SYMPLECTIC SINGULARITIES AND GEOMETRY OF GAUGE FIELDS
BANACH CENTER PUBLICATIONS, VOLUME 39
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

ON A THEOREM OF CHEKANOV

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Abstract. A proof of the Chekanov theorem is discussed from a geometric point of view. Similar results in the context of projectivized cotangent bundles are proved. Some applications are given.

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1. Definitions and statements of the results.

1.1. To state the problems considered here, some definitions are needed. In this paper, N always denotes a smooth compact manifold without boundary, and M stands for either the Euclidean space $\mathbb{R}^n$, a product $N \times \mathbb{R}^k$ or a closed smooth manifold. The *space of one-jets of functions on* M is the manifold $J^1(M,\mathbb{R}) = T^*M \times \mathbb{R}$, endowed with its standard contact form $\alpha = du - pdq$, where (q,p,u) are the canonical coordinates of $T^*M \times \mathbb{R}$. The *one-graph* $j^1 f$ of a function f on M is a Legendrian submanifold in $J^1(M,\mathbb{R})$. In general, the image $W \subset M \times \mathbb{R}$ of the restriction to a Legendrian submanifold $\lambda \subset J^1(M,\mathbb{R})$ of the projection $\pi : J^1(M,\mathbb{R}) \to J^0(M,\mathbb{R}) = M \times \mathbb{R}$ is called the *wave front* of λ. If W is a smooth hypersurface in $M \times \mathbb{R}$, then $\lambda = j^1 f$ for some function f. The image of the restriction to λ of the projection $J^1(M,\mathbb{R}) \to T^*M$ is an immersed Lagrangian submanifold L. A generic wave front W is a stratified singular hypersurface of $M \times \mathbb{R}$. At each point of W, a non-vertical tangent plane is well defined. A generic wave front completely determines the Legendrian manifold which is above. In

1991 *Mathematics Subject Classification*: Primary 53C; Secondary 58G.
The paper is in final form and no version of it will be published elsewhere.

the sequel, we will often identify these two objects. A *quasi-function* on M is a wave front in $M \times \mathbb{R}$ whose Legendrian lift is the time one of a compactly supported Legendrian isotopy of $J^1(M, \mathbb{R})$ starting from $j^1 0$, the one-jet extension of the zero function. Recall that a Legendrian isotopy is a one parameter family of embedded Legendrian manifolds, and that any Legendrian isotopy can be extended to a contact isotopy, that is a flow $J^1(M, \mathbb{R}) \times [0,1] \to J^1(M, \mathbb{R})$ preserving the contact structure. Note that this isotopy extension lemma (see e.g. [Ch]) is valid in any contact manifold having a globally defined contact form. A *critical point* of a wave front $W \subset M \times \mathbb{R}$ is the image by the projection $T^*M \to M$ of an intersection point $L \cap O_M$ of the corresponding Lagrangian immersion L with the zero section O_M of T^*M. This means that above this point W has an horizontal tangent hyperplane.

Let $E \to M$ be a vector bundle on M. A function $F : E \to \mathbb{R}$ is said to be a *generating family* if $j^1 F$ is transverse to the set $\mathcal{C}$ of all jets of functions constant along the fibres. If $E = M \times V$ is a product with coordinates (q, v), this means that $0 \in V^*$ is a regular value of $\frac{\partial f}{\partial v} : M \times V \to V^*$.

Proposition 1 (see [A-V-G], pp. 257–259). *Let F be a generating family. The natural projection $\mathcal{C} \to J^1(M, \mathbb{R})$, restricted to $j^1 F \cap \mathcal{C}$, is a Legendrian immersion. The image of this immersion will be referred in the sequel as the Legendrian manifold* induced *by F.*

