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EQUIVALENCE OF LAGRANGIAN GERMS IN THE PRESENCE OF A SURFACE

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Dedicated to Professor Stanisław Łojasiewicz for his 70th birthday

1. Introduction. Let ω be a closed two-form on a smooth manifold M. Most of interesting singular symplectic structures on M are given in the form of pull-back $\omega = \phi^\star\omega_0$ by a smooth map $\phi : M \to R^{2n}$ from a symplectic structure ω_0 on R^{2n} [7, 9, 5, 6]. The natural structure group of such singular symplectic manifolds is defined by integration of $\phi-$liftable Hamiltonian vector fields on (R^{2n}, ω_0). Action of this group on the space of Lagrangian varieties in (R^{2n}, ω_0) provides the classification of maximal isotropic varieties in pulled-back singular symplectic structure. Hence there is a natural question to ask: How does the local intersection data determine the orbits of the group of symplectomorphisms preserving singular values of ϕ and acting on the space of Lagrangian germs? This question may be reformulated as a generalization of the classical problem: Under which conditions the two Lagrangian germs are equivalent? In this paper we attempt to answer this question imposing the genericity conditions on the classified symplectic objects and assuming the set of critical values of ϕ is a smooth surface in R^{2n}. Another motivation to our study comes from the theory of symplectic triads [1, 10] and its generalization. However the local invariants considered in this paper do appear as the classified not necessary typical common positions of the considered two surfaces. Our aim is to construct all these invariants through the technique of finding the corresponding local normal forms. In Section 2 we classify the germs of generic pairs $(H, L; p)$, where H is a hypersurface in R^{2n} and L is a Lagrangian submanifold in (R^{2n}, ω_0). Then in Section 3 we consider the slightly different case if H is any coisotropic submanifold of (R^{2n}, ω_0). It is known that even-dimensional submanifolds in general position in the symplectic space

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are symplectic. In Section 4 we find the normal forms of pairs $(H, L; p)$ provided H is symplectic and H and L are in the transversal and nontransversal positions.

2. ***H*–relative Lagrangian submanifolds.** Let (M, ω_0) be a symplectic manifold, ω_0 be in the Darboux form. Let $H \subset M$ be a smooth hypersurface in M. By $(H; p)$ we denote the germ of H at p.

DEFINITION 1. A pair of germs $(H, L; p)$, where $(L; p)$ is a germ of a Lagrangian submanifold, is called an *H-relative Lagrangian submanifold of M.*

THEOREM 1. *If L is transversal to H at $p \in L \cap H$ then the pair $(H, L; p)$ is symplectically equivalent to the following one*

$$(\{(x, y) \in R^{2n} : x_1 = 0\},\ \{(x, y) \in R^{2n} : y_i = 0 \text{ for } i = 1, \ldots, n\}; 0).$$

Any two H-relative Lagrangian submanifolds which are transversal to H are H-relative symplectically equivalent, i.e. there exists a symplectomorphism Φ which preserves a hypersurface H and maps one Lagrangian manifold into the other.

Proof. It is easy to see that H is symplectically equivalent to

$$\{(x, y) \in R^{2n} : x_1 = 0\}.$$

L is H-relative symplectically equivalent to a Lagrangian submanifold generated by a function

$$S : R^n \ni x \mapsto S(x) \in R,$$

because L is transversal to H. A symplectomorphism

$$\phi(x, y) = (x, y - \operatorname{grad} S)$$

maps L into $\big(\{(x, y) \in R^{2n} : y_i = 0 \text{ for } i = 1, \ldots, n\}; 0\big)$. ∎

It is easy to prove the following lemma.

LEMMA 2. *If L is not transversal to H at $p \in H \cap L$ then the pair $(H, L; p)$ is symplectically equivalent to*

$$(H_0, L_0; 0) = (\{(x, y) \in R^{2n} : x_1 = 0\},$$
$$\{(x, y) \in R^{2n} : x_i = -\partial S/\partial y_i \text{ for } i = 1, \ldots, n\}; 0)$$

where S is a function-germ such that

$$d(\partial S/\partial y_1)|_0 = 0. \tag{1}$$

Proof. It is clear that H is symplectically equivalent to $H_0 = \{(x, y) : x_1 = 0\}$ and L may be simultaneously turned into

$$L_0 = \{(x, y) : x_I = -\partial S/\partial y_I(y_I, x_J),\ y_J = \partial S/\partial x_J(y_I, x_J)\},$$

where $I \cup J = \{1, \ldots, n\}$, $I \cap J = \emptyset$, $1 \in I$ ([3]). Then by the symplectomorphic change of variables

