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SYMPLECTIC CAPACITIES IN MANIFOLDS

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Abstract. Symplectic capacities coinciding on convex sets in the standard symplectic vector space are extended to any subsets of symplectic manifolds. It is shown that, using embeddings of non-smooth convex sets and a product formula, calculations of some capacities become very simple. Moreover, it is proved that there exist such capacities which are distinct and that there are star-shaped domains diffeomorphic to the ball but not symplectomorphic to any convex set.

1. Preliminaries. For an introduction to symplectic capacities, non-smooth Hamiltonian systems and characteristic differential inclusions we refer to a previous talk given at the Banach Center in October 93 [K93].

The aim of this note is to show that some calculations of symplectic capacities can be simplified through embeddings of *non-smooth convex sets*. No approximations by families of Hamiltonian functions are needed. We show that definitions of capacities of convex sets in the symplectic model space $(\mathbb{R}^{2n}, \omega)$ suffice to define and to calculate in some cases symplectic capacities for subsets in any symplectic manifolds. Moreover, some applications of the product formula for convex sets derived in [K90] are given.

To define the setting, let us consider the standard symplectic linear space $V := (\mathbb{R}^{2n}, \omega)$. The non-degenerate closed 2-form ω is expressed by the almost complex structure $J_0 : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$, which is described in standard coordinates by an n-fold tensor product of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cong i$. We write in these coordinates $x.y = \sum_{i=1}^{2n} x_i y_i$ for the scalar product and $\omega(x, y) = J_0 x.y$ for the symplectic form. A differentiable map $\varphi : V \to V$ is called symplectic if $\varphi^* \omega = \omega$, i.e. $d\varphi(x)^T J_0 d\varphi(x) = J_0$. We denote the set of symplectic embeddings of open subsets of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ by $\mathcal{E}_\omega(\mathbb{R}^{2n})$ and the symplectic diffeomorphisms of $\mathbb{R}^{2n}$ by $\mathcal{D}_\omega(\mathbb{R}^{2n})$.

Let $B(r) = B^{2n}(r) = \{x \in \mathbb{R}^{2n} \mid |x| < r\}$ be the ball and $Z(r) = B^2(r) \times \mathbb{R}^{2n-2} = \{x \in \mathbb{R}^{2n} \mid q_1^2 + p_1^2 < r^2\}$ be a cylinder with a symplectic base disc, where p_1, q_1 are the

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first two coordinates.

Let $\mathcal{K}$ be the set of possibly unbounded convex sets with perhaps empty interior. Given such a convex set K, let $n_K(x)$ be the section of elements of length 1 in the normal cone (see e.g. [A84]) at a point x. We study the periodic characteristic differential inclusion of a non-smooth convex set K which depends in fact only of the boundary of K:

$$\left.\begin{array}{rll} \text{(i)} & \dot\gamma(t) \in Jn_K\big(\gamma(t)\big) & \text{a.e.} \\ \text{(ii)} & \gamma(t) \in \partial K & \forall t \in [0, T_\gamma] \\ \text{(iii)} & \gamma(t + T_\gamma) = \gamma(t) & \forall t \in [0, T_\gamma] \\ & \multicolumn{2}{l}{\text{and } T_\gamma > 0 \text{ is the minimal period of } \gamma} \end{array}\right\} (*)$$

whose moduli space of solutions is called $\Gamma(K)$, which is in a well defined way equivalent to the periodic solutions of a non-smooth Hamiltonian system (see [K93]). The set of symplectic actions $A(\gamma) = \frac{1}{2}\int \dot\gamma.J_0\gamma\,dt$ of elements of $\Gamma(K)$ is called the action spectrum of K.

Definition 1. Let c be the map

$$\begin{aligned} c : \mathcal{K} &\longrightarrow [0, \infty] \\ K &\longmapsto c(K) = \inf\{A(\gamma) \mid \gamma \in \Gamma(K)\} \end{aligned}$$

assigning to K the *minimal characteristic action* of ∂K, using the convention that $\inf = \infty$ if $\Gamma(K)$ is empty.

It has been shown in [K90] that $c(K)$ (for a convex set K with non-empty interior) can be expressed with a simple formula through the minimum of the classical dual Hamiltonian functional introduced by Clarke and Ekeland [CE80] and that it satisfies the axioms of a capacity of convex sets in the standard symplectic vector space. This means that c coincides on *smooth* convex sets with the Ekeland-Hofer [EH89] and the Hofer-Zehnder capacity [HZ90] which are defined with the classical non-definite Hamiltonian functional and approximation by well chosen families of Hamiltonian functions. Moreover, c satisfies a useful formula for symplectic products [K90] which we will use later: $c(K_1 \times K_2) = \min\{c(K_1), c(K_2)\}$.

