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A HOMOTOPY CLASSIFICATION OF SYMPLECTIC IMMERSIONS

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1. Introduction. The main result of this paper is concerning to the C^0-dense parametric h-principle [1] of symplectic immersions. Let (N,σ) be a smooth symplectic manifold and M a manifold with a closed C^∞ 2-form ω on it. A smooth map $f : (M,\omega) \longrightarrow (N,\sigma)$ is called symplectic if f pulls back σ onto ω. Let $\mathrm{Symp}(M,N)$ denote the space of symplectic immersions of M into N with C^∞ compact-open topology and $\mathrm{Symp}_0(TM,TN)$ denote the space of bundle monomorphisms $F : TM \longrightarrow TN$ with C^0 compact-open topology where F satisfies $F^*\sigma = \omega$ and the underlying (continuous) map of F pulls back the cohomology class of σ, denoted by $[\sigma]$, onto the cohomology class $[\omega]$. Then the differential d maps $\mathrm{Symp}(M,N)$ into $\mathrm{Symp}_0(TM,TN)$. The main theorem may now be stated as follows.

Theorem 1.1. *If* $\dim M < \dim N$ *then* $d : \mathrm{Symp}(M,N) \longrightarrow \mathrm{Symp}_0(TM,TN)$ *is a weak homotopy equivalence. In particular, symplectic immersions satisfy the* C^0*-dense parametric h-principle in the space of continuous maps* $f : M \longrightarrow N$ *which pull back the cohomology class of* σ *onto that of* ω.

It is interesting to note that when $\dim N = 2\dim M$, taking ω equal to zero we obtain the following result of Lees [2].

Corollary 1.2. *The Lagrangian immersions satisfy* C^0*-dense parametric h-principle.*

Theorem 1.3. *Let* $F : T(\mathrm{Op}\, A) \longrightarrow TN$ *be a bundle monomorphism such that* $F^*\sigma = \omega$*, where* A *is a compact set in* M*, and let the underlying map* f *be a symplectic immersion on a neighbourhood of a compact set* $B \subset A$*. If the relative cohomology class* $[f^*\sigma - \omega]$ *vanishes in* $H^2(A,B)$ *then* F *can be homotoped to a symplectic immersion such that the homotopy remains constant in a neighbourhood of* B.

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It should be remarked that Gromov studied in [1, §3.4.2] a more general problem, namely the classification of σ-regular isometric immersions for an arbitrary closed 2-form σ. The general theorem arises from the h-principle of some auxiliary sheaf which comes as the solution sheaf of an infinitesimally invertible differential operator, and Gromov proved this by using sophisticated machinery, for example, the Nash-Moser Implicit Function Theorem. However, the situation becomes much simplified when we restrict ourselves to isometric immersions in a symplectic manifold.

Our proof of Theorem 1.1 is based on a comment of Gromov [1, p. 327]. The proof involves sheaf theoretic technique and Moser's Stability Theorem for symplectic forms. Gromov used this technique to prove the 'Open Extension Theorem', which gives h-principle for a large class of partial differential relations, namely, relations admitting of open extensions which are invariant under fibre-preserving diffeomorphisms. The main idea there was to find a class of diffeotopies that would 'sharply move a submanifold locally at hypersurfaces' [1, §2.2.3] and at the same time would keep the extension relation invariant under its action. In the Open Extension Theorem fibre-preserving diffeotopies serve this purpose. In this specific problem the role is played by exact diffeotopies [1, §3.4.2]. However, the relation corresponding to symplectic immersions is non-open and the sheaf of symplectic isometric immersions is not even microflexible [see Section 2]. Hence Theorem 1.1 does not follow immediately from the Open Extension Theorem. The difficulty has been bypassed here by passing to an auxiliary sheaf which is microflexible and which has the same homotopy type as the sheaf of those symplectic isometric immersions whose graphs lie in a certain predefined subspace. On the other hand, since the relation is not open, an infinitesimal solution is not necessarily a local solution. Nevertheless, Moser's Stability Theorem [4] tells us that an infinitesimal solution is isotopic to a local solution of the differential relation.

For any undefined term we refer to [1].

2. Brief review of the sheaf theoretic results. We now briefly describe the sheaf theoretic techniques to prove parametric h-principle. Let Φ denote the sheaf of solutions of some r-th order partial differential relation $\mathcal{R} \subset J^r(M,N)$ defined for C^r maps $M \longrightarrow N$, and Ψ the sheaf of sections of the r-jet bundle $J^r(M,N) \longrightarrow M$ with images in $\mathcal{R}$. The natural topologies on $\Phi(U)$ and $\Psi(U)$ are respectively the C^r and C^0 compact open topologies.

Definition 2.1. The solution sheaf Φ and the relation $\mathcal{R}$ are said to *satisfy parametric h-principle* if the r-jet map $j^r : \Phi \longrightarrow \Psi$ is a weak homotopy equivalence.

