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TRANSLATION FOLIATIONS OF CODIMENSION ONE ON COMPACT AFFINE MANIFOLDS

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Abstract. Consider two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, of dimension one and codimension one respectively, on a compact connected affine manifold (M, ∇). Suppose that $\nabla_{T\mathcal{F}_1} T\mathcal{F}_2 \subset T\mathcal{F}_2$; $\nabla_{T\mathcal{F}_2} T\mathcal{F}_1 \subset T\mathcal{F}_1$ and $TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2$. In this paper we show that either $\mathcal{F}_2$ is given by a fibration over S^1, and then $\mathcal{F}_1$ has a great degree of freedom, or the trace of $\mathcal{F}_1$ is given by a few number of types of curves which are completely described. Moreover we prove that $\mathcal{F}_2$ has a transverse affine structure.

Introduction. We work in the C^∞ category.

Consider a compact connected affine manifold (M, ∇), i.e. ∇ is a connection whose torsion and curvature vanish, of dimension n equipped with a finite family of foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$. We will say that $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are *translation foliations* (T.F.) if for any $1 \leq i < j \leq k$ we have $\nabla_{T\mathcal{F}_i} T\mathcal{F}_j \subset T\mathcal{F}_j$ and $\nabla_{T\mathcal{F}_j} T\mathcal{F}_i \subset T\mathcal{F}_i$, where $T\mathcal{F}_i$ and $T\mathcal{F}_j$ are the vector subbundles of tangent vectors to the leaves of $\mathcal{F}_i$ and $\mathcal{F}_j$ respectively.

This kind of structures appears, in a natural way, when we consider a bilagrangian fibration $\pi : (N, \omega, \omega_1) \to M$ whose fibres are tori $\mathbf{T}^n$. Then M is endowed with two integer affine structure $\mathcal{A}$ and $\mathcal{A}_1$ and a (1, 1) tensor field J which transforms $\mathcal{A}$ on $\mathcal{A}_1$. The eigenspaces of J give rise to a family of translation foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$ which are transverse (often with some singularities, see [1]).

Another example is given by Veronese webs when they are affine. Veronese webs have been introduced by Gelfand and Zakarevich for studying the bihamiltonian systems of odd dimension (see [5]).

Translation foliations on surfaces have been studied by Darboux (see [3]).

Here we will consider the case of *two translation foliations* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *which are transverse*, *i.e.* $TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2$, *such that* $\dim \mathcal{F}_1 = \operatorname{codim} \mathcal{F}_2 = 1$. For the sake of

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simplicity $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *shall be assumed orientable.* This last condition is always satisfied by taking a finite covering if necessary. Let us remark that $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversely orientable as well.

In affine coordinates the property of translation is equivalent to the following one: consider an open set A_1 of a leaf $\mathcal{F}_1$, an open set A_2 of a leaf of $\mathcal{F}_2$ and a point $p \in A_1 \cap A_2$; given $q_1 \in A_1$ and $q_2 \in A_2$ then $A_1 + q_2 - p$ is an open set of the leaf of $\mathcal{F}_1$ passing through q_2, and $A_2 + q_1 - p$ is an open set of the leaf of $\mathcal{F}_2$ passing through q_1 (obviously where it has a meaning). In other words, around p foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are completely determined by the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ passing through this point.

In this paper we show that $\mathcal{F}_2$ always has a transverse affine structure. Besides one of the three following possibilities holds:

1) Foliation $\mathcal{F}_2$ is given by a fibration over S^1. Then $\mathcal{F}_1$ has a great degree of freedom. Nevertheless $\mathcal{F}_1$ is spanned by an $\mathcal{F}_2$-foliate and $\mathcal{F}_2$-parallel vector field.