In this paper, we study the symplectic capacities extending c:

Definition 2. Let $\mathcal{M}^{2n}$ be the family of symplectic manifolds of given dimension $2n$ and $\mathcal{S}$ a family of symplectic embeddings defined on open domains of such manifolds. Let further $\mathcal{F}$ be an $\mathcal{S}$-invariant family of subsets of these manifolds containing $\mathcal{K}$. We denote by (D, ω) the set D with the symplectic form of the ambient manifold restricted to D (which may be degenerate on D). A *symplectic capacity for $\mathcal{F}$ and $\mathcal{S}$ extending c* is a map C of $\mathcal{F}$ to $\mathbb{R}_+$ satisfying

(a) $D, D' \in \mathcal{F},\ D \subset D' \Longrightarrow C(D) \le C(D')$,
(b) $D \in \mathcal{F},\ \varphi \in \mathcal{S} \Longrightarrow C\big(\varphi(D)\big) = C(D)$,
(c) if $K \in \mathcal{K}$, then $C(K) = c(K)$.

Capacities in V are therefore obtained by taking $\mathcal{M}^{2n} := \{\mathbb{R}^{2n}, \omega\},\ \mathcal{F} \subset \mathcal{P}(\mathbb{R}^{2n})$, where $\mathcal{P}(\mathbb{R}^{2n})$ is the set of all subsets of $\mathbb{R}^{2n}$, and we distinguish two cases: If $\mathcal{S} := \mathcal{D}_\omega$ we call C *diffeomorphism capacity* and if $\mathcal{S} := \mathcal{E}_\omega$ we call it *embedding capacity.*

The axioms are designed in the way that the existence of a symplectic capacity for V implies Gromov's squeezing theorem: The existence of a symplectic embedding of the ball of radius r into $Z(R)$ implies that $r \leq R$. However, to give a new proof of this theorem is not the aim of the present article.

2. Extensions in $\mathbb{R}^{2n}$. In order to control all extensions of c to any subset of $\mathbb{R}^{2n}$ at the same time, the idea is to consider the smallest and biggest functions satisfying monotonicity and $\mathcal{D}_\omega$-invariance for $D \in \mathcal{P}(\mathbb{R}^{2n})$:

DEFINITION 3.

$$\begin{aligned} \ell(D) &= \sup\{c(K) \mid K \in \mathcal{K} \quad \text{such that } \exists \varphi \in \mathcal{D}_\omega \text{ with } \varphi(K) \subset D\} \\ u(D) &= \inf\{c(K) \mid K \in \mathcal{K} \quad \text{such that } \exists \varphi \in \mathcal{D}_\omega \text{ with } D \subset \varphi(K)\}. \end{aligned}$$

Let analogously ℓ_e and u_e be defined with symplectic embeddings $\varphi \in \mathcal{E}_\omega$ with open domain of definition $\operatorname{dom}\varphi \supset \bar{K}$ instead of diffeomorphisms $\mathcal{D}_\omega$. As usual, we set 0 the supremum and ∞ the infimum on the empty set.

We call u and ℓ *upper and lower symplectic capacity in* $\mathbb{R}^{2n}$ respectively because any capacity extending c is estimated above and below by u and ℓ:

THEOREM 1.

(i) *All symplectic capacities* $C : \mathcal{F} \to [0, \infty]$ *coinciding on* $\mathcal{K}$ *with* c *are estimated by* u *and* ℓ: $\ell(D) \leq C(D) \leq u(D)$ *for every* $D \in \mathcal{F}$. *If* C_e *is moreover* $\mathcal{E}_\omega$*-invariant* (*an embedding capacity*), *it satisfies* $\ell \leq \ell_e \leq C_e \leq u_e \leq u$.

(ii) u *and* ℓ (*and also* u_e *and* ℓ_e) *are symplectic diffeomorphism capacities for* $\mathcal{P}(\mathbb{R}^{2n})$. ℓ_e *is moreover an embedding capacity*, *whereas* u_e *is not* $\mathcal{E}_\omega$*-invariant.*

(iii) *They all coincide on* $\mathcal{K}$ *with* c;

(iv) u *and* ℓ *are distinct*,

(v) *and* $u(D) = \inf_{\varphi \in \mathcal{D}_\omega} c\big(\operatorname{conv}\varphi(D)\big)$, *where* $\operatorname{conv} D$ *is the closed convex hull of* D.

Notation. We denote inward and outward approximation sets by

$$\begin{aligned} \mathcal{I}(D) &= \{K \in \mathcal{K} \mid \exists \varphi \in \mathcal{D}_\omega \quad \text{with } \varphi(K) \subset D\} \\ \mathcal{O}(D) &= \{K \in \mathcal{K} \mid \exists \varphi \in \mathcal{D}_\omega \quad \text{with } D \subset \varphi(K)\}, \end{aligned}$$

then the proofs for u and ℓ can simply be deduced from the properties of these sets.

Proof.