Before proceeding further we state some general definitions and results on topological sheaves.

Definition 2.2. Let $\mathcal{F}$ be a topological sheaf over M and A be a compact set in M. The symbol $\mathcal{F}(A)$ will denote the space of maps which are defined over some neighbourhood of A in M; in fact it is the direct limit of the spaces $\mathcal{F}(U)$ where U runs over all the open sets containing A. A map $f : P \longrightarrow \mathcal{F}(A)$ on a polyhedron P is called *continuous* if there exists an open set $U \supset A$ such that each f_p is defined over U and the resulting map $P \longrightarrow \mathcal{F}(U)$ is continuous with respect to the given topology on $\mathcal{F}(U)$.

DEFINITION 2.3. A sheaf $\mathcal{F}$ is called *flexible* if the restriction maps $\mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ are Serre fibrations for every pair of compact sets (A, B), $A \supset B$. The restriction map $\mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ is called a *microfibration* if given a continuous map $f_0' : P \longrightarrow \mathcal{F}(A)$ on a polyhedron P and a homotopy f_t, $0 \leq t \leq 1$, of $f_0'|_B$ there exists an $\varepsilon > 0$ and a homotopy f_t' of f_0' such that $f_t'|\operatorname{Op} B = f_t$ for $0 \leq t \leq \varepsilon$. If for any pair of compact sets the restriction morphism is a microfibration then the sheaf $\mathcal{F}$ is called *microflexible*.

THEOREM 2.4 (Sheaf Homomorphism Theorem [1, p. 77]). *Let $\mathcal{F}$ and $\mathcal{G}$ be two topological sheaves defined on a manifold M and let $f : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism. If both sheaves are flexible and if f is a local weak homotopy equivalence then f is a weak homotopy equivalence.*

So to prove parametric h-principle for a relation $\mathcal{R}$ it suffices to show that the sheaves Φ and Ψ (as defined above) are flexible and the r-jet map $\jmath^r : \Phi \longrightarrow \Psi$ is a local weak homotopy equivalence. For any partial differential relation $\mathcal{R}$ the sheaf Ψ is always flexible [1, p. 40]. But to prove flexibility of Φ we need to impose certain extensibility conditions on $\mathcal{R}$.

Let M be embedded in a higher dimensional manifold M' and let $\mathcal{R}'$ be a relation on M'. We denote the corresponding sheaf of solutions by Φ'.

DEFINITION 2.5. Φ' is said to be an *extension* of Φ if the inclusion of M in M' induces a restriction homomorphism $\alpha : \Phi'|_M \longrightarrow \Phi$; moreover, $\alpha(x)$ is a surjection for each $x \in M$.

This means that if we restrict a solution of $\mathcal{R}'$ to M we obtain a solution of $\mathcal{R}$ and moreover every local solution of $\mathcal{R}$ can be lifted to a local solution of $\mathcal{R}'$.

Now, for a pair of compact subsets (A, B) in M we define the space $\Gamma(A, B)$ of compatible pairs of solutions inside $\Phi'(B) \times \Phi(A)$. This set consists of all pairs (f', f) such that $\alpha(f') = f|\operatorname{Op} B$.

DEFINITION 2.6. The extension Φ' will be called a *microextension* if the obvious map $\gamma : \Phi'(A) \longrightarrow \Gamma(A, B)$ is a microfibration.

Now we explain the concept of *diffeotopy sharply moving M in M'*. It is worth recalling that the idea contained in this definition is a key point in the Smale-Hirsch Immersion Theorem.

DEFINITION 2.7. We fix a metric d on M. An open set in M will be called *small* if it is contained in a ball of small radius. A class of diffeotopy $\mathcal{D}$ on M' is said to *sharply move M in M'* if given any hypersurface S lying in a small open set of M and given any two positive numbers r and ε we can obtain a diffeotopy $\{\delta_t\}$ in $\mathcal{D}$ which satisfies the following conditions:

(a) δ_0 is the identity map,
(b) each δ_t is identity outside an ε-neighbourhood of S,
(c) $d(\delta_1(S), M) > r$.

Gromov gives the following sufficient condition for flexibility of Φ in his Main Lemma [1, p. 82] and Microextension Theorem [1, p. 85].