2) Foliation $\mathcal{F}_2$ has trivial holonomy and all its leaves are dense. Then there exist real numbers $a_0, \dots, a_{k-1}$ and a non-singular vector field X, tangent to $\mathcal{F}_1$, which is both $\mathcal{F}_2$-parallel and $\mathcal{F}_2$-foliate, such that $\nabla^k_X X = \sum_{j=0}^{k-1} a_j \nabla^j_X X$. Moreover $k \leq \operatorname{rank} M + 1$.

Therefore, in affine coordinates, $\mathcal{F}_1$ is described by a curve $\gamma(t)$ which is a solution of the equation $\gamma^{(k+1)} = \sum_{j=1}^{k} a_{j-1} \gamma^{(j)}$.

3) Foliation $\mathcal{F}_2$ has non-trivial holonomy. Then $\mathcal{F}_2$ has a finite number of minimal sets, all of them with non-trivial holonomy, and there exist natural numbers $r_1 = 1 < r_2 < \ldots < r_k$ such that, around each point, we may find affine coordinates on which the polynomial curve $\gamma(t) = (t, t^{r_2}, \dots, t^{r_k}, 0, \dots, 0)$ describes $\mathcal{F}_1$.

Moreover:

If $r_k > k$ then all the non-compact leaves of $\mathcal{F}_2$ have trivial holonomy; on the other hand the compact ones are just the only minimal sets.

If the affine manifold (M, ∇) is complete then $r_j = j$, $j = 1, \dots, k$ and all the leaves of $\mathcal{F}_2$ are dense.

1. Examples.

(a) Consider two imbeddings $f_1 : S^1 \to \mathbf{T}^n$; $f_2 : \mathbf{T}^{n-1} \to \mathbf{T}^n$, where $\mathbf{T}^k$ is the torus of dimension k, and a point $p \in \mathbf{T}^n$. Assume that:

(I) There exists $\alpha_0 \in S^1$ and $\beta_0 \in \mathbf{T}^{n-1}$ such that $p = f_1(\alpha_0) = f_2(\beta_0)$ and $\{f_{1*}\pi_1(S^1, \alpha_0), f_{2*}\pi_1(\mathbf{T}^{n-1}, \beta_0)\}$ spans $\pi_1(\mathbf{T}^n, p)$.

(II) For any $(\alpha, \beta) \in S^1 \times \mathbf{T}^{n-1}$ the subspaces $f_{1*}(T_\alpha S^1)$ and $f_{2*}(T_\beta \mathbf{T}^{n-1})$, after being carried to $0 \in \mathbf{T}^n$ by means of the canonical connection, are transverse.

Then the map $F : (\alpha, \beta) \in S^1 \times \mathbf{T}^{n-1} \to f_1(\alpha) + f_2(\beta) \in \mathbf{T}^n$ is a diffeomorphism and the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, defined by the submersions $\pi_2 \circ F^{-1} : \mathbf{T}^n \to \mathbf{T}^{n-1}$ and $\pi_1 \circ F^{-1} : \mathbf{T}^n \to \mathbf{S}^1$, are T.F.

(b) On $\widetilde{M} = \mathbf{R} \times \mathbf{R}^+$ we consider the equivalence relation $x\mathcal{R}y$ if and only if $y_1 = x_1 + k_1$ and $y_2 = \exp(k_2) x_2$ where $k_1, k_2 \in \mathbf{Z}$, and the foliations $\widetilde{\mathcal{F}}_1$, given by the curves $x_2 \exp(-x_1) =$ constant, and $\widetilde{\mathcal{F}}_2$, associated to vector field $\frac{\partial}{\partial x_1}$. Set $M = \widetilde{M}/\mathcal{R}$. By projecting $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ manifold M is endowed with two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ which are

T.F. with respect to the projected connection.

(c) Let $\{e_1, e_2\}$ be a basis of the Lie algebra of $\mathbf{T}^2$. Now equip $\mathbf{T}^2$ with the affine connection given by $\nabla_{e_1} e_1 = e_2$; $\nabla_{e_i} e_j = 0$ otherwise. Then the foliations $\mathcal{F}_1$, associated to e_1, and $\mathcal{F}_2$, associated to e_2, are T.F. Moreover $\mathcal{F}_1$ is parabolic, i.e. around each point there exist affine coordinates on which the leaf of $\mathcal{F}_1$ passing through this point can be written (t, t^2), and $\mathcal{F}_2$ is geodesic.