(i) We show only $\ell \leq C$. If $\ell = 0$, there is nothing to prove since any capacity C is non-negative. We may therefore suppose that there is $K \in \mathcal{K}$ and $\varphi \in \mathcal{D}_\omega$ with $\varphi(K) \subset D$; then

$$C(D) \overset{(a)}{\geq} C\big(\varphi(K)\big) \overset{(b)}{=} C(K) \overset{(c)}{=} c(K),$$

therefore $C(D) \geq \sup c(K) = \ell(D)$. An analogous argument yields $u(D) \geq C(D)$. The other inequalities can be proved in a similar way.

(ii) Monotonicity: $D_1 \subset D_2 \Longrightarrow \mathcal{I}(D_1) \subset \mathcal{I}(D_2)$, $\mathcal{O}(D_1) \supset \mathcal{O}(D_2)$, therefore

$$\ell(D_1) = \sup_{\mathcal{I}(D_1)} c \le \sup_{\mathcal{I}(D_2)} c = \ell(D_2)$$
$$u(D_1) = \inf_{\mathcal{O}(D_1)} c \le \inf_{\mathcal{O}(D_2)} c = u(D_2).$$

Symplectic invariance: Let $\psi \in \mathcal{D}_\omega$. For $K \in \mathcal{I}\big(\psi(D)\big)$, there is $\varphi(K) \subset \psi(D) \Longrightarrow \psi^{-1} \circ \varphi(K) \subset D \Longrightarrow K \in \mathcal{I}(D)$, by the group property of $\mathcal{D}_\omega$, thus $\mathcal{I}\big(\psi(D)\big) = \mathcal{I}(D)$. Analogously, $\mathcal{O}\big(\psi(D)\big) = \mathcal{O}(D)$, from where

$$\ell\big(\psi(D)\big) = \ell(D)$$
$$u\big(\psi(D)\big) = u(D).$$

The function $u_e(D) := \inf\{c(K) \mid K \in \mathcal{K} \text{ such that } \exists \varphi \in \mathcal{E}_\omega \text{ with } D \subset \varphi(K)\}$ satisfies immediately $u_e(D) \le u(D)$. But u_e is not $\mathcal{E}_\omega$-invariant (only $\mathcal{D}_\omega$-invariant):

$$\psi(D) \subset \varphi(K) \quad \psi, \varphi \in \mathcal{E}_\omega \not\Rightarrow D \subset \psi^{-1} \circ \varphi(K)$$

as ψ^{-1} may not be defined on $\varphi(K)$. But ℓ_e is $\mathcal{E}_\omega$-invariant:

$$\varphi(K) \subset \psi(D) \quad \psi, \varphi \in \mathcal{E}_\omega \Longrightarrow \psi^{-1}\varphi(K) \subset D$$

since ψ^{-1} is defined on the (smaller) set $\varphi(K)$.

(iii) To show $\ell(K) = c(K) = u(K)$ for all $K \in \mathcal{K}$, first note that

$$\ell(K) \ge c(K) \ge u(K)$$

because we can take $\varphi = id$ in the definition of ℓ and u. For the reverse inequality, we need the monotonicity of a symplectic capacity on smooth convex domains such as c_{EH}: For all $\varphi(K_1) \subset K \subset \psi(K_2)$ one gets $c(K_1) \le c(K) \le c(K_2)$ and therefore the claim by taking the infimum respectively the supremum on K_i.

(iv) We prove this by exhibiting an example: Consider the shell $A^{2n} = B(R) \setminus B(r)$, $r < R$. To calculate $u(A^{2n})$, observe that all images of convex sets by diffeomorphisms containing A^{2n} contain $B(R)$, which is itself convex; therefore $u(A) = c\big(B(R)\big) = \pi R^2$. For ℓ, look first at an area-preserving embedding $\varphi_0 \in \mathcal{E}_\omega$ in 2 dimensions $\varphi_0 : K := (0, 2\pi) \times \big(0, \frac{R^2 - r^2}{2}\big) \longrightarrow A^2$. Its image $\mathring{A}^2 \setminus \{(p,q) \mid p = 0, q > 0\}$ has the same area as K:

$$c(K) = \pi(R^2 - r^2) = \ell_e\big(\varphi_0(K)\big),$$

and fills out $B(R) \setminus B(r)$ with respect to the area measure. Therefore, the lower embedding capacity $\ell_e(A^2) := \sup\{c(K) \mid K \in \mathcal{K} \text{ such that } \exists \varphi \in \mathcal{E}_\omega \text{ with } \varphi(K) \subset A^2\}$ equals $\pi(R^2 - r^2)$. But $\ell(A^2)$ is less than $\ell_e(A^2)$ because $\mathcal{D}_\omega \subset \mathcal{E}_\omega$, from where we get the claim for dimension 2:

$$\ell(A^2) \le \ell_e(A^2) = \pi(R^2 - r^2) < \pi R^2 = u(A^2).$$

The product formula for the symplectic product $P = A^2 \times \cdots \times A^2$ yields finally $\ell(P) < u(P)$ for arbitrary dimensions.