THEOREM 2.8. *If Φ admits of a microextension Φ' which is microflexible and if there exists a class of acting diffeotopy on Φ' which sharply moves M in M' then Φ is a flexible sheaf.*

3. Defining an extension. Let (M,ω) and (N,σ) be as in Section 1. Then the symplectic immersions $(M,\omega) \longrightarrow (N,\sigma)$ correspond to the partial differential relation $\mathcal{R} \subset J^{(1)}(M,N)$ consisting of all 1-jets $j^1_x f$, $x \in M$, of local immersions f such that $f^*\sigma = \omega$ at x. Let Ψ denote the sheaf of bundle maps $F : TM \longrightarrow TN$ which pull back the form σ onto ω. This may be identified with the sheaf of sections of $\mathcal{R}$. To obtain an extension of $\mathcal{R}$, we will first embed (M,ω) isometrically into a symplectic manifold (M',ω'). We start with an $F : TM \longrightarrow TN$ in the sheaf $\Psi(M)$. Let $f : M \longrightarrow N$ be the underlying continuous map. We consider the bundle f^*TN/TM over M. Observe that the total space of the bundle, say X, has the same dimension as N. Now we can construct a symplectic form ω' on it. We first extend the bundle map F to a bundle morphism $F' : TX|M \longrightarrow TN$ such that F' maps fibres of $TX|M$ isomorphically onto the fibres of TN. The form ${F'}^*\sigma$ restricts to ω on M and hence can be extended to a closed form ω' on some neighbourhood M' of M in X. M' may be taken to be a tubular neighbourhood of M in X so that the inclusion $i : M \longrightarrow M'$ is a homotopy equivalence. Since ${F'}^*\sigma$ is non-degenerate so is ω' [5]. So (M,ω) is isometrically embedded in the symplectic manifold (M',ω').

We denote the sheaf of symplectic isometric immersions of (M,ω) in (N,σ) by $\mathcal{S}$ and that of (M',ω') in (N,σ) by $\mathcal{S}'$. Let $\mathcal{R}'$ denote the space of 1-jets of germs of symplectic immersions of (M',ω') in (N,σ) and Ψ' the sheaf of section of $\mathcal{R}'$.

PROPOSITION 3.1. *$\mathcal{S}'$ is an extension of $\mathcal{S}$.*

Proof. It is easy to see that the isometric embedding of (M,ω) in (M',ω') induces a morphism $\alpha : \mathcal{S}'|_M \longrightarrow \mathcal{S}$. To prove that $\alpha(x) : \mathcal{S}'(x) \longrightarrow \mathcal{S}(x)$ is onto we start with a local symplectic immersion f at a point $x \in M$. Let $\bar{f}$ be any extension of f to a local immersion in M'. Then, since dimension of M' is the same as the dimension of N, the form $\bar{\omega} = \bar{f}^*\sigma$ is a symplectic form. Now the two linear symplectic forms $\bar{\omega}_x$ and ω'_x defined on T_xM' coincide on the subspace T_xM. Hence there exists a linear isomorphism l of T_xM' which pulls back $\bar{\omega}_x$ onto ω'_x and keeps T_xM pointwise fixed. We consider the germ of a local map f' whose 1-jet at x equals to $j^1_x\bar{f}\circ l$ so that $j^1_x f' \in \mathcal{R}'$. By construction the jet $j^1_x f'$ projects onto $j^1_x f \in \mathcal{R}$. Moreover we may assume without loss of generality that f' extends f. So we have the following:

- ${f'}^*\sigma = \omega'$ at x.
- f' equals f on $U \cap M$, where U is the domain of f. Hence, pullbacks of both the forms ${f'}^*\sigma$ and ω' to M are the same.

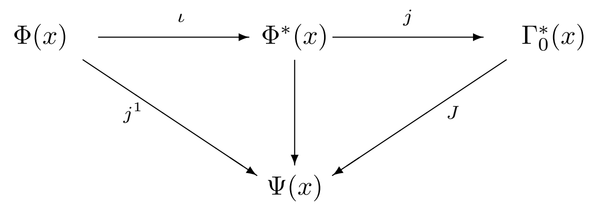
Therefore, by the Relative Poincaré Lemma, we obtain a 1-form φ on a neighbourhood, say $\widetilde{U}$, of x in U such that $d\varphi = {f'}^*\sigma - \omega'$ and $\varphi|\widetilde{U} \cap M = 0$. Now, by applying Moser's Theorem [4] we get a diffeomorphism δ on a neighbourhood, say U', of x in $\widetilde{U}$, such that

$\delta^*({f'}^*\sigma) = \omega'$, $\delta|U' \cap M$ is identity, and $d\delta_x = \mathrm{id}$. Then $f' \circ \delta$ is the required extension of f. ∎

PROPOSITION 3.2. *The* 1*-jet map* $j^1 : \mathcal{S} \longrightarrow \Psi$ *is a local weak homotopy equivalence.*