$$\Psi(x_I, x_J, y_I, y_J) = (x_I, -y_J, y_I, x_J),$$

we get the desired formula. ∎

Now we need the group G_H of germs of symplectomorphisms which preserve the fibration $(x,y) \to y$ and the hypersurface $H = \{(x,y) \in R^{2n} : x_1 = 0\}$ for the representation of $(H,L;p)$. Every element Φ of this group can be defined as a lifting of a diffeomorphism $\phi : R^n \ni y \mapsto \phi(y) \in R^n$ which preserve the fibration over $(y_2,\dots,y_n)$, i.e. $y = (y_1,y_2,\dots,y_n) \mapsto \bar{y} = (y_2,\dots,y_n)$ with adding the gradient of a function f which depends on $\bar{y}$

$$\Phi(x,y) = ((\phi^*)^{-1}(y)x + df(\bar{y}), \phi(y)).$$

Using the action of the group G_H on the Lagrangian germs we obtain the following result.

THEOREM 3. *A generic pair $(H,L;p)$ such that H is not transversal to L at p is symplectically equivalent to the pair $(H_0,L_0;0)$ obtained in Lemma 2, where*

$$S(y) = \pm y_1^k + \sum_{i=2}^{k-2} y_i y_1^{k-i} + \big(g(y_2,\dots,y_{k-2}) \pm \sum_{i=k-1}^{n} y_i^2\big) y_1, \tag{2}$$

provided $k = \dim_R \mathcal{E}_{y_1}/\Delta(S(y_1,0)) + 1 \le n+2$, where $g \in \mathbf{m}^2_{y_2,\dots,y_{k-2}} \setminus \mathbf{m}^3_{y_2,\dots,y_{k-2}}$, and $\mathbf{m}_{y_2,\dots,y_{k-2}}$ denotes the maximal ideal of the ring of smooth function-germs $\mathcal{E}_{y_2,\dots,y_{k-2}}$.

Proof. S is a deformation of the function $R \ni y_1 \mapsto S(y_1,0) \in R$ which satisfies (1). Therefore S is equivalent to the pull-back of the universal deformation of $\pm y_1^k$ ([3, 8]):

$$S(y) = \pm y_1^k + \sum_{i=2}^{k} \phi_i(y_2,\dots,y_n) y_1^{k-i},$$

where $\phi_{k-1} \in \mathbf{m}^2$. By the symplectomorphism

$$\Phi(x,y) = (x, y - \mathrm{grad}(\phi_k))$$

we reduce S to the form

$$S(y) = \pm y_1^k + \sum_{i=2}^{k-1} \phi_i(y_2,\dots,y_n) y_1^{k-i}.$$

We may assume that $(\phi_2,\dots,\phi_{k-2})$ is a submersion and

$$\phi_{k-1}(0,y_{k-1},\dots,y_n) \in \mathbf{m}^2_{y_{k-1},\dots,y_n} \setminus \mathbf{m}^3_{y_{k-1},\dots,y_n},$$

because $(H,L;0)$ is generic. Therefore S is equivalent to the germ

$$S(y) = \pm y_1^k + \sum_{i=2}^{k-2} y_i y_1^{k-i} + \phi(y_2,\dots,y_n) y_1,$$

where ϕ is deformation of a Morse function-germ $\phi(0,y_{k-1},\dots,y_n)$. Therefore by a lifting of a diffeomorphism of the form $\Psi(y) = (y_1,\dots,y_{k-2},\psi(y_2,\dots,y_n))$, we obtain (2). ∎

EXAMPLE 1. Let $n = 3$, then we have

$$\begin{aligned}
&S(y_1,y_2,y_3) = y_1^3 + y_2^2 y_1 \quad \text{(simple)},\\
&S(y_1,y_2,y_3) = \pm y_1^4 + y_2 y_1^2 + (g(y_2) \pm y_3^2) y_1,\ g \in \mathbf{m}^2,\\
&S(y_1,y_2,y_3) = y_1^5 + y_2 y_1^3 + y_3 y_1^2 + g(y_2,y_3) y_1,\ g \in \mathbf{m}^2.
\end{aligned}$$

COROLLARY 4. *If $k > n+2$ then the normal form of S under natural genericity conditions may be written in the following way*

$$S = \pm y_1^k + y_{i_1} y_1^{k-j_1} + \ldots + y_{i_l} y_1^{k-j_l} + \phi_{s_1}(\bar{y}) y_1^{k-s_1} + \ldots + \phi_{s_r}(\bar{y}) y_1^{k-s_r} + \psi(\bar{y}) y_1,$$

where $\{i_1, \ldots, i_l\} = \{2, \ldots, n\}$, $\{j_1, \ldots, j_l, s_1, \ldots, s_r\} = \{2, \ldots, k-2\}$, $\phi_{s_n} \in \mathbf{m}$, $\psi \in \mathbf{m}^2$.