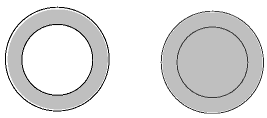

Fig. 1. Existence of distinct symplectic capacities.

(v) $\inf_{\varphi\in\mathcal{D}_\omega} c\big(\operatorname{conv}\varphi(D)\big) = \inf_\varphi \inf_{K\in\mathcal{K}}\{c(K) \mid \varphi(D)\subset K\}$ by the definition of the convex hull and monotonicity for convex sets. This is equal to $\inf_\varphi \inf_K\{c(K) \mid D\subset \varphi^{-1}(K)\} = u(D)$. ∎

Remark. To complete the calculation for the example in (iv), consider an area-preserving diffeomorphism $\psi_\varepsilon \in \mathcal{E}_\omega$:

$$\psi_\varepsilon : \mathring{A}^2 = \mathring{B}(R)\setminus B(r) \longrightarrow \mathring{B}(\sqrt{R^2-r^2+\varepsilon^2})\setminus B(\varepsilon)$$

which yields, together with the above result $\ell_e(A^2) = \pi(R^2-r^2)$ that all $\mathcal{E}_\omega$-invariant capacities of A^2 are $\pi(R^2-r^2)$.

This example shows that $\mathcal{E}_\omega$-invariant capacities C_e do not distinguish between annuli of the same area whereas u does. On the other hand, u does not distinguish between discs and annuli of the same (outer) radius, whereas C_e might.

3. Applications to closed characteristics and action inequalities.

As $C_{HZ}(D) \le u(D)$ for all D, one can draw a consequence of Theorem 4 in [HZ90]: *If $u(D)$ is finite* and ∂D admits a foliation $S_\varepsilon \in [0,1]$ by hypersurfaces such that $S_0 = \partial D$, then there exists a periodic solution on S_ε for almost every ε in $[0,1]$. This contains the almost existence theorem of Hofer and Zehnder in [HZ87] which generalized Viterbo's proof [V87] that every hypersurface of contact type carries at least one periodic orbit.

On the other hand, for a given D, the characterization of c as a minimum of the dual Hamiltonian action functional together with Theorem 1(v) may be useful to show that $u(D)$ is finite.

For convex sets with $B(r)\subset K\subset B(R)$, a theorem by Croke–Weinstein and a theorem by Ekeland (see [E90] for both) state

(a) $\forall\gamma\in\Gamma(K)\quad A(\gamma)\ge \pi r^2$ (Croke–Weinstein)
(b) $\exists\gamma\in\Gamma(K)\quad A(\gamma)\le \pi R^2$ (Ekeland)

These estimates can now be understood naturally in terms of capacities and are readily generalized:

PROPOSITION 1. *Consider $K\in\mathcal{K}$. If $D_1\subset K\subset D_2$ for two sets $D_i\in\mathcal{P}(\mathbb{R}^{2n})$, then for any extensions C_1, C_2 of c one gets*

(a) $\forall\gamma\in\Gamma(K)\quad A(\gamma)\ge C_1(D_1)$,
(b) $\exists\gamma\in\Gamma(K)\quad A(\gamma)\le C_2(D_2)$.

Proof. Monotonicity and $C_1(K) = C_2(K) = c(K)$ imply

$$C_1(D_1)\le c(K) = \min_{\gamma\in\Gamma(K)} A(\gamma)\le C_2(D_2). \ \blacksquare$$

As concrete example, one can improve the inequalities already by taking for D_1 and D_2 two radially deformed ellipsoids [K90]. They are symplectomorphic to standard ellipsoids and have therefore known capacity.

4. Star-shaped domains need not be symplectomorphic to any convex set. Theorem 1 together with the definition of c by closed characteristics on any set has an immediate

COROLLARY. *Consider a subset D_0 of $\mathbb{R}^{2n}$ with non-empty interior. Let $C(D_0)$ be its value for any symplectic capacity extending c. Then all sets $D \supset D_0$ carrying a characteristic loop on their boundary ∂D with action strictly less than $C(D_0)$* cannot *be symplectomorphic to a convex set. Consequently there are star-shaped domains which are not symplectomorphic to any convex set.*

Proof. Assume $D = \varphi(K)$ for $K \in \mathcal{K}$, $\varphi \in \mathcal{D}_\omega$, and show that this leads to a contradiction. On the one hand

$$C(D_0) \leq C(D) = C\big(\varphi(K)\big) = c(K) = \inf\{A(\gamma) \mid \gamma \in \Gamma(K)\};$$

but on the other, φ induces a bijection between characteristic curves leaving the actions invariant, because K and $\varphi(K)$ are simply connected, implying that for all characteristic loops on $\partial\varphi(K)$, $A(\gamma) \geq c(K) = C(D)$, contradiction. For $C(D_0) = \infty$ the theorem means: If ∂D carries a characteristic loop with finite action, then D cannot be symplectically diffeomorphic to a convex set. ■

As examples, consider $D_0 = B(r)$; then all sets $D \supset B(r)$ with a "neck loop" γ as in the theorem are not symplectomorphic to a convex set. In particular, there are star-shaped domains which are not symplectomorphic to any convex set.