Proof. The main ingredient of the proof is Moser's Theorem on the stability of symplectic forms in a cohomology class. Consider the map $\rho = e \circ j^1 : \Phi(x) \longrightarrow \Psi(x) \longrightarrow \mathcal{R}_x$, where the space $\mathcal{R}_x$ consists of 1-jets of symplectic immersions at x, and e is the evaluation map at x. We shall prove that ρ induces an injective map between homotopy groups. It will then imply that the induced map j^1_* on homotopy groups is also injective. Injectiveness of ρ_* may be proved proceeding as in du Plessis [3] and using the following observation. Let $\varphi_p : M \longrightarrow N$, $p \in P$, be a continuous family of smooth maps parametrized by a polyhedron P such that $\varphi_p^*\sigma = \omega$ at x. By the above lemma we can extend φ_p to a neighbourhood of x in M' as φ'_p such that ${\varphi'}_p^*\sigma = \omega'$ at x. We set $\omega'_p = {\varphi'}_p^*\sigma$. Then $\omega'_p = \omega'$ at x for each $p \in P$. Now by Moser's Theorem we get a family of diffeomorphisms δ_p (homotopic to the identity), defined on a neighbourhood of x such that $\delta_p^*\omega'_p = \omega'$, $\delta_p(x) = x$ and $d\delta_p|T_xM' = \mathrm{id}$. Define $\bar\varphi_p = \varphi'_p \circ \delta_p|_M$ on $\mathrm{Op}\, x$. Then $\bar\varphi_p$'s are isometric immersions on $\mathrm{Op}\, x$ and $j^1_x\varphi_p = j^1_x\bar\varphi_p$. Moreover, if some φ_p is isometric on a neighbourhood of x, we may get $\bar\varphi_p = \varphi_p$ on $\mathrm{Op}\, x$ in M.

We now prove that j^1_* is surjective. Let Γ denote the sheaf of smooth maps from M to N. This is a sheaf over M. Consider the parametric sheaf Γ^M over $M \times M$ which is defined as follows: For open subsets $U, V \subset M$ we set $\Gamma^M(U \times V)$ equal to $\Gamma(U)^V$, which is the space of continuous maps $V \longrightarrow \Gamma(U)$. Now take the restriction of Γ^M to the diagonal. We shall denote this sheaf by Γ^*, and call it the *associated sheaf* of Γ. Observe that $\Gamma^*(x)$ is the direct limit of the spaces $\Gamma(U)^U$ where U runs over open neighbourhoods of x in M. Thus a local section in Γ^* can be conceived as a continuous family of germs $\varphi_x \in \Gamma(U)$, $x \in U$. It can be proved that the canonical inclusion of Γ in Γ^*, given by $\varphi \mapsto \{u \mapsto \varphi\}$, is a weak homotopy equivalence (see [1, p. 76]). (The above construction is equally true for an arbitrary sheaf.) Returning to the proof of surjectiveness of j^1_*, we split j^1 in the following way:



where Φ^* is the associated sheaf of Φ and $\Gamma_0^*(x)$ is the subspace of $\Gamma^*(x)$ consisting of all those families of germs $\{\varphi_u : u \in \mathrm{Op}\, x\}$ for which φ_u is a local immersion and $\varphi_u^*\sigma = \omega$ at u, in other words $j^1_u\varphi_u \in \mathcal{R}$. Thus it is easy to see that any section in $\Psi(x)$ gives rise to a section in $\Gamma_0^*(x)$. Hence J is onto. (The same technique can be applied to show that J_* is onto at each homotopy level.) Now we shall show that the map j induces surjective homomorphism in the homotopy, which will complete our proof. To prove this, it is enough to consider the zeroeth homotopy level. To this end, take a family $\{\varphi_u : u \in U\}$ as above where U is an open neighbourhood of x. We may suppose without loss of generality that each φ_u is defined on the same open subset U. Now, using Moser's Theorem we can deform the family $\{\varphi_u\}$ to a family $\{\bar{\varphi}_u\}$ of symplectic immersions in $\Phi(\widetilde{U})$ such that $j^1_u\bar{\varphi}_u = j^1_u\varphi_u$ for all $u \in \widetilde{U}$, where $\widetilde{U}$ is an open neighbourhood of x in U. Moreover, for each u, the homotopy between $j^1\bar{\varphi}_u$ and $j^1\varphi_u$ is constant at u. The family $\{\bar{\varphi}_u\}$ defines a section in $\Phi^*(x)$. So we have proved that every path component of $\Gamma^*_0(x)$ intersects $\Phi^*(x)$. Thus j_* is onto at the zeroeth homotopy level. ∎

However, the extension sheaf $\mathcal{S}'$ is not microflexible, as it can be seen from the following example.

EXAMPLE 3.3. Consider the standard embedding of the closed unit disc in $\mathbf{R}^2$. If we deform it near the boundary by pushing it inside then it (the homotopy) cannot be extended symplectically on the whole of the disc.

This phenomenon may be explained as follows: If f_0 is a symplectic immersion over $\operatorname{Op} A$ and f_t a homotopy of f_0 such that $f_t|\operatorname{Op} B$ is a symplectic immersion, then the relative cohomology class of $f_t^*\sigma - \omega$ in $H^2(A, B)$ determines the obstruction to extending $f_t|\operatorname{Op} B$ to $\operatorname{Op} A$ as a symplectic immersions. If the cohomology class $[f_t^*\sigma - \omega] = 0 \in H^2(A, B)$ then there exists a smooth of 1-forms α_t such that α_t vanishes on $\operatorname{Op} B$ and $f_t^*\sigma - \omega = d\alpha_t$. Then Moser's Stability Theorem applies and we can lift $f_t|\operatorname{Op} B$ over A as symplectic immersion.

