1. A point $p$ on an embedded $C^\infty$-smooth real surface $M \hookrightarrow \mathbb{C}^2$ is said to have a complex tangent if $T_pM$ is a complex line in $T_p\mathbb{C}^2$. If this is the case, one can choose complex coordinates $z, w = u + iv$ on $\mathbb{C}^2$ such that $T_pM$ is the $z$-axis and $p = (0, 0)$. Then $M$ is locally a graph $w = f(z)$ where $f$ is a smooth complex valued function that vanishes of second order at the origin. If $\partial^2f/\partial z\partial \bar{z}(0) \neq 0$, then a quadratic holomorphic substitution of $z, w$ normalizes the second-order part of $f$ at the origin to

$$q(z) = \gamma(z^2 + \bar{z}^2) + z\bar{z}$$

where $\gamma \geq 0$ is Bishop’s invariant [B]. Then we have

$$f = q + h + ik$$

where $h, k$ are real functions that vanish of third order at 0. Let $\pi : \mathbb{C}^2 \rightarrow \{v = 0\} \subseteq \mathbb{C}^2$, $\pi(z, w) = (z, u)$, $M_0 = \pi M$, and assume that $q : \mathbb{C} \rightarrow \mathbb{R}$ is positive definite, i.e.

$$0 \leq \gamma < 1/2.$$  

In this case $0 \in M$ is called an elliptic point. By rescaling $z, w$ we may assume that

$$h, k \in C^\infty(\bar{\Delta}; \mathbb{R}), \quad h, k = O(|z|^3),$$

$$q + h \quad \text{is convex on } \bar{\Delta},$$

$$1 \leq (q + h)(z) \text{ on } \partial\bar{\Delta}.$$ 

Now $M_0$ bounds the hypersurface $D_0 = \{(q + h)(z) < u, \ v = 0\}$ which in turn is foliated by the holomorphic discs $\{(q + h)(z) < t, \ w = t, \ 0 < t < 1\}$ whose boundaries lie in $M_0$. 


The paper is in final form and no version of it will be published elsewhere.
This family of holomorphic discs survives in a possibly smaller neighbourhood of $0 \in M$ by Bishop’s Theorem [B].

In this note we show that this foliated hypersurface depends smoothly on $M$ in $C^\infty$ Fréchet topology and closely follow [KW].

Stability of holomorphic discs with boundary in totally real surfaces has been studied in [Be], [F], [K], [O]. It is relevant in constructions for fillable contact structures on $S^3$, see [EK], [H]. The present paper is organized as follows. In the second section we state in [Be], [F], [K], [O]. It is relevant in constructions for fillable contact structures on $S^3$ Fréchet topology and closely follow [KW]. This family of holomorphic discs survives in a possibly smaller neighbourhood of $0 \in W$. KLINGENBERG

2. For $l \in \mathbb{N}, l \geq 3$ let $\mathcal{M}_l = \{w = (q + h + ik)(z); q, h, k$ satisfying $(1) - (5)$ and $k = O(|z|^l)\}$ and let $\mathcal{M}_0 = \{w = (q + h)(z); q, h$ satisfying $(1) - (5)\}$ both equipped with the $C^\infty(\Delta)$ Fréchet topology. We have $\pi : \mathcal{M}_l \to \mathcal{M}_0$, $\pi M = M_0$, $M_0 = M_0 \cap \{u < \epsilon\}$, and $M_0$ bounds $D_0^l = \{(q + h)(z) < u < \epsilon, v = 0\}$. Here and later $M \in \mathcal{M}_l$ is identified with its graph function $f = q + h + ik$. The main result of [KW] now reads as follows.

Theorem 1 [KW]. Let $M \in \mathcal{M}_3$ then there exists $\epsilon > 0$ and a $C^\infty$ embedding $G : (D_0^l, M_0) \to (\mathbb{C}^2, M)$ such that

a) $G(0) = 0$, $dG(0) = [id]$ where $\text{Id} : \{v = 0\} \to \{v = 0\}$ is the identity.
b) $u \circ G(0, u) = u$
c) $G|w = r^2$ is holomorphic for all $r^2 < \epsilon$.