Remark 1. All simple cases for $n = 2$ are classified by A_k-singularities

$$A_{l-1} : S(y_1, y_2) = y_1^3 \pm y_2^l y_1,\ l \geq 2.$$

3. Coisotropic pairs. Now we slightly generalize the notion of a pair $(H, L; p)$ taking instead of a hypersurface H any coisotropic submanifold, say V. In the generic transversal case we have

PROPOSITION 5. *Let V^{2n-k} be a coisotropic submanifold of M, let L be a Lagrangian submanifold of M and $p \in V \cap L$. If L is transversal to V at p, then the pair $(V, L; p)$ is symplectically equivalent to*

$$(V_0, L_0; 0) = (\{(x,y) \in R^{2n} : x_1 = \ldots = x_k = 0\}, \{(x,y) \in R^{2n} : y_1 = \ldots = y_n = 0\}; 0).$$

Proof. We use the same method as in the proof of Theorem 1. ■

LEMMA 6. *If L is not transversal to V at p then the pair $(V, L; p)$ is symplectically equivalent to*

$$(V_0, L_0; 0) = (\{(x,y) \in R^{2n} : x_1 = \ldots = x_k = 0\},$$
$$\{(x,y) \in R^{2n} : x_i = -\partial S/\partial y_i(y) \text{ for } i = 1, \ldots, n\}; 0),$$

where S is a function-germ such that the rank at 0 of the derivative of the mapping

$$R^n \ni y \mapsto (-\partial S/\partial y_1(y), \ldots, -\partial S/\partial y_k(y))$$

is not maximal.

Proof. If L is transversal at 0 to the submanifold

$$\{(x,y) \in R^{2n} : y_1 = \ldots = y_k = 0\}$$

then we can find the generating function S, which depends on (x_I, y_J), where $I \cup J = \{1, 2, \ldots, n\}$, $I \cap J = \emptyset$, such that

$$\{1, \ldots, k\} \subset J \tag{3}$$

and the rank at 0 of the derivative of the mapping

$$R^n \ni (x_I, y_J) \mapsto (-\partial S/\partial y_1(x_I, y_J), \ldots, -\partial S/\partial y_k(x_I, y_J)).$$

is not maximal. Otherwise L is generated by a function F which depends on (x_I, y_J) such that condition (3) is not satisfied. Let $I_1 = I \cap \{1, \ldots, k\}$ and $I_2 = I \setminus \{1, \ldots, k\}$. We can find F such that the rank at 0 of the derivative of the mapping

$$R^{|I_1|} \ni x_{I_1} \mapsto \partial F/\partial x_{I_1}(x_{I_1}, 0) \in R^{|I_1|}$$

is 0. By a symplectomorphism

$$\Phi(x, y_{I_1}, y_{I_2}, y_J) = (x, y_{I_1} - x_{I_1}, y_{I_2}, y_J),$$

which preserves V_0, we get transversality of L to $\{(x, y) \in R^{2n} : y_1 = \ldots = y_k = 0\}$ at 0. Finally we use a symplectomorphism

$$\Psi(x_J, x_I, y_J, y_I) = (x_J, -y_I, y_J, x_I),$$

which completes the proof. ∎

Now we use the group of germs of symplectomorphisms which preserve the fibration $(x, y) \mapsto y$ and the submanifold V_0. This approach leads to the classification of families of functions of k variables.

THEOREM 7. *If V^{2n-k} is not transversal to L at p then the generic pair $(V^{2n-k}, L; p)$ is symplectically equivalent to the pair*

$$(\{(x, y) \in R^{2n} : x_1 = \ldots = x_k = 0\}, L_0; 0),$$

where L_0 is generated by the function

$$S(y) = f(y_1, \ldots, y_k) + \sum_{i=1}^{\mu-l-1} \phi_i(y_{k+1}, \ldots, y_n)\rho_i(y_1, \ldots, y_k) + \sum_{i=1}^{l} \psi_i(y_{k+1}, \ldots, y_n)y_i,$$

$\mu = \dim_R(\mathcal{E}_{y_1,\ldots,y_k}/\langle \partial f/\partial y_1, \ldots, \partial f/\partial y_k\rangle) < \infty$, $\rho_i \in \mathbf{m}^2_{y_1,\ldots,y_k}$, $\rho_1, \ldots, \rho_{\mu-l-1}$, $y_1, \ldots, y_l$ *generate* $\mathbf{m}_{y_1,\ldots,y_k}$ *modulo* $\langle \partial f/\partial y_1, \ldots, \partial f/\partial y_k\rangle$ *and*

$$\operatorname{rank} D_0(\psi_1, \ldots, \psi_l) < l.$$

4. Equivalence of Lagrangian submanifolds in the presence of a generic even-dimensional submanifold. First we formulate the result on the normal form of a symplectic surface in (M, ω_0).