Since $\mathcal{S}'$ is not microflexible we cannot apply the sheaf theoretic techniques (described in Section 2) on it. However, we shall see in the following section that there exists a topological sheaf on M' naturally associated to a subspace of the space of symplectic immersions which do satisfy microflexibility and has the same homotopy type as $\mathcal{S}'$.

4. Construction of the Auxiliary Sheaf. Since both the differential 2-forms σ and ω' are symplectic, the product form $p_2^*\sigma - p_1^*\omega'$ on $M' \times N$ is a symplectic form, where p_1 and p_2 are respectively the projection maps of $M' \times N$ onto the first and the second factor. We shall denote this product symplectic form by $\sigma - \omega'$. If $f : M \longrightarrow N$ is a symplectic isometric immersion then its graph map $g = (1, f) : M' \longrightarrow M' \times N$ is a Lagrangian section of $(M' \times N, \sigma - \omega')$, and this correspondence is bijective.

In the rest of this section we assume that *the symplectic form* $\sigma - \omega'$ is *exact* (which is equivalent to saying that σ and ω are exact). Let τ be a 1-form such that $\sigma - \omega' = d\tau$. We construct the sheaf of exact Lagrangian sections as follows: This consists of pairs (g, φ), where $g : M' \longrightarrow M' \times N$ is a section such that the underlying map $f = p_2 \circ g : M' \longrightarrow N$ is an immersion, and φ is a function on M' such that $g^*\tau = d\varphi$. We denote the sheaf of such pairs by $\mathcal{E}'$ and call it the sheaf of τ-exact Lagrangian sections. Observe that $\mathcal{S}'$ and $\mathcal{E}'$ are locally homotopically equivalent since the germ of a Lagrangian section at a point denotes a germ of an exact Lagrangian section; moreover the space of primitives φ for a τ-exact Lagrangian section g (meaning that φ satisfies the relation $g^*\tau = \varphi$) is isomorphic to $\mathbf{R}$. Consequently, the sheaf of sections corresponding to the relation, of which $\mathcal{E}'$ is the solution sheaf, has the same homotopy type as Ψ'. We now prove

PROPOSITION 4.1. *The sheaf* $\mathcal{E}'$ *of* τ*-exact Lagrangian sections is microflexible.*

Proof. Let (A, B) be a pair of compact sets in M'. Let g' be a τ-exact Lagrangian section over a A (meaning that it is defined on a neighbourhood of A) such that $g'^*\tau = d\varphi'$ for a 0-form φ', and (g_t, φ_t) a homotopy of $(g', \varphi')|\operatorname{Op} B$ in $\mathcal{E}'$.

We first prove the following simple lemma.

LEMMA 4.2. *Let g_t be a homotopy of τ-exact Lagrangian sections. If g_0 is τ'-exact Lagrangian for a 1-form τ' satisfying $\sigma - \omega' = d\tau'$, then g_t is also τ'-exact Lagrangian for each t.*

Proof. Two such forms τ and τ' differ by a closed 1-form c on $M' \times N$. So, we have the following relation

$$g_t^*\tau' = g_t^*\tau + g_t^*c$$

for every t. Then, by hypothesis, g_0^*c is an exact form. Since c is closed, g_t^*c is also exact. Consequently $g_t^*\tau'$ is exact for each t. ∎

Proof of Proposition 4.1 (continued). Now, by the standard theory of Lagrangian submanifolds [4], there exists a neighbourhood W of the Lagrangian submanifold $L' = \operatorname{Im} g'$ such that $(W, d\tau)$ is symplectomorphic to a neighbourhood of the zero section $Z_{L'}$ in the cotangent bundle $(T^*L', d\theta_{L'})$ with the standard symplectic form $d\theta_{L'}$ on it. Under this correspondence, the Lagrangian submanifolds in W are mapped onto the closed forms (near $Z_{L'}$), whereas the $\tau' = \delta^*\theta_{L'}$ -exact Lagrangians correspond to exact forms on L'. Clearly the sheaf of exact 1-forms is microflexible. Hence we can obtain lifts g'_t of g_t (for t small enough) which are τ'-exact Lagrangian sections. By the Lemma above they are also τ-exact. Moreover, for small t, the underlying maps will be immersions on $\operatorname{Op} A$. Now, we can choose a homotopy φ'_t on $\operatorname{Op} A$ such that $g_t'^*\tau = d\varphi'_t$. On $\operatorname{Op} B$, we have $d\varphi'_t = d\varphi_t$. Hence $\varphi'_t - \varphi_t = c_t$, where c_t is a closed 0-form, that is a constant. So we may replace φ'_t by $\varphi'_t - c_t$. The homotopy $(g'_t, \varphi'_t - c_t)$ is the required lift. ∎