To formulate the smooth dependence of $G$ from $M$ we introduce more notation. Let $\mathcal{G} : \mathcal{M}_3 \to \mathcal{M}_3$ be the locally trivial fibre bundle over $\mathcal{M}_3$ whose fibre over $M \in \mathcal{M}_3$ is given by $\mathcal{G}_M = \{G : (D_0^l, M_0) \hookrightarrow (\mathbb{C}^2, M); C^\infty$-smooth, some $\epsilon > 0\}$ and equip it with the $C^\infty$ Fréchet topology.

Theorem 2. There exists a continuous function $\epsilon : \mathcal{M}_3 \to \mathbb{R}^+$ and a $C^\infty$-smooth section $p$ of $\mathcal{G}$ whose values satisfy a), b), c) from Theorem 1.

3. Let $q + h \in \mathcal{M}_0$, then denote by $\Delta_{q+h} = \{z \in \Delta : (q + h)(z) < 1\}$. We associate to $\Delta_{q+h}$ the diffeomorphism $g(q + h) : \Delta \to \Delta_{q+h}$ that maps $\{|z|^2 = t\}$ to $\{(q + h)(z) = t\}$ for $0 \leq t \leq 1$ and preserves the rays through 0. The existence and smoothness of $g$ in $z$ and smooth dependence on $q + h$ follows from Morse Lemma and (5). We have $g : \mathcal{M}_0 \times (0, 1) \to C^\infty(S^3, \mathbb{C})$ defined by $g(q + h, r) = g(q + h)(\cdot, r)$, where we introduced polar $(0, r)$ coordinates on $\Delta$, and $g_0(q + h, r) := g(q + h, r) - g(q, r)$. In this regard one has the following

Proposition 3. The map $g : \mathcal{M}_0 \times (0, 1) \to C^\infty(S^3; \mathbb{C})$ is smooth and there exists $K = K(q + h, j, s)$ which is locally bounded in $q + h$ such that

a) $|g|, |g_0/r| \leq K \cdot r$
b) $|\partial^r g|, |\partial^r g_0/r| \leq K$

Here $g = g(q + h, r) \in C^\infty(S^3; \mathbb{C})$ and $|\cdot|_j$ denotes the $C^j$-sup norm.
For \((q + h, r) \in \mathfrak{M}_0 \times (0, 1), (j, \alpha) \in \mathbb{N} \times (0, 1)\) let \(H(q + h, r) \in \text{Hom}C^{j,\alpha}(S^1, \mathbb{R})\) be the Hilbert transform operator on the closed convex curve \(\{(q + h)(z) = r^2\} \subseteq \Delta\), parametrized by \(\theta \in S^1\). It maps \(\xi \in C^{j,\alpha}(S^1, \mathbb{R})\) to its harmonic conjugate with mean zero. The main technical point in the proof of Theorem 1 and 2 lies in the analysis of the asymptotic behaviour of \(H\) on the curves \(\{(q + h)(z) = r^2\}\) as \(r^2 \to 0\). The functions \(g, g_0\) enter in the integral operator representations of \(H\), see (2.7) of [KW].

**Proposition 4.** The map \(H : \mathfrak{M}_0 \times (0, 1) \to \text{Hom}C^{j,\alpha}(S^1, \mathbb{R})\) is continuous and there exists \(K = K(q + h, j, s) > 0\), locally bounded in the first argument, such that

a) \(|H(q + h, r) - H(q, r)|_{j, \alpha} \leq K \cdot r^2\)

b) \(|\partial_r^s H(q + h, r)|_{j, \alpha} \leq K\).

Given Proposition 3, the above propositions follow immediately from Lemma 2.2 and Theorem 2.5 of [KW].