In Example (b) $\mathcal{F}_2$ is geodesic as well and the leaves of $\mathcal{F}_1$ are written (t, ae^t) in suitable affine coordinates.

(d) *Hopf structure.* Given positive natural numbers $r_2, \ldots, r_n$, set $X = \frac{\partial}{\partial x_1} + \sum_{j=2}^n x_1^{r_j - 1} \frac{\partial}{\partial x_j}$ and $Y = x_1 \frac{\partial}{\partial x_1} + \sum_{j=2}^n r_j x_j \frac{\partial}{\partial x_j}$; then $[X, Y] = X$. On $\widetilde{M} = \mathbf{R}^n - \{0\}$, $n \geq 2$, the foliations $\widetilde{\mathcal{F}}_1$, associated to X, and $\widetilde{\mathcal{F}}_2$, defined by $dx_1 = 0$, are T.F. with respect to the canonical connection of $\mathbf{R}^n$. On the other hand they are preserved by the flow ϕ_t of Y.

On $\widetilde{M}$ we define the equivalence relation $x\mathcal{R}y$ if and only if $\phi_\ell(x) = y$ for some $\ell \in \mathbf{Z}$. As the vector field Y is both affine and foliate the quotient manifold M, i.e. $S^1 \times S^{n-1}$, is affine and the projected foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F. Foliation $\mathcal{F}_2$ has non-trivial holonomy and, in suitable affine coordinates, each leaf of $\mathcal{F}_1$ is given by the curve $(t, t^{r_2}, \ldots, t^{r_n})$.

Other foliations which are T.F. may be constructed in the same way. For example on $\mathbf{R}^3 - \{0\}$ we can set $Y = 2x_1 \frac{\partial}{\partial x_1} + 4x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$ and consider the foliations $\widetilde{\mathcal{F}}_1$, given by $\frac{\partial}{\partial x_1} + (x_1 + x_3^2) \frac{\partial}{\partial x_2}$, and $\widetilde{\mathcal{F}}_2$ associated to $\frac{\partial}{\partial x_2}$; $-2x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$.

(e) On $\mathbf{T}^3$ consider the affine connection obtained by setting $\nabla_{\frac{\partial}{\partial \alpha_1}} \frac{\partial}{\partial \alpha_1} = \frac{\partial}{\partial \alpha_3}$; $\nabla_{\frac{\partial}{\partial \alpha_2}} \frac{\partial}{\partial \alpha_2} = 2 \frac{\partial}{\partial \alpha_3}$; $\nabla_{\frac{\partial}{\partial \alpha_i}} \frac{\partial}{\partial \alpha_j} = 0$ otherwise, and the translation foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ spanned by $\frac{\partial}{\partial \alpha_1} + 2^{-\frac{1}{2}} \frac{\partial}{\partial \alpha_2}$ and $\frac{\partial}{\partial \alpha_1} - 2^{-\frac{1}{2}} \frac{\partial}{\partial \alpha_2}$; $\frac{\partial}{\partial \alpha_3}$ respectively. Now the affine structure is integer, $\mathcal{F}_1$ is parabolic and all the leaves of $\mathcal{F}_2$ are dense.