We shall now describe a class of diffeotopy which would act on the sheaf $\mathcal{E}'$ and at the same time sharply move a submanifold of M' of positive codimension. Since ω' is symplectic we have an isomorphism $I_{\omega'} : \mathcal{X}(M') \longrightarrow \Lambda^1(M')$ from the space of vector fields $\mathcal{X}(M')$ onto the space of 1-forms $\Lambda^1(M')$. A C^∞ diffeotopy δ_t of M' is called exact if δ_0 is identity and if $\delta'_t = \frac{d\delta_t}{dt}$ is a Hamiltonian vector field for each t. So we can write $\delta'_t.\omega' (= I_{\omega'}(\delta'_t)) = d\alpha_t$ for some smooth family of exact 1-form $d\alpha_t$ on M'. If α_t can be chosen to be identically zero on the open subset where δ_t is constant then such a diffeotopy is called a *strictly exact diffeotopy*.

PROPOSITION 4.3. *The strictly exact diffeotopies of M' act on the sheaf $\mathcal{E}'$.*

Proof. Let δ_t be a strictly exact diffeotopy on M'. We define a diffeotopy $\bar{\delta}_t$ on $M' \times N$ by $\bar{\delta}_t(x, y) = (\delta_t(x), y)$, where $x \in M'$ and $y \in N$. It follows that $\bar{\delta}'_t.(\sigma - \omega')$ is exact for each t. Let α_t be a smooth family of 0-forms on $M' \times N$ satisfying $\bar{\delta}'_t.(\sigma - \omega') = d\alpha_t$. Then,

$$\begin{aligned}\frac{d}{dt}(\bar{\delta}_t{}^*\tau) = \mathcal{L}_{\bar{\delta}'_t}\tau = d(\bar{\delta}'_t.\tau) + \bar{\delta}'_t.d\tau &= d(\bar{\delta}'_t.\tau) + \bar{\delta}'_t.\sigma - \bar{\delta}'_t.\omega' \\ &= d(\bar{\delta}'_t.\tau) + d\alpha_t = d(\bar{\delta}'_t.\tau + \alpha_t).\end{aligned}$$

If we define $\varphi_t = \int_0^t (\bar{\delta}'_t.\tau + \alpha_t)\, dt$ then $\bar{\delta}_t^*\tau = \tau + d\varphi_t$. Now we are in a position to define the action. For $(g, \varphi) \in \mathcal{E}'$ and δ_t as above, we set

$$\delta_t^*(g, \varphi) = (\delta_t^* g, (\delta_t^{-1})^*(\varphi + g^*\varphi_t)),$$

where $\delta_t^* g = \bar{\delta}_t \circ g \circ \delta_t{}^{-1}$. ■

Proposition 4.4. *The exact diffeotopies of the symplectic manifold* (M', ω') *sharply move* M *in* M'.

Proof. (Gromov) To move a closed hypersurface S lying in a small open set U of M we start with a vector $\partial_0 \in T_{x_0}(M')$ transversal to U in M'. This ∂_0 extends to an exact field $\partial = I_{\omega'}^{-1}(dH)$ on which is transversal to U, since U is chosen small. In order to make the corresponding exact isotopy δ_t sharply moves S, we take the union $S_\varepsilon = \cup_t \delta_t(S) \in M'$ over $t \in [0, \varepsilon]$ and then multiply the Hamiltonian H by a properly chosen C^∞ function a on M' which vanishes outside an arbitrarily small neighbourhood of $\operatorname{Op} S_\varepsilon$ and which equals one in a smaller neighbourhood of S_ε. This makes the diffeotopy corresponding to the field $I_{\omega'}(d(aH))$ as sharp as we want. ■

Now applying the Main Lemma of Gromov [1, p. 82] we may conclude from above that

Proposition 4.5. *The sheaf* $\mathcal{E}'|_M$ *is flexible.*

It then follows from the Sheaf Homomorphism Theorem that $\mathcal{E}'|_M$ satisfies parametric h-principle.

Let $\mathcal{E}$ be the sheaf of pairs (g, φ) on M, where $g : M \longrightarrow M' \times N$ is a section such that its underlying map is an immersions and φ is a function on M satisfying the relation $g^*\tau = d\varphi$. To descend h-principle from $\mathcal{E}'|_M$ to $\mathcal{E}$ we observe that

Proposition 4.6. $\mathcal{E}'$ *is a microextension of* $\mathcal{E}$.