For example, if F is a generating family on the product $E = M \times V$, then the set λ

$$\lambda = \{(q, p, u) \text{ such that } \exists v \ \frac{dF}{dv}(q, v) = 0, \ p = \frac{\partial F}{\partial q}(q, v), \ u = F(q, v)\}$$

is an immersed Legendrian submanifold. When M is compact, a generating family $F : E \to \mathbb{R}$ is said to be *quadratic at infinity* (g.f.q.i. in the sequel) if there exists a function $Q : E \to \mathbb{R}$ which is a non-degenerate quadratic form in each fibre and such that $(q, v) \to F(q, v) - Q(q, v)$ has compact support. From a g.f.q.i. $F : E \to \mathbb{R}$, one can construct a new generating family defined on a trivial bundle $M \times V$, which coincides with a fixed quadratic form $Q(v)$ on the parameter vector space V outside a compact set and which induces the same Legendrian submanifold. In the calculations below, E is always assumed to be trivial. When M is not compact, a g.f.q.i. is by definition a generating family defined on a product $M \times V$ which coincides with a fixed quadratic form $Q(v)$ outside a compact set. G.f.q.i. are an important tool in symplectic topology (see e.g. [V, Si, Th]).

Proposition 2 ([Ch-Z]). *Let M be a closed compact manifold and denote by $sb(M)$ the sum of the Betti numbers of M. A g.f.q.i. $F : M \times V \to \mathbb{R}$ has more than $sb(M)$ critical points provided that they are all non-degenerate.*

Theorem 3 (Chekanov Theorem, [C]). *Denote by λ_0 and λ_1 the extremities of a Legendrian isotopy of $J^1(M, \mathbb{R})$. If λ_0 is induced by a g.f.q.i., so is λ_1. In particular any quasi-function is induced by some g.f.q.i.*

By construction, the critical points of a generating family are in one to one correspondence with the critical points of the Legendrian it induces. Denote by $cs(M)$ the *stable minimal number of critical points of a function on M*, i.e. the minimal number of critical points of a g.f.q.i. on a vector bundle over M. It is known [Ch-Z] that $cs(M) \geq ls(M)$, where $ls(M)$ is the Lyusternik-Schnirelmann category of M.

COROLLARY 4. *Let M be a compact closed manifold. Any quasi-function on M has more than $cs(M)$ critical points, and at least $sb(M)$ if all of them are non-degenerate.*

Fig. 1 shows an example of a quasi-function on S^1 having two critical points, and another wave front having no critical point, which, therefore, is not Legendrian equivalent to the zero section.

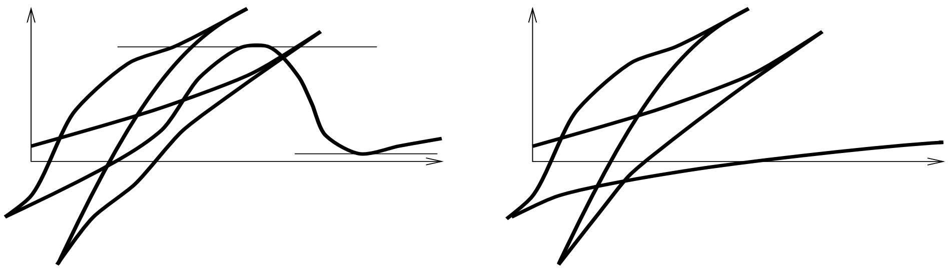

Fig. 1. Critical points of a quasi-function on the circle.

Chekanov's original proof of Theorem 3 takes several steps. It starts from the following fact: There exists an embedding $J^1(M,\mathbb{R}) \hookrightarrow J^1(\mathbb{R}^N,\mathbb{R})$ since M can be embedded in $\mathbb{R}^N$. A compactly supported Legendrian isotopy of $J^1(M,\mathbb{R})$ can be extended to a compactly supported Legendrian isotopy of $J^1(\mathbb{R}^N,\mathbb{R})$, and conversely, a generic g.f.q.i. on $\mathbb{R}^N$ induces a g.f.q.i. on M (see [Ch, B] for full details). Hence it is enough to prove the corresponding theorem in $\mathbb{R}^N$. The second step is to lift the problem to the symplectization of $J^1(\mathbb{R}^N,\mathbb{R})$. This symplectization is symplectomorphic to a cotangent bundle $T^*(\mathbb{R}^N \times (0,\infty))$. Chekanov has shown how to adapt the proof of the symplectic version [Si] of Theorem 3 to this situation, where it cannot be applied directly. J. C. Sikorav indicated to me a shorter proof based on the same symplectization idea.

In 1993, M. Chaperon produced a direct proof of Theorem 3. By Chekanov's trick, it is enough to consider the case $M = \mathbb{R}^n$.

