x, D) \leq \epsilon \} \). Because \( D_\epsilon \) is bounded, the norm \( \|d\varphi(x)\| \) is uniformly bounded from below and above on \( D_\epsilon \setminus D \). Then there exists a constant \( r \) such that \( u(D_\epsilon) = (1 + r\epsilon)u(D) \), which proves the Hausdorff-continuity of \( u \).
Both negations for \( \ell \) follow from the following counterexample: Consider a union
\[ D = \bigcup_{i=1,...,4} D_i \]
of four disjoint, juxtaposed open unit squares \( D_i \) such that \( D \) is a closed
square of length 2. Then \( D \) has capacity \( \ell(D) = 4 \), but \( \ell(D) = \ell(D_i) = 1 \). Moreover
\( D_\varepsilon \supset D \) for all \( \varepsilon > 0 \).

**Theorem 2.** For any capacity \( C \) extending \( c \) the generalized product formula holds:

(a) \( \min \{ \ell(D_1), \ell(D_2) \} \leq \ell(D_1 \times D_2) \leq C(D_1 \times D_2) \leq u(D_1 \times D_2) \leq \min \{ u(D_1), u(D_2) \} \).

(b) If \( \ell(D_i) = u(D_i) \) for \( i = 1, 2 \), then \( C(D_1 \times D_2) = \min \{ C(D_1), C(D_2) \} \).

**Proof.**

(a) Take a minimizing sequence \( (K_i^k, \varphi_i^k), k \in \mathbb{N} \) for each \( i \) and conclude: For \( u \),
assume \( D_i \subset \varphi_i^k(K_i^k) \) and \( u(D_i) = \inf_c c(K_i^k) \) for \( i = 1, 2 \). Clearly
\( D_1 \times D_2 \subset \varphi_i^k(K_i^k) \times \varphi_i^k(K_i^k) \) and therefore using the product formula for convex sets
\( u(D_1 \times D_2) \leq \inf_k e(K_i^k \times K_i^k) = \inf_k \min \{ c(K_i^k), c(K_i^k) \} = \min \{ u(D_1), u(D_2) \} \), and similarly for \( \ell \).

(b) follows immediately from (a).

**Remark.** It is easy to see that there are ‘many’ sets satisfying the hypotheses of (b)
which are not symplectomorphic to any convex set: Take for instance examples \( D \) similar
to the one in the Corollary to Theorem 1 such that moreover \( B(r) \subset D \subset Z(r) \), see
Figure 2. They all satisfy \( \ell(D) = u(D) \) and are not symplectomorphic to any convex set,
which shows that Theorem 2 is a true generalization of the product formula for \( K \).

Theorem 2 applies in particular to \( c_{EH} \) (using [Si90]) and \( c_{HZ} \).

**6. Extensions to general symplectic manifolds.** Now that extensions to \( \mathbb{R}^{2n} \)
have been studied, it is easy to generalize them analogously to manifolds.

**Definition 4.** For any subset of a symplectic manifold of given dimension \( 2n \), we
define the non-negative numbers
\[ u(D) = \inf_{\varphi \in \mathcal{F}_K} c(\text{conv} \varphi(D)), \]
\[ e(D) = \sup \{ c(K) \mid K \in \mathcal{K} \text{ such that } \exists \varphi \in \mathcal{S} \text{ with } \varphi(K) \subset D \}, \]
\[ k(D) = \sup \{ u(P) \mid P \subset D \text{ contractible} \}. \]

**Theorem 3.**

(i) \( e, k \) and \( u \) satisfy the axioms of Definition 2 for any subsets of all symplectic
manifolds and any family of embeddings.

(ii) All symplectic embedding capacities \( C \) coinciding on \( \mathcal{K} \) with \( c \) are estimated by \( e \)
and \( u \), \( e \leq C \leq u \).

**Proof.** The proof is analogous to the one for \( \ell \) and \( u \) and is therefore skipped. For \( k \),
one simply observes that every \( \varphi(K) \) is a contractible set, so that \( e \leq k \leq u \) immediately
follows.

**7. Surfaces.** Given any compact surface \( S \) of genus \( g \), consider the canonical system
of \( 2g \) non-dividing curves \( \alpha_i, \beta_i, i = 1, \ldots, g \). Then \( S \setminus A \) with \( A := \bigcup_{i=1}^g \alpha_i \cup \beta_i \) is
conformally equivalent to a 2g-gon, which is itself conformally equivalent to a disk $D$ in $\mathbb{C}$:

$$f : S \setminus A \to D$$

is a conformal map and is therefore symplectic:

$$\text{Area}(D) = \text{Area}(S \setminus A) = \text{Area}(S).$$

**Consequences.**

1. $P = S \setminus A$ is contractible. Every other contractible subset of $S$ has area less than $\text{Area}(S)$, therefore $k(S) = u(P) = \text{Area}(S)$.

2. $f^{-1}$ is a symplectic diffeomorphism $D \to S \setminus A$ from an open convex set into $S$, which realizes the maximum for area-preserving embeddings: $e(S) = \text{Area}(S)$.

This proves

**Proposition 3.** For any surface $S$ with or without boundary, all symplectic embedding capacities $C$ extending $c$ are equal to the area of $S$: $e(S) = \text{Area}(S) = k(S)$.