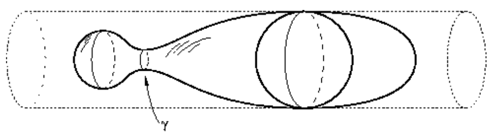

Fig. 2. A star-shaped domain which is not symplectomorphic to any convex set.

5. Further examples.

PROPOSITION 2.

(i) *If $D \subset \mathbb{R}^{2n-1} \subset \mathbb{R}^{2n}$ is bounded, then $C(D) = 0$ for all symplectic capacities C. For example $u(S^{2n-2}) = 0$, whereas $u(S^{2n-1}) = u\big(B(1)\big) = \pi$.*

(ii) *A Lagrangian plane L satisfies $u(L) = \infty$.*

(iii) *Let $\mathring{D}_1 \supset \bar{D}_2$, then $u(D_1 \setminus D_2) = u(D_1)$.*

(iv) *Let $T^d = \partial B_1 \times \cdots \times \partial B_d$ be a standard isotropic torus, where B_i are simply connected 2-dimensional domains in the standard symplectic 2-space. Put $B_i = 0$, $i = d+1, \ldots, n$. Then $u(T^d) = \min_{i=1,\ldots,n} \mathrm{Area}(B_i) < \infty$ for all $d \leq n$ which is 0 for all $d < n$. Moreover, $C_e(\Lambda) = 0$ for all $\mathcal{E}_\omega$-invariant capacities C_e and for* all Lagrangian *tori Λ.*

(v) *Let* $\{D_i \mid i \in I\}$ *be a collection of open bounded subsets with* $\bar{D}_i \cap \bar{D}_j = \varnothing$ *for* $i \neq j$ *and let* $D = \bigcup_{i\in I} D_i$. *Then* $u(D) \geq \sup\{u(D_i)\} \geq \ell(D)$.

(vi) $u(\bar{D}) = u(\mathring{D})$, *but* $\ell(\bar{D}) \neq \ell(\mathring{D})$ *in general.*

(vii) u *is Hausdorff-continuous on bounded domains*, *but* ℓ *is not.*

This illustrates how much *differently from measures* capacities behave.

Proof.

(i) Consider a vector $e \in \mathbb{R}^{2n}$ orthogonal to D and $e' = Je$ and the convex rectangle $K_\varepsilon := [-R, R]e' \times [-\varepsilon, \varepsilon]e \subset \operatorname{span}\{e', e\} =: E^\perp$. D is contained in the symplectic product of convex sets $K_\varepsilon \times E$. By the product formula for c, one gets $C(D) \leq c(K_\varepsilon \times E) = c(K_\varepsilon) = 2R \cdot 2\varepsilon \to 0$ for $\varepsilon \to 0$.

This is true for any capacity, not only for extensions of c, because K_ε is area-preserving diffeomorphic to a disc with area $2\varepsilon =: \pi r^2$, i.e. $K_\varepsilon \times E \sim B(r) \times \mathbb{R}^{2n-2}$.

In conclusion, all bounded subsets of $\mathbb{R}^{2n-1}$ have vanishing value for any capacity function C.

(ii) As L is an n-dimensional plane in $\mathbb{R}^{2n}$, its normal cone is an n-dimensional quadrant, whose image by J_0 is a quadrant in L. The differential inclusion $(*)$ has therefore no closed orbit, which means that $c(L) = \infty$.

(iii) $\varphi(K) \supset D_1$ if and only if $\varphi(K) \supset D_1 \setminus D_2$ for $\mathring{D}_1 \supset \bar{D}_2$, because $\varphi(K)$ is contractible. This implies $\mathcal{O}(D_1 \setminus \bar{D}_2) = \mathcal{O}(D_1)$ and therefore $u(D_1 \setminus D_2) = u(D_1)$. (*Remark*: A special case is the shell $\mathring{B}(R) \setminus B(r)$ we treated earlier.)

(iv) $T^d \subset \partial\big(\bigotimes_{i=1}^n B_i\big) =: \partial P$ where P is the symplectic product of B_i whose capacities can be estimated by the product formula for convex sets (with B_i area-preserving diffeomorphic to convex discs):

$$u(P) = \min\{u(B_i)\} = u(B_k),$$

for some k. As $u(B_k)$ is the area of the bounded set B_k, $u(T^d)$ is bounded. If $d < n$, it is even 0.