PROPOSITION 5 ([Ch]). *If a contact transformation h is $\mathcal{C}^1$-close enough to the identity, and equal to the identity outside a compact set, the image $h(\lambda)$ of a Legendrian manifold λ of $J^1(\mathbb{R}^n,\mathbb{R})$ is induced by a g.f.q.i. provided that λ is induced by a g.f.q.i.*

Since any compactly supported Legendrian isotopy can be extended to a compactly supported contact flow, which itself can be considered as the composition of finitely many "close to the identity" contact transformations (as shown in [Ch]), Proposition 5 implies Theorem 3.

The work presented here started from a question of V. Arnold asking for the geometrical meaning of the non-trivial-to-guess formulae on which Chaperon's proof of Proposition 5 is based. The proof of Proposition 5 presented below (Section 2) might be considered as an answer to this question since it consists in a (elementary) geometrical construction starting from the fact that, under the hypothesis of Proposition 5, the image by h of the one-graph of an affine function on $\mathbb{R}^n$ is the one-graph of a function, which is a crucial ingredient of Chaperon's formulae.

1.2. To describe the other goals of this paper, some further definitions are needed. A *contact element* of M is a point q of M, together with a hyperplane of T_qM. The set of all contact elements (resp. cooriented contact elements) of M can be identified with its projectivized (resp. spherized) cotangent bundle PT^*M (resp. ST^*M). Each of these sets carries a natural contact structure (see [A-V-G], p. 266). A Finsler metric on M being given, the Liouville form pdq of T^*M induces a contact form on the set of unitary covectors, which is contactomorphic to ST^*M. The image of an immersed Legendrian submanifold in PT^*M (resp. ST^*M) by the projection $PT^*M \to M$ (resp. $ST^*M \to M$) is also called a *wave front*. The fibrewise compactification of $J^1(N,\mathbb{R}) \to J^0(N,\mathbb{R})$ is $PT^*(N\times\mathbb{R}) \to N\times\mathbb{R}$ and the wave fronts defined in the $J^1(N,\mathbb{R})$ setting are a particular case of this definition. A smooth submanifold N in M is a wave front, since the set of all contact elements tangent to N forms a Legendrian embedding of N in PT^*M. A generic wave front in M is a stratified singular hypersurface, with a tangent hyperplane defined at every point. Hence, if the codimension of N is greater than 1, N is not a generic wave front. The set of all covectors which vanish on the tangent space of a wave front W is called the conormal bundle ν^*W of W. Observe that in general ν^*W is immersed but not embedded in T^*M. A wave front in $M = N\times\mathbb{R}$ which is the projection of the time one of a Legendrian isotopy of $PT^*(N\times\mathbb{R})$ starting from the Legendrian lift of $N\times\{0\}$ can be considered as a *generalized quasi-function* on N, multivalued and with its derivative going to infinity at some points. Generalized quasi-functions appear as geometric solutions of some first order non-linear PDE.

Let $p : E \to M$ be a vector bundle. Let H be a smooth hypersurface in E. Denote by $\mathcal{C} \subset PT^*E$ the set of all contact elements of E which contain the tangent space of the fibre. If the Legendrian lift of H is transverse to $\mathcal{C}$, then H is called a *generating hypersurface* (it generates a Legendrian submanifold of PT^*M, as shown by Proposition 6). For example if (q,v) are coordinates on the product $E = M\times V$, and $\mathrm{H} : E \to \mathbb{R}$ a smooth function, the set $H = \mathrm{H}^{-1}(c)$ is a generating hypersurface if it is non-empty and if the map $(q,v) \to (\mathrm{H}(q,v), \frac{\partial \mathrm{H}}{\partial v}(q,v))$ is transverse to $(c,0) \in \mathbb{R}\times V^*$. The graph (in $N\times V\times\mathbb{R}$) of a generating family is a generating hypersurface for the induced front in $N\times\mathbb{R}$. Any regular level of a g.f.q.i. is a generating hypersurface.