(f) Consider a compact connected affine manifold (M, ∇) equipped with two translation foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, such that $\dim \mathcal{F}_1 = \operatorname{codim} \mathcal{F}_2 = 1$ and $T\mathcal{F}_1 \oplus T\mathcal{F}_2 = TM$. Let G be the group of affine diffeomorphisms of (M, ∇) which preserve $\mathcal{F}_1$ and $\mathcal{F}_2$. Consider a second compact connected affine manifold W. Assume that W, $\mathcal{F}_1$ and $\mathcal{F}_2$ are orientable. If $\widetilde{W}$ is the universal covering of W then $\widetilde{W} \times M$ can be endowed with the translation foliations $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ given by $T\widetilde{\mathcal{F}}_1 = \{0\} \times T\mathcal{F}_1$ and $T\widetilde{\mathcal{F}}_2 = TW \times T\mathcal{F}_2$. Therefore, by suspending each morphism from $\pi_1(W)$ to G, we obtain a new example of translation foliations.

If $(M, \mathcal{F}_1, \mathcal{F}_2)$ is as in Example (d), then Y gives rise to a vector field Y', on M, whose flow ϕ'_t is included in G. Now by taking $W = S^1$ and ϕ'_b, where $b \notin \mathbf{Q}$, as image of a generator of $\pi_1(S^1)$, one constructs an example of translation foliations where the codimension one foliation only has one ($\dim M > 2$) or two ($\dim M = 2$) compact leaves. The other ones are locally dense (note that all the leaves of $\mathcal{F}_2$ were proper; obviously the number of compact leaves of this last foliation is the same as before).

(g) Given $B \in SL(\mathbf{Z}, 2)$ let $\varphi_B : \mathbf{T}^2 \to \mathbf{T}^2$ be the associated isomorphism. Consider an element $A \in SL(\mathbf{Z}, 2)$ with two distinct positive real eigenvalues λ_1, λ_2. Let $\{d_1, d_2\}$ be a basis of the Lie algebra of $\mathbf{T}^2$ such that $(\varphi_A)_* d_i = \lambda_i d_i$, $i = 1, 2$. Now endow

$\mathbf{T}^4 = \mathbf{T}^2 \times \mathbf{T}^2$ with the affine structure given by $\nabla_{e_1} e_1 = e_3$; $\nabla_{e_i} e_j = 0$ otherwise, where $e_1 = (d_1, 0)$, $e_2 = (d_2, 0)$, $e_3 = (0, d_1)$, $e_4 = (0, d_2)$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ the translation foliations spanned by e_1 and $\{e_2, e_3, e_4\}$ respectively.

By suspending the diffeomorphism $\phi = (\varphi_A, \varphi_{A^2})$ as in Example (f) ($W = S^1$) one constructs two translation foliations $\mathcal{F}'_1$ and $\mathcal{F}'_2$, on a compact affine 5-manifold M', the first one parabolic and the second one with non-trivial holonomy and dense leaves. Note that the affine manifold M' is complete (see Corollary 5.1).

2. The polynomial points of $\mathcal{F}_1$. If each leaf of $\mathcal{F}_2$ is compact and its holonomy is trivial, i.e. if $\mathcal{F}_2$ is given by a fibration over S^1, the leaves of $\mathcal{F}_1$ have a great degree of freedom as Example (a) shows. Nevertheless they are the orbits of a non-singular vector field X which is both $\mathcal{F}_2$-foliate and $\mathcal{F}_2$-parallel. Indeed, as $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F. all $\mathcal{F}_2$-foliate vector field is parallel. Conversely if a non-singular $\mathcal{F}_2$-foliate vector field X, tangent to $\mathcal{F}_1$, is $\mathcal{F}_2$-parallel then $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F. (even if the leaves of $\mathcal{F}_2$ are dense). Obviously $\mathcal{F}_2$ has transverse affine structures.

From now on we will suppose, if necessary, that $\mathcal{F}_2$ is not given by a fibration over S^1.

We will say that a non-singular curve $\gamma : I \to M$ *describes* $\mathcal{F}_1$ *at* p if $p \in \gamma(I)$ and γ lies on the leaf of $\mathcal{F}_1$ passing through p. A point $p \in M$ will be called *polynomial* (for $\mathcal{F}_1$) if there exists a curve γ, describing $\mathcal{F}_1$ at p, which is polynomial in affine coordinates. The set P of all polynomial points is open and saturated for $\mathcal{F}_2$.