Proof. From Proposition 3.1 and the discussion preceeding Proposition 4.1 it follows that $\mathcal{E}'$ is an extension of $\mathcal{E}$. To prove that $\mathcal{E}'$ is a microextension of $\mathcal{E}$ we consider a lifting problem

$$\begin{array}{ccc} P \times \{0\} & \xrightarrow{(G_0', \psi_0')} & \mathcal{E}'(A) \\ \big\downarrow i & & \big\downarrow \gamma \\ P \times I & \xrightarrow[(g', \varphi'), (g, \varphi)]{} & \Gamma(A, B) \end{array}$$

where $\alpha \circ (G_0', \psi_0') = (g_0, \varphi_0)$ and $(h_0', \psi_0')|\operatorname{Op} B = (g_0', \varphi_0')$ and where $\Gamma(A, B)$ is a subset of $\mathcal{E}'(B) \times \mathcal{E}(A)$ consisting of compatible solutions as defined in Section 2. (To avoid too many symbols we assume P to be a point and denote $g(t)$ by g_t and so on.) We shall denote the underlying maps of G_0', g_t' and g_t by F_0', f_t' and f_t. Since they are immersions (which correspond to an open differential relation), we can obtain a lift of the corresponding microextension problem for immersions. Let us denote the lift by F_t, where $0 \le t \le \varepsilon$ for some positive number $\varepsilon \le 1$. Now each F_t being immersion between equidimensional spaces, pulls back σ onto a symplectic form on a neighbourhood of A. Let us set $F_t^*\sigma = \omega_t'$. We denote the corresponding graph map by G_t. Then we have the relation $F_t^*\sigma - \omega' = dG_t^*\tau$. On the other hand we obtain a homotopy ψ'_t of ψ'_0 such that ψ'_t coincides with φ_t and φ'_t on the relevant spaces. The 1-form α_t defined by $\alpha_t = G_t^*\tau - d\psi'_t$ satisfies the following

(a) $\alpha_0 = 0$
(b) α_t vanishes on some open neighbourhood of A in M,
(c) α_t vanishes on an open neighbourhood of B in M'
(d) $F_t^*\sigma - \omega' = d\alpha_t$.

Consider the vector fields $X_t = I_{\omega_t}^{-1}(\frac{d\alpha_t}{dt})$. The vector field vanishes on $\mathrm{Op}_M A$ as well as on $\mathrm{Op}_{M'} B$. Hence it can be integrated on a neighbourhood of A in M' to obtain a family of diffeomorphisms $\{\delta_t;\ 0 \le t \le \widetilde{\varepsilon}\}$ for some $\widetilde{\varepsilon} \le \varepsilon$ such that

(e) δ_0 is identity on $\mathrm{Op}_{M'} A$,
(f) $\delta_t |\, \mathrm{Op}_M A = \mathrm{id}$,
(g) $\delta_t |\, \mathrm{Op}_{M'} B = \mathrm{id}$,
(h) $\delta_t^* \omega'_t = \omega'$.

The required partial lift of the original lifting homotopy problem can now be given by the graph map of $F'_t = F_t \circ \delta_t$. In fact, Since F'_t is a symplectic immersion ${G'}_t^*\tau$ is closed. On the other hand, $i : M \longrightarrow M'$ induces an isomorphism $i^* : H^2_{deR}(M') \longrightarrow H^2_{deR}(M)$, and we know from our initial data that $i^*{G'}_t^*\tau$ is exact. Hence, ${G'}_t^*\tau$ is also exact. It is now a trivial matter to fix ψ'_t. ∎

The Microextension Theorem of Gromov [1, p. 85] now implies that the sheaf $\mathcal{E}$ is flexible. We have already proved the local h-principle in Proposition 3.2. So again appealing to the Sheaf Homomorphism Theorem we may conclude that $\mathcal{E}$ satisfies parametric h-principle.

Finally we prove

PROPOSITION 4.7. *$\mathcal{E}(M)$ has the same homotopy type as the space $\mathcal{S}(M)$ of symplectic isometric immersions.*

Proof. Consider the following sequence of maps between the function spaces: $\mathcal{E}'|_M \xrightarrow{(p_2)_*} \mathcal{S}'|_M \xrightarrow{j^1} \Psi'|_M$. The C^0-dense parametric h-principle for $\mathcal{E}'|_M$ says that the composition is a weak homotopy equivalence. Hence $(p_2)_*$ induces injective maps between homotopy groups. On the other hand, given any symplectic immersion f near M in M' we can obtain a τ-exact Lagrangian section $g \in \mathcal{E}'|_M$ such that $p_2 \circ g$ is arbitrarily C^0-close to f. In particular we may choose g within the neighbourhood of graph f which is symplectomorphic to the neighbourhood of the zero section $Z_{M'}$ in $T^*(M')$ (see Proposition 4.1). Hence $p_2 \circ g$ can be homotoped within the space $\mathcal{S}'$ to f. In fact, g corresponds to a closed form whereas graph f corresponds to the zero section. We denote the corresponding forms by the same sympbols. The homotopy $(1-t)g$ brings g onto graph f within the space of Lagrangian sections as multiplication by t takes closed forms to closed forms, which correspond to Lagrangian sections of $M' \times N \longrightarrow M'$ provided they are sufficiently C^∞ close to the zero form. This observation proves that $(p_2)_*$ induces an isomorphism between the homotopy groups.

