SYMPLECTIC SINGULARITIES AND GEOMETRY OF GAUGE FIELDS
BANACH CENTER PUBLICATIONS, VOLUME 39
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

BIRKHOFF NORMAL FORMS AND ANALYTIC GEOMETRY

JEAN PIERRE FRANÇOISE

Département de Mathématiques, Université de Paris VI, BP 172
4, Place Jussieu, tour 46-45, 5e étage, 75252 Paris, France
E-mail: jpf@ccr.jussieu.fr

I. Introduction. It is natural to relate the center problem for polynomial planar vector fields ([F-P], [F-Po], [Z1], [Z2]) to normal form theory. Our purpose in this conference is to provide an extension to $2m$-dimensional vector fields ($m > 1$) of the center problem. We use normal form theory and methods of analytic geometry (Artin's approximation theorem [A], [P] and Gabrielov theorem [Ga]).

The center problem calls for finding the necessary and sufficient conditions for which a polynomial planar vector field, tangent to rotation at the origin, has all its orbits periodic in a neighborhood of this point. In such a case, the vector field is said to be a center. This problem is solved for quadratic vector fields (cf. [D], [F-Po], [Z1], [Z2]) and cubic vector fields with homogeneous nonlinearities (cf. [F-Po], [Z1], [Z2]). For $m > 1$, we need to distinguish between formal centers and analytic centers. We show that the set of polynomial vector fields with a formal center is an algebraic set and we give a characterization of the analytic centers.

The methods that we introduce here can be useful also in the study of Hamiltonian systems. We show for instance that the set of polynomial integrable Hamiltonian systems (of fixed degree) tangent to a rotation and such that the frequencies do not depend on the initial conditions (called the harmonic set here) is a real algebraic set. Let us now state more precisely our results and notations.

Let X be a germ of analytic vector field at $0 \in \mathbb{R}^{2m}$ with a linear part (multi-rotation):

$$j_1(X) = \sum_{i=1}^{m} \lambda_i \Big(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \Big).$$

1991 *Mathematics Subject Classification*: 58F14, 34C15, 32B20.

The paper is in final form and no version of it will be published elsewhere.

Let us assume that the coefficients $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ are independent over $\mathbb{Z}$:

$$(*)\qquad \text{If } \langle \lambda, \alpha\rangle = \sum_{i=1}^{m} \lambda_i \alpha_i = 0,\ \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m,\ \text{then } \alpha = 0.$$

Definition. We say that X is an *analytic center* if X has m analytic first integrals $(f_1, \ldots, f_m)$ which are tangent at the origin to $(x_1^2 + y_1^2, \ldots, x_m^2 + y_m^2)$ up to the order three.

We show the following

Theorem 1. *If the vector field X is an analytic center then there are germs of analytic coordinates $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_m)$, $\widetilde{y} = (\widetilde{y}_1, \ldots, \widetilde{y}_m)$ such that the vector field X preserves the functions $p_1 = \widetilde{x}_1^2 + \widetilde{y}_1^2$, $p_2 = \widetilde{x}_2^2 + \widetilde{y}_2^2$, $\ldots$, $p_m = \widetilde{x}_m^2 + \widetilde{y}_m^2$.*

Definition. We say that X is a *formal center* if X has m formal first integrals $(\widehat{f}_1, \ldots, \widehat{f}_m)$ which are tangent to $(x_1^2 + y_1^2, \ldots, x_m^2 + y_m^2)$ up to the order three.

Let X be a polynomial vector field on $\mathbb{R}^{2m}$:

$$X = \sum_{i=1}^{m} \Big[\lambda_i \Big(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\Big) + \sum_{2 \le |\alpha| + |\beta| \le d} f_{i;\alpha,\beta} x^\alpha y^\beta \frac{\partial}{\partial x_i} + f_{i+m;\alpha,\beta} x^\alpha y^\beta \frac{\partial}{\partial y_i}\Big],$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$ are multi-indices of length $|\alpha| = \alpha_1 + \ldots + \alpha_m$, $|\beta| = \beta_1 + \ldots + \beta_m$, and where x^α, y^β stand for $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, $y^\beta = y_1^{\beta_1} \cdots y_m^{\beta_m}$.