Proposition 6 (see [A-V-G], pp. 273–277). *Let H be a generating hypersurface. There exists a unique Legendrian immersion $l \subset PT^*M$ such that the set of critical values of the restriction of p to H is the wave front of l.*

For example, if the map $M\times V \to \mathbb{R}\times V^* :\ (q,v) \mapsto (\mathrm{H}(q,v), \frac{\partial \mathrm{H}}{\partial v}(q,v))$ is transverse to $(c,0)$, then the set

$$\{(q,p) \in PT^*M \text{ such that } \exists v\ \ \mathrm{H}(q,v) = c,\ \frac{\partial \mathrm{H}}{\partial v}(q,v) = 0,\ \exists \rho \in \mathbb{R}\ \ p = \rho \cdot \frac{\partial \mathrm{H}}{\partial q}(q,v)\}$$

is the image of a Legendrian immersion.

The starting point of this paper is the observation that if a wave front $W \subset M$ is given by a generating hypersurface H in $M\times V$, it is the *envelope* of a family of hypersurfaces in M, parametrized by $v \in V$. Near W, each of these hypersurfaces is smooth. From now on, M is an open manifold.

Definition. A generating hypersurface $H \subset E = M \times V$ is called a *good generating hypersurface* if $H = \mathrm{H}^{-1}(0)$ is a regular level of some function $\mathrm{H} : M \times V \to \mathbb{R}$ with no critical points, and which is of the form $H(q,v) = f(q) + F(q,v)$, where F is a g.f.q.i.

The following theorem is proved in Section 3.

Theorem 7. *Let l_0 and l_1 be the extremities of a Legendrian isotopy of ST^*M. If l_0 has a good generating hypersurface, then so has l_1.*

Corollary 8. *A generalized quasi-function has a good generating hypersurface.*

We are now interested in the generalization of Corollary 4 in the setting of projectivized cotangent bundles.

Definition. Let W be a wave front in M, and let f be a function $M \to \mathbb{R}$. A *critical point* of f on W is the image by the projection $T^*M \to M$ of an intersection point between the graph of the differential of f and the conormal bundle of W.

In the generic situation, a critical point of $f_{|W}$ corresponds to a tangency between a regular level of f and the generic strata of W.

Definition. A *quasi-hypersurface* in M is a wave front $W \subset M$ such that there exists a smooth immersed closed submanifold N of M and a Legendrian isotopy l_t of $PT^*(M)$ starting from the Legendrian lift of N so that W is the wave front of l_1.

The following theorem shows that quasi-hypersurfaces behave like submanifolds with respect to Morse theory. The reader is invited to compare with a result by Lalonde and Sikorav [L-S, Theorem 3 (1)] on conormal bundles in symplectic topology.

Theorem 9. *A function on M without critical points, when restricted to a quasi-hypersurface $W \subset M$ has more than $cs(N)$ critical points, and generically at least $sb(N)$. The same estimates hold for the lower bound of the number of intersections between the conormal bundle ν^*W of W and the Lagrangian projection of a Legendrian manifold λ of $J^1(M,\mathbb{R})$ provided that λ has no critical points and that it is induced by a generating family of the form $f(q) + F(q,v)$, where F is a g.f.q.i.*

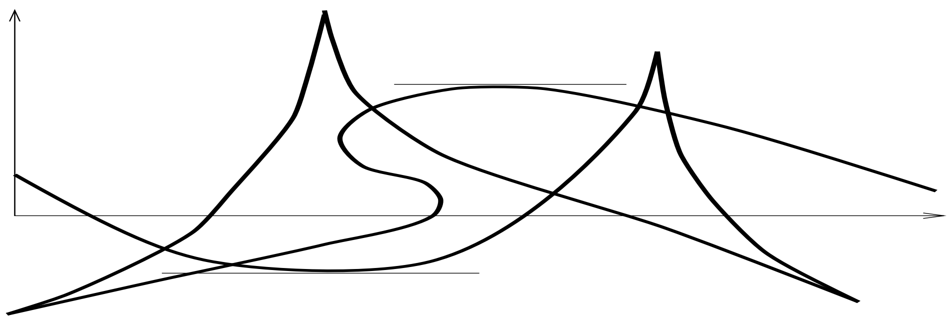

Fig. 2. Critical points of a generalized quasi-function on the circle.