THEOREM 1. *Let H be a leaf of $\mathcal{F}_2$ with non-trivial holonomy. Then the holonomy of H is linear and $H \subset P$.*

COROLLARY 1.1. *Assume that $\mathcal{F}_2$ has non-trivial holonomy. Then*:

(I) *$\mathcal{F}_2$ has a finite number of minimal sets, all of them with non-trivial holonomy.*

(II) *$P = M$.*

First we deduce Corollary 1.1 from Theorem 1. The number of exceptional minimal sets of $\mathcal{F}_2$ is finite and the union of all its compact leaves is a closed set (for $\operatorname{codim} \mathcal{F}_2 = 1$). By Theorem 1 the holonomy of each leaf is linear so there are a finite number of compact leaves, all of them with non-trivial holonomy. By Sacksteder's theorem each exceptional minimal set contains a leaf with non-trivial holonomy (see [8]). This proves (I).

On the other hand by Theorem 1 again, P contains every leaf with non-trivial holonomy, so it contains all the minimal sets. Therefore $P = M$; otherwise we could find a minimal set on ∂P. This proves (II).

Now for proving Theorem 1 we will study the holonomy of $\mathcal{F}_2$ referred to $\mathcal{F}_1$.

Consider a point $p \in M$. Let H_i, $i = 1, 2$, be the leaf of $\mathcal{F}_i$ passing through p and let τ be a loop at p on H_2. In affine coordinates, around p, the holonomy map φ_τ associated to τ, referred to the transversal H_1, is given by the restriction of an affine transformation A of $\mathbf{R}^n$ respecting the orientation and locally sending H_1 on H_1; moreover $A(p) = p$.

Indeed, first consider the case of an arc τ' on H_2 contained in a convex affine coordinate domain. As $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F., its holonomy map $\varphi_{\tau'}$, referred to the leaves of $\mathcal{F}_1$ passing through the ends of τ', is the restriction of a translation of $\mathbf{R}^n$. Now divide τ into small pieces contained each of them in a convex affine coordinate domain.

Consider an affine coordinate system with p as origin, i.e. $p \equiv 0$. In this case A is a linear transformation of $\mathbf{R}^n$. On the other hand, if γ describes $\mathcal{F}_1$ at p and $\gamma(0) = p$ then there exist two open intervals I' and I'', containing zero, and a diffeomorphism $\varphi : I' \to I''$ such that $\varphi(0) = 0$ and $A \circ \gamma = \gamma \circ \varphi$ on I'.

Deriving with respect to t yields $A(\gamma'(0)) = \varphi'(0) \cdot \gamma'(0)$, i.e. $\gamma'(0)$ is an eigenvector of A with eigenvalue $\varphi'(0) > 0$ since $\mathcal{F}_2$ is transversely orientable.

PROPOSITION 1. *If* $\varphi_\tau \neq \mathrm{Id}$ *then there exist an eigenspace* W *of* $\mathbf{R}^n$, *a basis* $\{e_1, \ldots, e_k\}$ *of it, natural numbers* $1 = r_1 < r_2 < \ldots < r_k$ *and a positive real* $\lambda \neq 1$ *such that*:

(I) $Ae_j = \lambda^{r_j} e_j$, $j = 1, \ldots k$.

(II) *The curve* $\gamma(t) = \sum_{j=1}^{k} t^{r_j} e_j$ *describes* $\mathcal{F}_1$ *at* p *and* $p = \gamma(0)$.

Moreover $\varphi_\tau(\gamma(t)) = \gamma(\lambda t)$, *i.e. the map* φ_τ *is linear.*

Remark. The isomorphism A regarded as a linear transformation of T_pM belongs to the holonomy group, at p, of the affine connection.