We denote more shortly as f the finite collection of all the coefficients $f_{i;\alpha,\beta}$ and $f_{i+m;\alpha,\beta}$. The integer d is called the degree of X. A polynomial vector field of degree less than d can be seen as a point f of a finite dimensional vector space F.

Obviously, any element of F defines a unique germ of analytic vector field at $0 \in \mathbb{R}^{2m}$ of the type that we have previously considered.

Theorem 2. *The set C of elements of F which have a formal center at $0 \in \mathbb{R}^{2m}$ is algebraic.*

Let us denote by Z the set of elements of F which have an analytic center at $0 \in \mathbb{R}^{2m}$.

If $m = 1$ then, by a theorem of B. Malgrange ([M]), $Z = C$. If $m = 1$ and $d = 2$, H. Dulac ([D]) gave explicitly the equations of the algebraic set C. The structure of C is nevertheless not so simple and Żołądek has recently given a complete description of it ([Z1], [Z2]). It seems much harder to do it if $d > 2$.

If $m > 1$, a well known theorem of C. L. Siegel [S] shows that $Z \neq C$.

The basic tool we use to show Theorem 2 is the theory of formal normal forms for finite dimensional families of vector fields. In the Hamiltonian case, this normal form is better known as the Birkhoff normal form.

II. Proof of Theorem 1.

II.1. *The normal form.* We review some of the standard facts on normal form theory. Let X be a germ of analytic vector field at $0 \in \mathbb{R}^{2m}$ with a linear part

$$j_1(X) = \sum_{i=1}^{m} \lambda_i \Big(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\Big), \quad \text{with the condition } (*),$$

then there exists a formal coordinate system (x'_i, y'_i) $(i = 1, \ldots, m)$ such that

$$X = \sum_{i=1}^{m} \Big[A_i \Big(x'_i \frac{\partial}{\partial x'_i} + y'_i \frac{\partial}{\partial y'_i} \Big) + B_i \Big(x'_i \frac{\partial}{\partial y'_i} - y'_i \frac{\partial}{\partial x'_i} \Big) \Big]$$

where the formal series A_i, B_i $(i = 1, \ldots, m)$ only depend on $p'_1 = {x'_1}^2 + {y'_1}^2$, $p'_2 = {x'_2}^2 + {y'_2}^2$, $\ldots$, $p'_m = {x'_m}^2 + {y'_m}^2$.

Remarks.

(i) Such a formal coordinate system is *not* unique. Any other coordinate system normalizing X is of the form

$$x'' = \xi_i(p'_1, \ldots, p'_m) \cdot x'_i, \quad y'' = \eta_i(p'_1, \ldots, p'_m) \cdot y'_i, \quad i = 1, \ldots, m.$$

(ii) Any formal series ψ which is preserved by X $(X \cdot \psi = 0)$ must depend only on $p'_1, \ldots, p'_m$.

(iii) If $m = 1$, the theorem entails a formal series $p' = p'_1 = {x'_1}^2 + {y'_1}^2$ such that

$$X \cdot p' = 2p' A_1(p').$$

The coefficients of A_1 are called the Lyapunov-Poincaré coefficients. They all vanish if and only if X has a formal first integral p' and by a theorem of B. Malgrange ([M]) p' is necessarily analytic.

Note that if one of the Lyapunov-Poincaré coefficients does not vanish then p' is not necessarily convergent.

(iv) If X is Hamiltonian relatively to

$$\omega = \sum_{i=1}^{m} dx_i \wedge dy_i,$$

then there is a formal coordinate system (x'_i, y'_i) $(i = 1, \ldots, m)$ such that

$$\omega = \sum_{i=1}^{m} dx'_i \wedge dy'_i$$

and

$$X = \sum_{i=1}^{m} \Big[B_i \Big(x'_i \frac{\partial}{\partial y'_i} - y'_i \frac{\partial}{\partial x'_i} \Big) \Big]$$

(namely $A_i = 0$). This is nothing else than the Birkhoff normal form of X.