Corollary 10. *A generalized quasi-function on a compact manifold N has more than $cs(N)$ critical points (points with horizontal tangent hyperplane), and at least $sb(N)$ in the generic case.*

Section 4 is devoted to the proof of Theorem 9 and related results. In Section 5, another construction is proposed, based on the hodograph transform. It yields some interesting consequences on the extrinsic geometry of wave fronts, typified by the following theorem:

Theorem 11. *Let W be a wave front such that there exists a Legendrian isotopy l_t of $ST^*\mathbb{R}^n$ starting from a Legendrian lift of the standard S^{n-1} in $\mathbb{R}^n$, such that W is the wave front of l_1. Then W has at least n pseudo-diameters, that is lines which are normal to W at two different points, in such a way that the two orientations of the line defined by the coorientation of W at the points of perpendicularity differ.*

In the case when W is embedded, Theorem 11 is a consequence of the results of F. Takens and J. White [T-W].

2. Chekanov theorem in the space of one-jets.

2.1. *Two lemmas.* Let $\lambda \subset J^1(\mathbb{R}^n, \mathbb{R})$ induced by a g.f.q.i. $F(q,v)$, and $W \subset \mathbb{R}^n \times \mathbb{R}$ the corresponding wave front. Recall that by definition there exists a non-degenerate quadratic form $Q(v)$, such that $F - Q$ has compact support. As mentioned above, W is the envelope of the family of smooth graphs of the functions $q \to F(q,v)$. Recall that the graph of a function is not the envelope of the tangent hyperplanes, except when the function is convex. However, we have the following lemma:

Lemma 12 (Stabilization). *We can suppose that W is the envelope of a family of hyperplanes i.e. there exists a generating function which is affine with respect to q.*

Proof. The function $F_1 : \mathbb{R}^n \times V \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $F_1(q,v,x,y) = F(y,v) + x \cdot (q-y)$ induces the same Legendrian submanifold λ, and is affine in q.

Remark. F_1 is not quadratic at infinity. However it coincides with $Q_1(q,v,x,y) = Q(v) + x \cdot (q-y)$ if y is large enough. We can deform F_1 into a function F_2 such that it induces the same λ, it is affine with respect to q when q is in a compact subset of $\mathbb{R}^n$ outside which W is identical to the graph of the 0 function, and it coincides with Q_1 outside a compact set: Let $\rho : [0,\infty) \to \mathbb{R}$ be a smooth function such that $\rho(x) = 1$ if $x \in [0,1]$ and $\rho(x) = 0$ if $x \in [2,\infty)$, and let $\phi_a(q,v,x,y) = \rho(\frac{\|q\|}{a})\rho(\frac{\|v\|}{a})\rho(\frac{\|x\|}{a})\rho(\frac{\|y\|}{a})$. There exists an appropriate choice of the constant a, such that the function $\phi_a(q,v,x,y)F_1(q,v,x,y) + (1-\phi_a(q,v,x,y))Q_1(q,v,x,y)$ is a possible choice for F_2.

Lemma 13 (Envelope preservation). *Consider a generating family $(q,v) \to F(q,v)$ and the family of smooth wave fronts (graphs) of $\mathbb{R}^n \times \mathbb{R}$ obtained from the family of functions $q \to F(q,v)$. Suppose that for each $v \in V$, the front of the image by h of $j^1F(\cdot,v)$ is smooth and given by the graph of $q \to G(q,v)$. Then G is a generating family, and the Legendrian manifold induced by G is the image by h of the one induced by F.*

Proof. Denote by $(Q(q,p,u), P(q,p,u), U(q,p,u))$ the components of the map h. Since it is a contact transformation, there exists $\mu(q,p,u) > 0$ such that $dU - PdQ =$

$\mu \cdot (du - pdq)$. The function G is defined by

$$G(Q(q,p,u),v) = U(q,p,u),\ p = \frac{\partial F}{\partial q}(q,v),\ u = F(q,v)$$

Lemma 13 follows from the fact that the relation $\frac{\partial G}{\partial v}(Q,v) = \mu \cdot \frac{\partial F}{\partial v}(q,v)$ holds when $Q = Q(q, \frac{\partial F}{\partial q}(q,v), F(q,v))$.