Now we can apply Moser's homotopy argument to show that all Lagrangian tori are symplectically equivalent, i.e. for all Lagrangian tori Λ, there is a $\varphi \in \mathcal{E}_\omega$ such that $\varphi(\Lambda) = T^n$ is a standard torus. Consequently

$$C_e(\Lambda) = C_e\big(\varphi(\Lambda)\big) = C_e(T^n) \leq u(B_k).$$

In particular, for all $\varepsilon > 0$, there is a standard torus T^n with $u(T^n) = \varepsilon$, i.e. $C_e(\Lambda) = 0$ for all Λ and C_e.

(v) $\varphi(K) \supset D \Rightarrow \varphi(K) \supset D_i : \mathcal{O}(D) \subset \mathcal{O}(D_i)$, implying $u(D) \geq \sup_{i\in I}\{u(D_i)\}$. If $\varphi(K) \subset D$, then it must be contained in one of the D_i and conversely: $\mathcal{I}(D) = \bigcup_{i\in I} \mathcal{I}(D_i)$, yielding $\ell\big(\bigcup_{i\in I} D_i\big) = \sup_{i\in I}\{\ell(D_i)\}$.

(vi) For any symplectic diffeomorphism φ defined on $\mathbb{R}^{2n}$, one infers

$$\mathring{D} \subset \varphi(K) \Longleftrightarrow \mathring{D} \subset \varphi(\mathring{K}) \Longleftrightarrow \bar{D} \subset \varphi(\bar{K}),$$

from where $u(\bar{D}) = u(\mathring{D})$.

(vii) Consider $D_\varepsilon = \{x \in \mathbb{R}^{2n} \mid \operatorname{dist}(x, D) \leq \varepsilon\}$. Because D_ε is bounded, the norm $\|d\varphi(x)\|$ is uniformly bounded from below and above on $D_\varepsilon \setminus D$. Then there exists a constant r such that $u(D_\varepsilon) = (1 + r\varepsilon)u(D)$, which proves the Hausdorff-continuity of u.

Both negations for ℓ follow from the following counterexample: Consider a union $D = \bigcup_{i=1,\ldots,4} D_i$ of four disjoint, juxtaposed open unit squares D_i such that $\bar{D}$ is a closed square of length 2. Then $\bar{D}$ has capacity $\ell(\bar{D}) = 4$, but $\ell(D) = \ell(D_1) = 1$. Moreover $D_\varepsilon \supset \bar{D}$ for all $\varepsilon > 0$. ■

THEOREM 2. *For any capacity C extending c the generalized product formula holds*:
(a) $\min\{\ell(D_1), \ell(D_2)\} \le \ell(D_1 \times D_2) \le C(D_1 \times D_2) \le u(D_1 \times D_2) \le \min\{u(D_1), u(D_2)\}$.
(b) *If $\ell(D_i) = u(D_i)$ for $i = 1, 2$, then $C(D_1 \times D_2) = \min\{C(D_1), C(D_2)\}$.*

Proof.

(a) Take a minimizing sequence $(K_i^k, \varphi_i^k), k \in \mathbb{N}$ for each i and conclude: For u, assume $D_i \subset \varphi_i^k(K_i^k)$ and $u(D_i) = \inf_k c(K_i^k)$ for $i = 1, 2$. Clearly $D_1 \times D_2 \subset \varphi_1^k(K_1^k) \times \varphi_2^k(K_2^k)$ and therefore using the product formula for convex sets $u(D_1 \times D_2) \le \inf_k c(K_1^k \times K_2^k) = \inf_k \min\{c(K_1^k), c(K_2^k)\} = \min\{u(D_1), u(D_2)\}$, and similarly for ℓ.

(b) follows immediately from (a). ■

Remark. It is easy to see that there are 'many' sets satisfying the hypotheses of (b) which are not symplectomorphic to any convex set: Take for instance examples D similar to the one in the Corollary to Theorem 1 such that moreover $B(r) \subset D \subset Z(r)$, see Figure 2. They all satisfy $\ell(D) = u(D)$ and are not symplectomorphic to any convex set, which shows that Theorem 2 is a true generalization of the product formula for $\mathcal{K}$.

Theorem 2 applies in particular to c_{EH} (using [Si90]) and c_{HZ}.

6. Extensions to general symplectic manifolds. Now that extensions to $\mathbb{R}^{2n}$ have been studied, it is easy to generalize them analogously to manifolds.

DEFINITION 4. For any subset of a symplectic manifold of given dimension $2n$, we define the non-negative numbers

$$\underline{u}(D) = \inf_{\varphi \in \mathcal{E}_\omega} c(\operatorname{conv} \varphi(D)),$$
$$e(D) = \sup\{c(K) \mid K \in \mathcal{K} \quad \text{such that } \exists \varphi \in \mathcal{S} \text{ with } \varphi(K) \subset D\},$$
$$k(D) = \sup\{\underline{u}(P) \mid P \subset D \text{ contractible}\}.$$

THEOREM 3.