II.2. *Simplification of the analytic integrals.* We prove now Theorem 1.

Proof. (Compare to ([V]) for Hamiltonian systems.) We consider a formal coordinate system (x'_i, y'_i) in which X is in a normal form. We see that $(f_1, \ldots, f_m)$ are necessarily functions of

$$p'_1 = {x'_1}^2 + {y'_1}^2, \ldots, p'_m = {x'_m}^2 + {y'_m}^2.$$

Let Σ be the analytic set defined as the critical locus of $f_1, \ldots, f_m$:

$$\Sigma = \{(x, y) / df_1 \wedge \ldots \wedge df_m = 0\}.$$

The analytic equations of the critical locus have formal solutions $x'_i = y'_i = 0$. Artin's approximation theorem ([A]) yields analytic coordinates $(\overline{x}_i, \overline{y}_i)$ $(i = 1, \dots, m)$ so that $\Sigma = \bigcup_i \Sigma_i$,

$$\Sigma_i = \{(x, y)/\overline{x}_i = \overline{y}_i = 0\}.$$

We consider now the analytic function f_i restricted to the analytic set Σ_i. The set Σ_i is alternatively given by the formal equations $x'_i = y'_i = 0$. Each function f_j $(j \neq i)$ is tangent to ${x'_j}^2 + {y'_j}^2$ and

$$f_i = f_i(p'_1, \dots, p'_m).$$

In restriction to Σ_i this entails a formal relation $\widehat{\phi}_i$ between the f_j $(j = 1, \dots, m)$

$$f_i = \widehat{\phi}_i(f_1, \dots, \widehat{f_i}, \dots, f_m).$$

By a theorem of Gabrielov [Ga], it yields an analytic relation on Σ_i

$$f_i = \phi_i(f_1, \dots, \widehat{f_i}, \dots, f_m).$$

We now define the analytic functions

$$g_i = f_i - \phi_i(f_1, \dots, \widehat{f_i}, \dots, f_m).$$

Note that they vanish identically on the set Σ_i.

In restriction to Σ_i, we have

$$d(f_i|\Sigma_i) = \sum_{j \neq i} \frac{\partial \phi_i}{\partial y_j} d(f_j|\Sigma_i).$$

But on Σ_i, there must be a linear relation between df_i and df_j $(j \neq i)$. On Σ_i, the functions f_j $(j \neq i)$ are generically independent. Thus the only possible relation is

$$df_i = \sum_{j \neq i} \frac{\partial \phi_i}{\partial y_j} df_j$$

and we conclude that g_i is critical on Σ_i.

If we use the analytic coordinates $(\overline{x}_i, \overline{y}_i)$, we have obtained that

$$g_i = \overline{x}_i^2 + \overline{y}_i^2 + \dots$$

is zero and critical on $\overline{x}_i = \overline{y}_i = 0$.

The Morse lemma with parameters easily implies the existence of analytic coordinates $(\widetilde{x}_i, \widetilde{y}_i)$ such that

$$g_i = \widetilde{x}_i^2 + \widetilde{y}_i^2$$

and the result of the theorem follows. ■

III. Proof of Theorem 2.

III.1. *The normal form with parameters.* Let X be a polynomial vector field

$$X = \sum_{i=1}^m \Big[\lambda_i \Big(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\Big) + \sum_{2 \leq |\alpha| + |\beta| \leq d} f_{i;\alpha,\beta} x^\alpha y^\beta \frac{\partial}{\partial x_i} + \sum_{2 \leq |\alpha| + |\beta| \leq d} f_{i+m;\alpha,\beta} x^\alpha y^\beta \frac{\partial}{\partial y_i}\Big].$$