2.2. *Proof of Proposition* 5. We will show that it is possible to apply Lemma 13 on the stabilization given by Lemma 12.

Recall that h is the identity outside a compact set and is uniformly $\mathcal{C}^1$-close to the identity. We use a crucial fact of Chaperon's construction. Denote by $(Q,P,U)(q,p,u)$ the components of h. If the differential of h is close enough to the identity, then det $\frac{\partial(Q,U)}{\partial(q,u)}$ does not vanish at any point (q,p,u), and hence, when p is fixed, the map $(q,u) \mapsto (Q,U)(q,p,u)$ is a diffeomorphism of $J^0(\mathbb{R}^n,\mathbb{R})$. On the one-graph of an affine function f on $\mathbb{R}^n$, the coordinate p is fixed, and then $h(j^1 f)$ has a smooth front, hence it is the one-graph of a function g which coincides with f outside a compact set. This shows that the hypothesis of Lemma 13 are fulfilled by h and by the generating family $F_1(q,v,x,y) = F(y,v) + x \cdot (q-y)$ given by Lemma 12. Since h has compact support, the function G obtained by Lemma 13 coincides with F_1 if $\|q\|$ or $\|x\|$ are large enough. As remarked before, F_1 is "not too far" from Q_1, and so is G because of the hypothesis on h. One can deform G such that it coincides with Q_1 outside a compact set: There exists a choice of the constant a such that the function $G_1(q,v,x,y) = \phi_a(q,v,x,y)G(q,v,x,y) + (1 - \phi_a(q,v,x,y))Q_1(q,v,x,y)$ induces the same Legendrian submanifold than G, that is $h(\lambda)$. This is similar to what is done in the papers [B (Lemma 4)], [Ch (Lemma 3)], [Th] to make the functions standard at infinity, so we do not reproduce the estimates which lead to the choice of the constant a here. The function $G_2(q,v,x,Y) = G_1(q,v,x,q-Y)$ is a g.f.q.i. which induces $h(\lambda)$ and coincides with the non-degenerate quadratic form $Q(v) + xY$ outside a compact set.

3. The ST^*M case.

3.1. *Proof of Theorem* 7. The proof of Proposition 5 presented above is based on an envelope construction (stabilization and envelope preservation), a geometrical fact that can be adapted in the more general context of ST^*M. Through the identification of hypersurfaces with the regular levels of functions, this would be very similar to the calculation of Section 2. But is not necessary to carry out this program, because Theorem 7 is a consequence of Theorem 3, via the following construction:

Let W_0 and W_1 be the fronts associated to the extremities of a Legendrian isotopy of ST^*M. Let q be coordinates on M and (q,p) the associated coordinates on T^*M in which we see ST^*M as the hypersurface $\|p\| = 1$. The contact form on ST^*M is pdq, restricted to the unitary covectors. By the isotopy extension lemma, W_1 is the image of W_0 by a contact flow $(q,p,t) \to (Y,X)$. There exists a positive function $\mu : ST^*\mathbb{R}^n \times [0,1] \to \mathbb{R}$ such that $XdY = \mu(q,p,t)pdq$. Here, p and X denote unitary covectors, but this contact flow extends to a Hamiltonian flow of $T^*M \setminus O_M$:

$$(q,p,t) \to \big(Y(q,\frac{p}{\|p\|},t), \frac{\|p\|}{\mu(q,\frac{p}{\|p\|},t)} \cdot X(q,\frac{p}{\|p\|},t)\big).$$

Set $U(q,p,u,t) = u$ and lift this flow to a contact flow $\Phi : \mathcal{U} \times [0,1] \to \mathcal{U}$, where $\mathcal{U} = J^1(M,\mathbb{R}) \setminus \{p=0\}$. Recall that, by hypothesis, W_0 is the zero-set of a wave front in $J^0(M,\mathbb{R})$ whose Legendrian lift $\lambda_0 \subset J^1(M,\mathbb{R})$ does not meet $\{p=0\}$. Hence $\lambda_0 \subset \mathcal{U}$. The flow Φ is not compactly supported, but it induces on $\lambda_t = \Phi(\lambda_0,t)$ a compactly supported Legendrian isotopy. By hypothesis, there exists a function f such that $\lambda_0 - j^1 f$ is induced by a g.f.q.i. By Theorem 3, so is $\lambda_1 - j^1 f$. The Legendrian submanifold λ_1 cannot meet $\{p=0\}$ since it is the image by a flow of $\mathcal{U}$ of λ_0. By construction, Φ preserves the coordinate u, hence the image of W_0 by the original contact flow of $ST^*\mathbb{R}^n$ is the zero-set of λ_1.