Proceeding as in Proposition 4.6 we may observe that both the restriction maps

$\mathcal{S}'(M) \longrightarrow \mathcal{S}(M)$ and $\mathcal{E}'(M) \longrightarrow \mathcal{E}(M)$ are fibration. Moreover, for any $g \in \mathcal{E}$, the fibres in $\mathcal{E}'(M)$ and $\mathcal{S}'(M)$ over g and $p_2 \circ g$ respectively are homotopically equivalent. We have proved above that $\mathcal{E}'(M)$ and $\mathcal{S}'(M)$ are of the same weak homotopy type. Hence using homotopy exact sequence of fibrations we conclude that $\mathcal{E}(M)$ and $\mathcal{S}(M)$ are also of the same weak homotopy type. ■

This leads us to the following intermediate theorem.

THEOREM 4.8. *If the differential forms σ and ω are exact then the space of symplectic immersions of M into N satisfies parametric h-principle.*

5. Proof of the main theorem. Let us now go back to our case where $\sigma - \omega$ is not necessarily exact on $M \times N$. However, if $f : M \longrightarrow N$ is a continuous map such that $f^*[\sigma] = [\omega]$ then f can be extended to a map $f' : M' \longrightarrow N$ such that ${f'}^*[\sigma] = [\omega']$. Then in a neighbourhood, say W, of graph f there exists a 1-form τ such that $\sigma - \omega' = d\tau$. We shall denote the sheaf of symplectic immersions $M \times N$ whose graphs lie in W by the symbol $\mathcal{S}_W$. Then from the discussion of the previous section it follows that $\mathcal{S}_W$ satisfies parametric h-principle. We now come to the proof of Theorem 1.1.

Proof of Theorem 1.1. It remains only to prove the injectivity of the maps $d_* : \pi_i(\mathcal{S}(M)) \longrightarrow \pi_i(\mathrm{Symp}_0(TM, TN))$ for each integer i. Let f_0 and f_1 be two symplectic immersions on M such that their differentials df_0 and df_1 are homotopic in $\mathrm{Symp}_0(TM, TN)$; that is, there exists a homotopy $F_t : TM \longrightarrow TN$ such that $F_t^*\sigma = \omega$ for each t and the underlying maps $f_t : M \longrightarrow N$ satisfies $f_t^*[\sigma] = [\omega]$. For each t we can choose a neighbourhood W_t of graph f_t on which $\sigma - \omega$ is exact. Then the sheaves $\mathcal{S}_t$ $(= \mathcal{S}_{W_t})$ satisfy the parametric h-principle. We can cover the set $\bigcup_t f_t(M)$ by finitely many such W_t's such that any two consecutive ones (ordered by the real number) intersect in a set which contains completely the graph of some f_t. Without any loss of generality we may assume that the neighbourhoods $\{W_1,\ W_2\}$ have this property. Let, for some t_0, the graph of f_{t_0} lie in $W_1 \cap W_2$. Then by h-principle for the sheaf $\mathcal{S}_{W_1 \cap W_2}$ we obtain a symplectic immersion f C^0-close to f_{t_0} such that the differentials df and F_{t_0} are homotopic within $\mathrm{Symp}_0(TM, TN)$ and the underlying maps of the homotopy have their graphs in $W_1 \cap W_2$. Then applying parametric h-principle for $\mathcal{S}_1$ we conclude that f and f_0 are homotopic within the space $\mathcal{S}_1$. On the other hand f is homotopic to f_1 within the space $\mathcal{S}_2$. Joining these two homotopies we obtain a homotopy between f_0 and f_1 in the space of symplectic immersions. This proves that the differential d induces an isomorphism between the homotopy groups at the zero level.

Working with a family of such maps parametrized by spheres S^i, we can similarly prove the isomorphism between the higher homotopy groups of the relevant spaces which gives the desired h-principle. ■

We now prove the relative or extension version of h-principle for symplectic immersions.

Proof of Theorem 1.3. Since $[f^*\sigma - \omega]$ vanishes in $H^2(A, B)$, there is a 1-form φ vanishing on $\mathrm{Op}\, B$ such that $f^*\sigma - \omega = d\varphi$. Hence, for a proper choice of W and τ, $\sigma - \omega = d\tau$ on W and $g = (1, f)|\, \mathrm{Op}\, B$ is in $\mathcal{E}_{W,\tau}(B)$. Now consider the following diagram where the horizontal arrows are weak homotopy equivalences and the vertical ones are fibrations.

$$\begin{array}{ccc} \mathcal{E}_W(A) & \longrightarrow & \Psi_W(A) \\ \downarrow & & \downarrow \\ \mathcal{E}_W(B) & \longrightarrow & \Psi_W(B) \end{array}$$

Hence the fibres over $g|_B$ and $df|_{TB}$ are also weak homotopy equivalent. The theorem follows as F lies in the fibre over $df|_{TB}$. ∎

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