If the coefficients $(\lambda_1,\dots,\lambda_m)$ satisfy condition $(*)$, then there exists a formal coordinate system (x'_i,y'_i) $(i=1,\dots,m)$ such that

$$X=\sum_{i=1}^m A_i\Big(x'_i\frac{\partial}{\partial x'_i}+y'_i\frac{\partial}{\partial y'_i}\Big)+B_i\Big(x'_i\frac{\partial}{\partial y'_i}-y'_i\frac{\partial}{\partial x'_i}\Big)$$

where the formal series A_i,B_i $(i=1,\dots,m)$ only depend on

$$p'_1={x'_1}^2+{y'_1}^2,\dots,p'_m={x'_m}^2+{y'_m}^2.$$

Proposition III.1. *The formal coordinate system* (x'_i,y'_i) $(i=1,\dots,m)$ *can be chosen so that the coefficients of the formal series* (A_i,B_i) *are polynomial functions of the coefficients*

$$f=(f_{i;\alpha,\beta},f_{i+m;\alpha,\beta}).$$

This proposition extends the fact that, for $m=1$, the Lyapunov-Poincaré coefficients are polynomial functions of the coefficients of the vector field.

Proof. It is more convenient to use the complex coordinates

$$z_i=x_i+\sqrt{-1}\,y_i,\quad \overline{z}_i=x_i-\sqrt{-1}\,y_i$$

and to write the vector field as

$$X=\sum_{i=1}^m\Big[\sqrt{-1}\,\lambda_i\Big(z_i\frac{\partial}{\partial z_i}-\overline{z}_i\frac{\partial}{\partial\overline{z}_i}\Big)+\sum_{2\le|\alpha|+|\beta|\le d} g_{i;\alpha,\beta}z^\alpha z^{-\beta}\frac{\partial}{\partial z_i}+g_{i+m;\alpha,\beta}z^\alpha z^{-\beta}\frac{\partial}{\partial\overline{z}_i}\Big]$$

A formal coordinate system (x'_i,y'_i) (or alternatively $z'_i,\overline{z}'_i$) can be produced by induction as follows. The system must be so that

$$X\cdot z'_i=z'_i\psi_i,\quad X\cdot\overline{z}'_i=\overline{z}'_i\overline{\psi}_i$$

where $\psi_i=A_i+\sqrt{-1}\,B_i$ is a function of

$$p'_1=z'_1\cdot\overline{z}'_1,\dots,p'_m=z'_m\cdot\overline{z}'_m.$$

The k-jet of z'_j can be obtained inductively from the $k-1$-jet by an equation of the form

$$\sum_{i=1}^n\lambda_i\Big(z_i\frac{\partial}{\partial z_i}-\overline{z}_i\frac{\partial}{\partial\overline{z}_i}\Big)\cdot {z'_j}^{(k)}=\Theta_j$$

where Θ_j is known and depends polynomially of the coefficients f.

Let us denote by P_k the set of polynomial functions in $(z_1,\dots,z_m;\overline{z}_1,\dots,\overline{z}_m)$ and f, of degree less than k in $(z,\overline{z})$.

Let $j_1(X):P_k\to P_k$ be the mapping induced by the derivative along the vector field: $j_1(X)=\sum_{i=1}^m\lambda_i\Big(z_i\frac{\partial}{\partial z_i}-\overline{z}_i\frac{\partial}{\partial\overline{z}_i}\Big)$.

Given $G\in P_k$, there is a unique decomposition

$$G=N+R$$

where $N=\sum_\alpha N_\alpha(F)p^\alpha$ belongs to the kernel of $j_1(X)$ and $R=\sum_{\substack{\alpha,\beta\\ \alpha\ne\beta}}R_{\alpha\beta}(f)z^\alpha\overline{z}^\beta$ belongs to the image of $j_1(X)$.

N contributes to the normal form and we can write

$$R = j_1(X) \cdot F \quad \text{with} \quad F = \sum_{\alpha \neq \beta} \frac{R_{\alpha\beta}(f)}{\alpha - \beta} z^\alpha \overline{z}^\beta$$

which contributes to the change of coordinates.