3.2. *Generalized functions.* The main application of this construction is the situation when $M = N \times \mathbb{R}$, and where W is obtained from $N \times \{0\}$ by a Legendrian isotopy of $PT^*(N \times \mathbb{R})$. In this case we call W a generalized quasi-function. Since $N \times \{0\}$ is the zero set of the projection on the second factor of $N \times \mathbb{R}$, we get Corollary 8.

4. Critical points of a function on a wave front.

4.1. *Proof of Theorem* 9. Let W be the quasi-hypersurface under consideration. By definition, there exists a Legendrian isotopy of PT^*M starting from the Legendrian lift of W and finishing to the Legendrian lift of an immersed closed submanifold $N \subset M$ (i.e. the projectivized conormal bundle of N). Lift this isotopy to ST^*M, and extend it to a contact flow. This contact flow can be chosen so that it projects to a contact flow of PT^*M. (This means that the isotopy extension lemma is valid in PT^*M.) Recall that $\mathcal{U} = J^1(M,\mathbb{R}) \setminus \{p=0\}$, and denote by $\Phi : \mathcal{U} \times [0,1] \to \mathcal{U}$ the flow obtained by a procedure similar to the one of Section 2. Let $f : M \to \mathbb{R}$ be the function with no critical points under consideration. The one graph $j^1 f$ is a Legendrian submanifold λ of $\mathcal{U}$. As in Section 2, $\lambda_1 = \Phi(\lambda,1)$ is induced by a generating family which is the sum of f and a g.f.q.i. The restriction of this generating family to N is quadratic at infinity. As observed in [L-S, Theorem 3], Proposition 2 implies that the Lagrangian projection of λ_1 intersects the conormal bundle of N in more than $cs(N)$ points, and generically in more than $sb(N)$ points. By construction, the flow Φ, which is homogeneous in the fibre (and not only positively homogeneous), sends ν^*W to ν^*N. The intersections of the Lagrangian projection of λ_1 restricted to N are in one to one correspondence with the critical points of λ restricted to W, which proves the first assertion of Theorem 9. In the proof above, one can replace $\lambda = j^1 f$ by a Legendrian submanifold of $J^1(M,\mathbb{R})$ with no critical points and which is induced by a generating family of the form $f(q) + F(q,v)$ where F is a g.f.q.i. Theorem 9 applied in the case when $M = N \times \mathbb{R}$ and f is the projection to the second factor gives Corollary 10.

4.2. As shown by Fig. 3, there is no similar result in the ST^*M context. Fig. 3 shows five steps of a path in the space of wave fronts between the "eight curve" and the "lips" front, which is the projection of a Legendrian isotopy of $ST^*\mathbb{R}^2$. The projection to the horizontal axis has no critical points on the "lips" front. The third step is the time when it fails to be the projection of a Legendrian isotopy of $PT^*\mathbb{R}^2$. Observe also that the "lips" front is a regular level of a quasi-function.

Fig. 3. A path between the "eight" curve and the "lips" front.

5. Geometry of wave fronts via the hodograph transform. In this section we are concerned with wave fronts in $\mathbb{R}^n$ only. The following transformation, introduced by Arnold in [A, p. 48], contains as a particular case the identification of convex bodies of $\mathbb{R}^n$ with their support functions. In a sense to be precised, some wave fronts of $\mathbb{R}^n$ might have *support g.f.q.i.*

PROPOSITION (Hodograph transform [A]). *$ST^*\mathbb{R}^n$ is contactomorphic to $J^1(S^{n-1}, \mathbb{R})$.*