On the other hand, if $\widetilde{\tau}$ is another loop at p on H_2 then $\varphi_{\widetilde{\tau}}(\gamma(t)) = \gamma(\widetilde{\lambda} t)$ because $\varphi_{\widetilde{\tau}}(\gamma(t)) = \widetilde{A}(\gamma(t))$ for some $\widetilde{A} \in GL(\mathbf{R}^n)$. Therefore the holonomy of each leaf of $\mathcal{F}_2$ is linear.

It is clear that Theorem 1 follows from Proposition 1. For proving this last result we shall examine all the possible cases.

First case: The real Jordan canonical form of A only has one block. Then there exists a basis $\{e_1, \ldots, e_n\}$ of $\mathbf{R}^n$ such that $Ae_j = \lambda e_j + e_{j-1}$, $j = 2, \ldots n$, and $Ae_1 = \lambda e_1$. Naturally $\lambda = \varphi'(0)$. *It will be shown that near the origin* γ *lies on the line* $\mathbf{R}\{e_1\}$. We will do it for $t \geq 0$; the other side is analogous.

A point $t_0 \in I'$ is called *stationary* if $\varphi(t_0) = t_0$. When $\lambda \neq 1$ the only stationary point close to $0 \in I'$ is the zero itself (mean value theorem).

LEMMA 1. *If* $t_0 > 0$ *is stationary then* $\gamma([0, t_0]) \subset \mathbf{R}\{e_1\}$.

Proof. The map $\varphi : [0, t_0] \to [0, t_0]$ is a diffeomorphism and $A^k \circ \gamma = \gamma \circ \varphi^k$ where $\varphi^k = \varphi \circ \ldots \circ \varphi$. Set $\gamma(t) = \sum_{j=1}^{n} \gamma_j(t) e_j$. If $\gamma_n(t) \neq 0$ for some $t \in [0, t_0]$ then $\lambda = 1$ since otherwise $A^k(\gamma(t)) = \sum_{j=1}^{n-1} f_{jk}(t) e_j + \lambda^k \gamma_n(t) e_n$ tends to infinity (if $\lambda < 1$ take negative k) and the set $\gamma_n([0, t_0])$ is not compact.

But if $\lambda = 1$ the $(n-1)$-th coordinate of $A^k(\gamma(t))$ equals $\gamma_{n-1}(t) + k\gamma_n(t)$ which again tends to infinity unless $\gamma_n(t) = 0$. In other words $\gamma_n = 0$. Now the same reasoning shows that $\gamma_{n-1} = \ldots = \gamma_2 = 0$. ∎

By replacing A and φ with A^{-1} and φ^{-1} respectively if necessary, Lemma 1 allows us to suppose $\varphi(t) < t$ for any $t > 0$. Therefore $0 < \lambda \leq 1$ and $\lim_{k\to\infty}\{\varphi^k(t)\} = 0$.

1.a) *First assume* $\lambda = 1$. The $(n-1)$-th coordinate of $A^k(\gamma(t))$ equals $\gamma_{n-1}(t) + k\gamma_n(t)$ which tends to infinity etc... In short $\gamma_2 = \ldots = \gamma_n = 0$.

1.b) *Now assume* $0 < \lambda < 1$.

LEMMA 2. *Let* g *be a function defined around zero such that* $g(\varphi(t)) = \lambda g(t)$. *If* $g(0) = g'(0) = 0$ *then* $g(t) = 0$ *for any* $t > 0$ *close to zero.*

P r o o f. There exist $t_0 > 0$, a constant $B > 0$ and a positive integer number k such that $|\lambda^{-k}(\varphi^k)'(t)| \le B$ for all $t \in [0, t_0]$. Indeed, as $\varphi'(0) = \lambda < 1$ we can find $0 < \lambda' < 1$ and $t_0 > 0$ such that $\varphi(t) \le \lambda' t$ on $[0, t_0]$. Therefore $\varphi^k(t) \le (\lambda')^k t$.