We easily get the result of the proposition since at each step of the induction process the coefficients of the polynomials depend polynomially of the coefficients f. ■

III.2. *The ideal of the coefficients of the normal form.* We have already noticed that the normalizing transformation is not unique, we prove now

LEMMA III.2.1. *The ideal, generated by the coefficients of $\psi_i = A_i + \sqrt{-1}\, B_i$, in the ring $\mathbb{C}[f]$ of polynomials in the coefficients f, is independent of the normalizing transformation.*

Proof. From the previous remark ((i) in II.1), we deduce, given another normalizing coordinate system $(z_i'', \overline{z}_i'')$, that we have

$$z_i'' = \chi_i(p_1', \ldots, p_m') z_i'$$
$$z_i'' = \overline{\chi}_i(p_1', \ldots, p_m') \overline{z}_i'$$

so that

$$X \cdot z_i'' = \psi_i \cdot \chi_i z_i' = \psi_i z_i''.$$

This yields

$$X = \sum_i \psi_i'' \Big(z_i'' \frac{\partial}{\partial z_i''} - \overline{z}_i'' \frac{\partial}{\partial \overline{z}_i''} \Big)$$

with

$$\psi_i'' = \psi_i \circ T$$

where

$$T : \mathbb{R}_+^m \to \mathbb{R}_+^m$$
$$T : (p_1', \ldots, p_m') \mapsto (p_1'', \ldots, p_m'') = (p_1' |\psi_1|^2, \ldots, p_m' |\psi_m|^2). \ ■$$

We get now

PROPOSITION III.2.2. *The ideal generated by the coefficients of the real power series (A) (resp. (B)) is independent of the choice of the normalizing transformation.*

Proof. The coefficients A_i (resp. B_i) are the real parts (resp., imaginary parts) of ψ_i. It is obvious that the ideal generated by the coefficients of $\psi \circ T$ is contained in the ideal generated by the coefficients of ψ. But T is invertible and so it is clear that the ideal does not depend of the normalizing transform. ■

III.3. *The center set and the harmonic set.*

DEFINITION III.3.1. The set of f such that $A(f) = 0$ is called the *center set*. It is denoted by C. The elements of C have a formal center at the origin.

The set of f such that $B(f) = 0$, is called the *harmonic set*.

We now prove Theorem 2.

Proof. f is a formal center if and only if X has m formal first integrals which are tangent to $x_1^2 + y_1^2, \ldots, x_m^2 + y_m^2$. They are necessarily functions of $p_1', \ldots, p_m'$ and so $X \cdot p_1' = \ldots = X \cdot p_m' = 0$ and $A(f) = 0$. The set C is exactly given by the zero-set of the ideal generated by the coefficients of A. ■

Theorem 2 is a generalization of a theorem of H. Dulac ([D]) for planar vector fields.

We observe now that the harmonic set is the zero set of the ideal generated by the coefficients of B and so it is an algebraic set.

This set is of particular interest in the case of Hamiltonian systems.

IV. Application to Hamiltonian systems. We have already observed that in the Hamiltonian case $A(f) = 0$. (The Hamiltonian vector fields are contained in the center set C.) J. Vey ([V]) proved that a Hamiltonian vector field in F has an analytic center if and only if it is analytically conjugated to its Birkhoff normal form.

Let us consider a Hamiltonian vector field X in Z. It is well known that X has invariant tori $p_1 = c_1, \ldots, p_m = c_m$ on which the vector field X is linear.

The values $\lambda_1 + B_1(c), \ldots, \lambda_m + B_m(c)$ are the frequencies of X on the corresponding tori.

Definition IV.1. We say that X is *harmonic* if all the frequencies of X are constants independent of the invariant tori (and so equal to λ).

It is sometimes interesting to detect in parameter families of Hamiltonian systems those which are harmonic. For instance on a Riemannian manifold, it is interesting to find the metrics which are of Zoll type ([G]). We obtain as a corollary of our analysis:

Corollary. *In the set of Hamiltonian systems contained in Z, the conditions for which X is harmonic are algebraic.*

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