Proof. In all the sequel, we denote by the same θ a point in S^{n-1} and the unitary vector of $\mathbb{R}^n$ obtained via the standard embedding $S^{n-1} \hookrightarrow \mathbb{R}^n$. Denote by (Q, θ) the coordinates of $ST^*\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$. A point in $J^1(S^{n-1}, \mathbb{R}) = T^*S^{n-1} \times \mathbb{R}$ is denoted by (θ, p, u). View p as a tangent vector to the unitary sphere S^{n-1} centered at the origin. Denote by $\langle \cdot\ ,\ \cdot \rangle$ the Euclidean scalar product of $\mathbb{R}^n$ and by Q_θ the projection of Q to the tangent plane of S^{n-1} at the point θ. The hodograph transform is the mapping $(Q, \theta) \to (\theta, p = Q_\theta, u = \langle Q, \theta \rangle)$. One can check that this is a contactomorphism.

The fibre of $ST^*\mathbb{R}^n$ over the origin is sent to the one-jet extension of the zero function on the sphere, and the constant functions on the sphere correspond to spheres centered at the origin in $\mathbb{R}^n$. For the sake of brevity, a wave front $W \subset \mathbb{R}^n$ whose Legendrian lift $l \subset ST^*\mathbb{R}^n$ is Legendrian isotopic to a Legendrian lift of the sphere will be called a *pseudo-sphere*. By Theorem 3, to each pseudo-sphere in $\mathbb{R}^n$, there corresponds a *support g.f.q.i.* defined on S^{n-1}. Observe that if a g.f.q.i. $F : S^{n-1} \times V \to \mathbb{R}$ induces a wave front W in $\mathbb{R}^n$ via the hodograph transform, the generating family $F + d$ corresponds to the front W_d equidistant at distance d from W in the direction of its coorientation.

Pick a pseudo-sphere induced by a g.f.q.i. F and consider the function $G : S^{n-1} \times V \times V \to \mathbb{R}$ defined by $G(\theta, v, w) = F(\theta, v) + F(\theta^a, w)$, where θ^a denotes the point of S^{n-1} which is antipodal to θ. The generating function G can be deformed into a g.f.q.i. G_1 with the same critical points, and G_1 induces a g.f.q.i. g on $E = \frac{(S^{n-1} \times V \times V)}{(\theta, v, w) \sim (\theta^a, w, v)}$ which is a vector bundle over $\mathbb{R}P^{n-1}$. Since the Lyusternik-Schnirelmann category of $\mathbb{R}P^{n-1}$ is n, g has at least n critical points. Such a critical point corresponds to unordered couple (θ, θ^a) such that there exists $(v, w) \in V \times V$ with the following property: $\frac{\partial F}{\partial v}(\theta, v) = \frac{\partial F}{\partial v}(\theta^a, w) = 0$ and $\frac{\partial F}{\partial \theta}(\theta, v) + \frac{\partial F}{\partial \theta}(\theta^a, w) = 0$. This means that the direction of θ is parallel to a pseudo-diameter, which proves Theorem 11.

To finish with, let us emphasize the link between the support g.f.q.i. of a pseudo-sphere and the envelope construction of Section 2, now in the case of a compact front in $\mathbb{R}^n$. Let W be a pseudo-sphere and $F : S^{n-1} \times V \to \mathbb{R}$ a support g.f.q.i. of W. One can check that the hypersurface in $E = \mathbb{R}^n \times S^{n-1} \times V$ defined by the regular equation $\langle q, \theta \rangle - F(\theta, v) = 0$ is a generating hypersurface, quadratic at infinity with respect to the non-compact parameters. W is hence the envelope of a family of hyperplanes. We call a wave front W *locally convex* if there exist neighbourhoods of every point of its generic strata where W is a piece of convex hypersurface. A locally convex pseudo-sphere is the envelope of its tangent hyperplanes, or, equivalently, the support g.f.q.i. of a locally convex pseudo-sphere can be chosen to be a function.

The study of the extrinsic geometry of wave fronts is the subject of a paper in preparation with J. C. Alvarez, where Theorem 11 and related results will be extended to quasi-hypersurfaces. It will also be shown that one has the same lower bound for the number of geodesics which are twice normal to the quasi-hypersurface when $\mathbb{R}^n$ is endowed with an appropriate Finsler metric.

I am deeply grateful to V. Arnold who initiated this work, and to F. Laudenbach, for his interest and help. I wish to express my warmest thanks to the organizers of the workshop.

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