Set $\mu = |\max\{\varphi''(t) \mid t \in [0, t_0]\}|$. Then

$$|\varphi'(\varphi^{k-1}(t))| \le |\varphi'(0)| + \mu\varphi^{k-1}(t) \le \lambda + \mu \cdot (\lambda')^{k-1} t$$

Hence

$$\begin{aligned}|\lambda^{-k}(\varphi^k)'(t)| &= |\lambda^{-1}\varphi'(\varphi^{k-1}(t))| \cdot |\lambda^{1-k}(\varphi^{k-1})'(t)| \\ &\le |1 + \mu\lambda^{-1}(\lambda')^{k-1} t| \cdot |\lambda^{1-k}(\varphi^{k-1})'(t)| \le \ldots \\ &\le \prod_{j=1}^{k-1}(1 + \mu\lambda^{-1}(\lambda')^j t)|\lambda^{-1}\varphi'(t)| \le \exp(\mu\lambda^{-1}\lambda'(1-\lambda')^{-1} t)|\lambda^{-1}\varphi'(t)| \le B.\end{aligned}$$

On the other hand $g(t) = \lambda^{-k} g(\varphi^k(t))$ when $t \in [0, t_0]$; so

$$|g'(t)| \le |\lambda^{-k}(\varphi^k)'(t)| \cdot |g'(\varphi^k(t))| \le B|g'(\varphi^k(t))| \to 0,$$

because $\{\varphi^k(t)\} \to 0$ and $g'(0) = 0$. Therefore $g = 0$ on $[0, t_0]$, since $g(0) = 0$. ■

Consider the curve γ again. If $n \ge 2$ then $\gamma_2'(0) = \ldots = \gamma_n'(0) = 0$ as $\gamma'(0)$ is an eigenvector. Moreover $\gamma_n(\varphi(t)) = \lambda\gamma_n(t)$ for $A(\gamma(t)) = \gamma(\varphi(t))$, whence $\gamma_n = 0$. But then $\gamma_{n-1}(\varphi(t)) = \lambda\gamma_{n-1}(t)$ etc... To sum up $\gamma_2 = \ldots = \gamma_n = 0$.

Finally by changing the parametrization of the curve if necessary, we may suppose $\gamma(t) = te_1$.

Second case: The real Jordan canonical form of A has two or more blocks. Consider a decomposition $V = \bigoplus_{\ell=1}^m V^\ell$ where each V^ℓ is an eigenspace and each linear map $A_{|V^\ell}$ has one block only.

Set $\gamma = (\gamma^1, \ldots, \gamma^m)$. Then every component $(\gamma^\ell)'(0)$ is an eigenvector of $A_{|V^\ell}$ and at least one of them does not vanish, for example $(\gamma^1)'(0)$. The first case applied to γ^1 and $A_{|V^1}$, allows us to find a basis $\{e_1^1, \ldots, e_{n_1}^1\}$ of V^1 and a parametrization of γ such that $Ae_j^1 = \lambda_1 e_j^1 + e_{j-1}^1$, $j = 2, \ldots, n_1$, $Ae_1^1 = \lambda_1 e_1^1$, $\gamma^1(t) = te_1^1$ and $\varphi(t) = \lambda_1 t$.

If $\lambda_1 = 1$ then $\varphi_\tau = \mathrm{Id}$. *Therefore assume* $0 < \lambda_1 < 1$ (if $\lambda_1 > 1$ take A^{-1} and φ^{-1} instead of A and φ).

First consider the subspaces V^j such that $A_{|V^j}$ has a real eigenvalue. For the sake of simplicity suppose that it is the case of V^2. Choose a basis $\{e_1^2, \ldots, e_{n_2}^2\}$ such that $Ae_j^2 = \lambda_2 e_j^2 + e_{j-1}^2$, $j = 2, \ldots n_2$, and $Ae_1^2 = \lambda_2 e_1^2$. Set $\gamma^2 = \sum_{j=1}^{n_2} h_{n_2-j} e_j^2$. Then $\lambda_2 h_0(t) = h_0(\lambda_1 t)$ as $A \circ \gamma = \gamma \circ \varphi$. So $h_0(t) = \lambda_2^{-r} h_0(\lambda_1^r t)$ and $h_0^{(k)}(t) = \lambda_1^{kr}\lambda_2^{-r} h_0^{(k)}(\lambda_1^r t)$.