Proposition 3 has first been proved by Siburg [Si93] for embedding capacities (which he called Hofer-Zehnder capacities) by construction of an adapted Hamiltonian function.

This is in contrast to the diffeomorphism capacity $u$ which is different from the area:

Recall that the annulus $S = B(R) \setminus \bar{B}(r)$ satisfies $e(S) = \text{Area}(S) = k(S) = \pi(R^2 - r^2)$ but $u(S) = u(S) = \pi R^2$, see Figure 1.

8. Symplectic 4-tori and the Herman-Zehnder example. Following [HZ94], we consider $(\mathbb{R}^4, \omega_\alpha)$ with the symplectic structure $\omega_\alpha(x, y) = A_\alpha X.Y$ defined by

$$A_\alpha = \begin{pmatrix} 0 & -1 & \alpha_2 & 0 \\ 1 & 0 & -\alpha_1 & 0 \\ -\alpha_2 & \alpha_1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -A_\alpha^T$$

(which satisfies $\det(A_\alpha) = 1$ but not $A_\alpha^2 = -I$). This form induces a symplectic structure on the manifold $M = T^3 \times [0, d] = \mathbb{R}^3/\mathbb{Z}^3 \times [0, d]$ denoted again $\omega_\alpha$. For $\alpha_1, \alpha_2 = 0$, one gets the standard almost complex structure $J_0$. For $d < 1$, $(M, \omega_\alpha)$ is embedded in the torus $(T^4, \omega_\alpha)$.

Functions $H$ on $\mathbb{R}^4$ which are 1-periodic in the first three variables pass to the quotient as well as their Hamiltonian vector fields

$$\xi_H := -A_\alpha^{-1} H'(x),$$

where $H'(x)$ is the Euclidean gradient of $H$. As

$$A_\alpha^{-1} = \begin{pmatrix} 0 & -1 & 0 & -\alpha_1 \\ 1 & 0 & 0 & -\alpha_2 \\ 0 & 0 & 0 & -1 \\ \alpha_1 & \alpha_2 & 1 & 0 \end{pmatrix},$$

we get for the Hamiltonian function $H_0(x) = x_4$ a constant vector field

$$\xi_{H_0} = (\alpha_1, \alpha_2, 1, 0) =: (\alpha, 0).$$
which integrates to an affine flow preserving all 3-tori $T^3 \times \{s\}$. If $\alpha = (\alpha_1, \alpha_2, 1)$ is rationally independent, i.e. $\alpha z \neq 0 \ \forall z \in \mathbb{Z}^3 \setminus 0$, this flow is dense and has no periodic orbits. Therefore it represents an example of a Hamiltonian flow whose energy levels $T^3 \times \{s\}$ are all regular and compact but none of them carries a periodic orbit.

M. Herman proved in [H91] that $H_0$ is dynamically stable under perturbations if $\alpha$ is irrational satisfying a diophantine condition: This represents a counter-example against the $C^k$-closing conjecture for $k$ sufficiently large.

In [HZ94], it has been showed that $c_{HZ}(M, \omega_\alpha)$ is infinite if $\alpha$ is irrational. Here we estimate $C(M, \omega_\alpha)$ for any $C$ extending $c$ by exhibiting a convex set contained in $M$:

$$C(M, \omega_\alpha) \left\{ \begin{array}{ll} = \infty & \text{if } \alpha \text{ irrational} \\ \geq d & \text{if } \alpha \text{ rational} \end{array} \right.$$  

**Proof.** Consider the linear map $(\mathbb{R}^4, \omega) \to (\mathbb{R}^4, \omega_\alpha)$ given by the matrix

$$N_\alpha = \begin{pmatrix} 1 & 0 & \alpha_1 & 0 \\ 0 & 1 & \alpha_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is symplectic: $N_\alpha^T AN_\alpha = J_0$. Denote the canonical basis by $e_k$.

(i) If $\alpha$ is irrational, then the Lagrangian plane $L$ spanned by $e_1$ and $e_3$ is embedded by $N_\alpha$ into $\tilde{M} = \mathbb{R}^3 \times [0, d]$. The quotient of this map onto $M$ winds $L$ in the 3-torus densely around itself. But a Lagrangian plane has infinite capacity, from where the first part of the claim.

If $\alpha$ is rational, then there are relatively prime $n_i \in \mathbb{Z}$, $i = 1, 2$ such that $\alpha_i = \frac{n_i}{n_3}$. Then:

(ii) $N_\alpha$ embeds the standard unit 3-cube into a fundamental domain of the action of $\mathbb{Z}^3$ on $\tilde{M}$. Therefore $C(M, \omega_\alpha) \geq c([0, 1]^3 \times [0, d], \omega_0) = d$ by the product formula, which proves the second claim.

(iii) But the map $N_\alpha$ also sends then the parallelogram $P$ spanned by $e_1, \frac{1}{n_3}e_2, n_3e_3, e_4$ into a fundamental domain of the action of $\mathbb{Z}^3$ on $\tilde{M}$. Therefore $C(M, \omega_\alpha) \geq c(P, \omega_0)$, which is equal to $\min\{\frac{1}{n_3}, d\}$ again by the product formula for $c$.

This last observation shows the relation to (i), but also prompts a question concerning fundamental domains of in $\tilde{M}$ (which would determine $e(M)$): What is the biggest capacity a fundamental domain in $\tilde{M}$ can have?

**References**


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