(i) *e, k and $\underline{u}$ satisfy the axioms of Definition 2 for any subsets of all symplectic manifolds and any family of embeddings.*

(ii) *All symplectic embedding capacities C coinciding on $\mathcal{K}$ with c are estimated by e and $\underline{u}$: $e \le C \le \underline{u}$.*

Proof. The proof is analogous to the one for ℓ and u and is therefore skipped. For k, one simply observes that every $\varphi(K)$ is a contractible set, so that $e \le k \le \underline{u}$ immediately follows. ■

7. Surfaces. Given any compact surface S of genus g, consider the canonical system of $2g$ non-dividing curves $\alpha_i, \beta_i, i = 1, \ldots, g$. Then $S \setminus A$ with $A := \bigcup_{i=1}^g \alpha_i \cup \beta_i$ is conformally equivalent to a $2g$-gon, which is itself conformally equivalent to a disk D in $\mathbb{C}$:

$$f : S \setminus A \to D$$

is a conformal map and is therefore symplectic:

$$\text{Area}(D) = \text{Area}(S \setminus A) = \text{Area}(S).$$

CONSEQUENCES.

(1) $P = S \setminus A$ is contractible. Every other contractible subset of S has area less than Area(S), therefore $k(S) = \underline{u}(P) = \text{Area}(S)$.

(2) f^{-1} is a symplectic diffeomorphism $D \to S \setminus A$ from an open convex set into S, which realizes the maximum for area-preserving embeddings: $e(S) = \text{Area}(S)$.

This proves

PROPOSITION 3. *For any surface S with or without boundary, all symplectic embedding capacities C extending c are equal to the area of S: $e(S) = \text{Area}(S) = k(S)$.*

Proposition 3 has first been proved by Siburg [Si93] for embedding capacities (which he called Hofer-Zehnder capacities) by construction of an adapted Hamiltonian function.

This is in contrast to the diffeomorphism capacity u which is different from the area: Recall that the annulus $S = B(R) \setminus \bar{B}(r)$ satisfies $e(S) = \text{Area}(S) = k(S) = \pi(R^2 - r^2)$ but $\underline{u}(S) = u(S) = \pi R^2$, see Figure 1.

8. Symplectic 4-tori and the Herman-Zehnder example. Following [HZ94], we consider $(\mathbb{R}^4, \omega_\alpha)$ with the symplectic structure $\omega_\alpha(x, y) = A_\alpha X.Y$ defined by

$$A_\alpha = \begin{pmatrix} 0 & -1 & \alpha_2 & 0 \\ 1 & 0 & -\alpha_1 & 0 \\ -\alpha_2 & \alpha_1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -A_\alpha^T$$

(which satisfies $\det(A_\alpha) = 1$ but not $A_\alpha^2 = -I$). This form induces a symplectic structure on the manifold $M = T^3 \times [0, d] = \mathbb{R}^3/\mathbb{Z}^3 \times [0, d]$ denoted again ω_α. For $\alpha_1, \alpha_2 = 0$, one gets the standard almost complex structure J_0. For $d < 1$, (M, ω_α) is embedded in the torus (T^4, ω_α).

Functions H on $\mathbb{R}^4$ which are 1-periodic in the first three variables pass to the quotient as well as their Hamiltonian vector fields

$$\xi_H := -A_\alpha^{-1} H'(x),$$

where $H'(x)$ is the Euclidean gradient of H. As

$$A_\alpha^{-1} = \begin{pmatrix} 0 & -1 & 0 & -\alpha_1 \\ 1 & 0 & 0 & -\alpha_2 \\ 0 & 0 & 0 & -1 \\ \alpha_1 & \alpha_2 & 1 & 0 \end{pmatrix},$$

we get for the Hamiltonian function $H_0(x) = x_4$ a constant vector field

$$\xi_{H_0} = (\alpha_1, \alpha_2, 1, 0) =: (\alpha, 0),$$

which integrates to an affine flow preserving all 3-tori $T^3 \times \{s\}$. If $\alpha = (\alpha_1, \alpha_2, 1)$ is rationally independent, i.e. $\alpha.z \neq 0 \quad \forall z \in \mathbb{Z}^3 \setminus 0$, this flow is dense and has no periodic orbits. Therefore it represents an example of a Hamiltonian flow whose energy levels $T^3 \times \{s\}$ are all regular and compact but none of them carries a periodic orbit.

M. Herman proved in [H91] that H_0 is dynamically stable under perturbations if α is irrational satisfying a diophantine condition: This represents an counter-example against the C^k-closing conjecture for k sufficiently large.