LEMMA 8. *If $X^{2(n-k)}$ is a symplectic submanifold of (M, ω_0) then at each point $p \in X^{2(n-k)}$ the germ $(X^{2(n-k)}; p)$ is symplectically equivalent to*

$$(X_0; 0) = (\{(x, y) \in R^{2n} : x_i = y_i = 0 \textit{ for } i = 1, \ldots, k\}; 0).$$

Proof. By induction using the Moser method and the procedure described in [4]. ∎

Let us consider a symplectic pair $(X^{2(n-k)}, L; 0)$. First we consider the transversal case. Since L is transversal to the submanifold

$$\{(x, y) \in R^{2n} : x_1 = \ldots = x_k = 0\}$$

we can express L by a generating function $S : R^n \ni x \mapsto S(x) \in R$. The transversality assumption is equivalent to the following condition

$$\operatorname{rank} D_0\big(R^{n-k} \ni (x_{k+1}, \ldots, x_n) \mapsto (\partial S/\partial x_1(0, x_{k+1}, \ldots, x_n), \ldots, \partial S/\partial x_k(0, x_{k+1}, \ldots, x_n)) \in R^k\big) = k.$$

Now we restrict our problem to the case $k = 1$.

THEOREM 9. *If L is transversal to $X^{2(n-1)}$ at p then the symplectic pair $(X^{2(n-1)}, L; p)$ is symplectically equivalent to*

$$(X_0, L_0; 0) = \big(\{(x, y) \in R^{2n} : x_1 = y_1 = 0\}, \{(x, y) \in R^{2n} : y_1 = x_2,\ y_2 = x_1,\ y_i = 0 \textit{ for } i = 3, \ldots, n\}; 0\big).$$

Proof. By Lemma 8 and the transversality assumption we can represent X^{2n-2} in the form

$$\{(x,y) \in R^{2n} : x_1 = y_1 = 0\}$$

and we can generate L by a function

$$S : R^n \ni x \mapsto S(x) \in R,\ S \in \mathbf{m}^2$$

which satisfies the following condition: there exists $j \in \{2, \ldots, n\}$ such that

$$\partial^2 S/\partial x_j \partial x_1(0) \neq 0. \tag{4}$$

Now we expand S in the form

$$S(x) = x_1^2 h(x) + x_1 g(x_2, \ldots, x_n) + f(x_2, \ldots, x_n).$$

By a symplectomorphism

$$\Phi(x,y) = (x, y + \operatorname{grad}(x_1^2 h(x) + f(x_2, \ldots, x_n))),$$

which preserves X^{2n-2}, we transform L and consequently S to the form

$$S(x) = x_1 g_1(x_2, \ldots, x_n).$$

By condition (4) the mapping

$$\phi : R^n \ni x \mapsto (x_1, g_1(x_2, \ldots, x_n), x_3, \ldots, x_{j-1}, x_2, x_{j+1}, \ldots, x_n) \in R^n$$

is a diffeomorphism. Now we use the symplectomorphism $\phi^\star : R^{2n} \to R^{2n}$, which completes the proof. ∎

THEOREM 10. *If L is not transversal to $X^{2(n-1)}$ at p then the generic symplectic pair $(X^{2(n-1)}, L; p)$ is symplectically equivalent to the following pair*

$$(\{(x,y) \in R^{2n} : x_1 = y_1 = 0\},$$
$$\{(x,y) \in R^{2n} : y_1 = \pm x_1 \pm x_2^2 \pm \ldots \pm x_n^2, y_2 = \pm 2x_1x_2, \ldots, y_n = \pm 2x_1x_n\}; 0).$$

Proof. L is transversal to

$$\{(x,y) \in R^{2n} : x_1 = 0\} \text{ or } \{(x,y) \in R^{2n} : y_1 = 0\}.$$

Otherwise $T_0L \subset T_0X$, which is impossible, because L is an n-dimensional Lagrangian submanifold and $X^{2(n-1)}$ is a $(2n-2)$-dimensional symplectic manifold. We may assume that L is transversal to $\{(x,y) \in R^{2n} : x_1 = 0\}$ at 0. Then $(X, L; 0)$ is symplectically equivalent to

$$(\{(x,y) \in R^{2n} : x_1 = y_1 = 0\}, \{(x,y) \in R^{2n} : y = \partial S/\partial x(x)\}; 0),$$

where S does not satisfy condition (4). By genericity of $(X, L; p)$ we get

$$\partial^2 S/\partial x_1^2(0) \neq 0.$$

S is a deformation of a function $R \ni x_1 \mapsto S(x_1, 0)$ on the manifold with boundary $\{x \in R^n : x_1 = 0\}$ ([2]). Therefore it has the form

$$S(x) = \pm x_1^2 + g(x_2, \ldots, x_n)x_1,$$

and $dg_0 = 0$. We may assume that g is a Morse function, which completes the proof. ∎

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