4. For the proof of Theorem 2 one represents \(G = p(M) \in \mathfrak{S}_M\) by the ansatz

\[
G(z, w) = (z + zA(z, r), w + B(z, r)),
\]

where \(u = r^2\), \(A, B\) are holomorphic in \(z\) on \(\{(q + h)(z) < r^2\} \subseteq \Delta\), \(\text{Re} B(0, r) = 0\), \(\text{Im} A(0, r) = 0\), and one stipulates that \(G\) maps \(\{(q + h)(z) = r^2, u = r^2\} \subseteq M_0\) into \(M = \{w = (q + h + ik)(z)\}\), which may be written as

\[
r^2 + B(z, r) = (q + h + ik)(z + zA(z, r))
\]

for \((q + h)(z) = r^2\). This functional equation needs to be solved for \(A, B\), these extended holomorphically to the interior of \((q + h)(z) = r^2\), and shown to depend smoothly on \(q + h + ik \in \mathfrak{M}_3\). Using Hilbert transform one may eliminate \(B\) from (7) and is lead to the following:

\[
F(q + h + ik, r, \xi) = 0,
\]

where \(F : \mathfrak{M}_3 \times (0, 1) \times C^{j,\alpha}(S^1) \to C^{j,\alpha}(S^1)\) is defined by

\[
F = (q + h)(z + zA(q + h, r)[\xi]) - r^2 + H(q + h, r)[k(z + zA(q + h, r)[\xi])].
\]

Here,

\[
A(q + h, r)[\xi] = \xi + iH(q + h, r)[\xi]
\]

and \((q + h)(z) = r^2\). By Propositions 3 and 4, \(F\) is smooth. As in [KW, §1] it follows that if \(\xi_r \in C^{j,\alpha}(S^1), r \in (0, 1)\) satisfies (8), then

\[
A(z, r) = A(q + h, r)[\xi]
\]

\[
B(z, r) = -r^2 + (q + h + ik)(z + zA(z, r))
\]

satisfy (7) and its holomorphic extensions in (6) will give a point \(G \in \mathfrak{S}_M\) satisfying a), b), c) of Theorem 1. Applying a Picard iteration scheme to solve (8) for \(\xi_r\) leads to the following existence result [KW, 3.3]:

**Proposition 5.** Let \((j, \alpha) \in \mathbb{N} \times (0, 1), l \geq 3\). Then there exists a continuous function \(\epsilon : \mathfrak{M}_l \to \mathbb{R}^+\) and a smooth map \(\{M \times (0, \epsilon(M)); M \in \mathfrak{M}_3\} \to C^{j,\alpha}(S^1) : q + h + ik \to \xi_r, r^2 > \epsilon^2(M)\) with

a) \(F(q + h + ik, r, \xi_r) = 0\)
b) \(|\partial^r \xi r| \leq K r^{l-2-s}\)

where \(K = K(l, j, s, q + h + ik) > 0\) is locally bounded in \(\mathcal{M}_3\).

Now by extending \(\xi r\) to \{\((q + h)(z) < r^2\)\} harmonically and by (6), (10) one concludes the following as in Theorem 4.2, [KW].

**Theorem 6.** Let \(l \geq 7, s = (l - 7)/3\), then there exists a continuous section \(p(l)\) of \(\mathcal{G}^* \to \mathcal{M}_l\) satisfying a), b), c) of Theorem 1, where \(\mathcal{G}^*\) is as in the second section but with \(C^s\)-regularity instead of \(C^\infty\)-regularity.

Now Theorem 2 follows from Theorem 6, boundary regularity of holomorphic discs in totally real surfaces, and the following

**Proposition 7.** Let \(\text{Hol}(\mathbb{C}^2, 0)\) be the manifold of germs at 0 of holomorphic invertible maps of \((\mathbb{C}^2, 0)\) fixing the origin, endowed with Fréchet topology. Then for any \(l \geq 3\) there exists a continuous map \(h : \mathcal{M}_3 \to \text{Hol}(\mathbb{C}^2, 0), M \to h_M\), such that \(h_M(M) \in \mathcal{M}_l\).

**References**