In [HZ94], it has been showed that $c_{HZ}\big(M, \omega_\alpha\big)$ is infinite if α is irrational. Here we estimate $C(M, \omega_\alpha)$ for any C extending c by exhibiting a convex set contained in M:

$$C(M, \omega_\alpha) \begin{cases} = \infty & \text{if } \alpha \text{ irrational} \\ \geq d & \text{if } \alpha \text{ rational} \end{cases}$$

Proof. Consider the linear map $\big(\mathbb{R}^4, \omega\big) \to \big(\mathbb{R}^4, \omega_\alpha\big)$ given by the matrix

$$N_\alpha = \begin{pmatrix} 1 & 0 & \alpha_1 & 0 \\ 0 & 1 & \alpha_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is symplectic: $N_\alpha^t A N_\alpha = J_0$. Denote the canonical basis by e_k.

(i) If α is irrational, then the Lagrangian plane L spanned by e_1 and e_3 is embedded by N_α into $\tilde{M} = \mathbb{R}^3 \times [0, d]$. The quotient of this map onto M winds L in the 3-torus densely around itself. But a Lagrangian plane has infinite capacity, from where the first part of the claim.

If α is rational, then there are relatively prime $n_i \in \mathbb{Z}$, $i = 1, 2$ such that $\alpha_i = \frac{n_i}{n_3}$. Then:

(ii) N_α embeds the standard unit 3-cube into a fundamental domain of the action of $\mathbb{Z}^3$ on $\tilde{M}$. Therefore $C(M, \omega_\alpha) \geq c([0,1]^3 \times [0, d], \omega_0) = d$ by the product formula, which proves the second claim.

(iii) But the map N_α also sends then the parallelogram P spanned by $e_1, \frac{1}{n_3} e_2, n_3 e_3, e_4$ into a fundamental domain of the action of $\mathbb{Z}^3$ on $\tilde{M}$. Therefore $C(M, \omega_\alpha) \geq c(P, \omega_0)$, which is equal to $\min\{\frac{1}{n_3}, d\}$ again by the product formula for c.

This last observation shows the relation to (i), but also prompts a question concerning fundamental domains of in $\tilde{M}$ (which would determine $e(M)$): What is the biggest capacity a fundamental domain in $\tilde{M}$ can have?

References

[A84] J. P. Aubin, *L'analyse non linéaire et ses motivations économiques*, Masson, Paris, 1984.

[CE80] F. H. Clarke, I. Ekeland, *Hamiltonian trajectories with prescribed minimal period*, Comm. Pure Appl. Math. 33 (1980), 103–116.

[E90] I. Ekeland, *Convexity methods in Hamiltonian mechanics*, Springer, Berlin-Heidelberg, 1990.

[EH89] I. Ekeland, H. Hofer, *Symplectic Topology and Hamiltonian Dynamics I*, Math. Z. 200 (1989), 355–378. See also C. R. Acad. Sci. Paris Sér. I Math. 307 (1988), 37–40.

[EH90] I. Ekeland, H. Hofer, *Symplectic Topology and Hamiltonian Dynamics II*, Math. Z. 203 (1990), 553–567.

[G85] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307–347.

[H91] M. Herman, *Exemples de flots Hamiltoniens dont aucune perturbation en topologie C^∞ n'a d'orbites périodiques sur un ouvert de surfaces d'énergies*, C. R. Acad. Sci. Sér. I Math. 312 (1991), 989–994.

[HZ87] H. W. Hofer, E. Zehnder, *Periodic solutions on hypersurfaces and a result by C. Viterbo*, Invent. Math. 90 (1987), 1–9.

[HZ90] H. W. Hofer, E. Zehnder, *A new capacity for symplectic manifolds*, in: Analysis et cetera, Academic Press, 1990, 405–429.

[HZ94] H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, Basel-Boston-Berlin, 1994.

[K90] A. F. Künzle, *Une capacité symplectique pour les ensembles convexes et quelques applications*, Ph.D. thesis, Université Paris IX Dauphine, June 1990.

[K91] A. F. Künzle, *The least characteristic action as symplectic capacity*, preprint, Forschungsinstitut für Mathematik, ETH Zürich, May 1991.

[K93] A. F. Künzle, *Singular Hamiltonian systems and Symplectic Capacities*, in: Singularities and Differential Equations, Banach Center Publ. 33 (1996), 171–187.

[R70] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton N.J., 1970.

[Si93] K. F. Siburg, *Symplectic capacities in two dimensions*, Manuscripta Math. 78 (1993), 149–163.

[Si90] J. C. Sikorav, *Systèmes Hamiltoniens et topologie symplectique*, Lecture Notes, Dipartimento di Matematica dell'Università di Pisa, August 1990.

[V87] C. Viterbo, *A proof of the Weinstein conjecture in $\mathbb{R}^{2n}$*, Ann. Inst. H. Poincaré Anal. Non Linéaire 4 (1987), 337–357.