From some positive integer number on $|\lambda_1^k \lambda_2^{-1}| < 1$, and $h_0^{(k)}(t) = 0$ because $\{\lambda_1^r t\} \to 0$. In other words, h_0 is a polynomial.

On the other hand $h_0^{(s)}(0) = \lambda_1^s \lambda_2^{-1} h_0^{(s)}(0)$. If λ_2 is not a positive power of λ_1 then $h_0 = 0$. Doing the same with h_1, then with h_2 and so on, yields $h_0 = h_1 = \ldots = h_{n_2-1} = 0$; i.e. $\gamma^2 = 0$.

If $\lambda_2 = \lambda_1^k$, where $k \in \mathbf{N} - \{0\}$, then $h_0(t) = at^k$. Moreover $h_1(\lambda_1 t) = \lambda_2 h_1(t) + h_0(t)$ since $A \circ \gamma = \gamma \circ \varphi$. Hence $h_1^{(k+1)}(t) = \lambda_1^{k+1}\lambda_2^{-1} h_1^{(k+1)}(\lambda_1 t) = \lambda_1 h_1^{(k+1)}(\lambda_1 t)$ and $h_1^{(k+1)}(t) = \lambda_1^r h_1^{(k+1)}(\lambda_1^r t)$. Therefore $h_1^{(k+1)} = 0$ because $\{\lambda_1^r\} \to 0$.

In a word $h_1(t) = \sum_{j=0}^{k} b_j t^j$. Now the relation $h_1(\lambda_1 t) = \lambda_2 h_1(t) + h_0(t)$ implies that $h_0 = 0$ and $h_1(t) = b_k t^k$.

By a similar argument $h_1 = \ldots = h_{n_2-2} = 0$ and $\gamma^2(t) = ct^k e_1^2$.

For the other blocks with real eigenvalues we do the same. If $A_{|V^\ell}$ has no real eigenvalue by complexifying it we obtain two blocks with non-real eigenvalues λ' and $\bar{\lambda}'$ respectively. Obviously λ' and $\bar{\lambda}'$ are not powers of λ_1 therefore $\gamma^\ell = 0$.

By rearranging according to powers of t we obtain a family $\{e_1, \ldots, e_k\}$ of eigenvectors, with eigenvalues λ^{r_j}, $j = 1, \ldots k$, where $\lambda = \lambda_1$ and $1 = r_1 < r_2 < \ldots < r_k$, such that $\gamma(t) = \sum_{j=1}^{k} t^{r_j} e_j$. That completes the proof of Proposition 1. ∎

3. The degree of flatness of $\mathcal{F}_1$. Given a curve γ on M, by definition $\gamma^{(1)}$ is its velocity, $\gamma^{(2)}$ its acceleration, i.e. the covariant derivative of $\gamma^{(1)}$ along γ, $\gamma^{(3)}$ the covariant derivative of $\gamma^{(2)}$ along γ etc… In affine coordinates $\gamma^{(k)}$ is just the k-th derivative of γ with respect to the parameter. The maximum number of linearly independent successive derivatives $\gamma^{(1)}(t_0)$, $\gamma^{(2)}(t_0), \ldots, \gamma^{(k)}(t_0)$ at a point $p = \gamma(t_0)$ does not depend on the parametrization. When γ describes $\mathcal{F}_1$ at p we denote this number by $s(p)$. That defines a locally increasing function $s : M \to \mathbf{N}$ which is constant along the leaves of $\mathcal{F}_2$ since $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F.

LEMMA 3. *Suppose that $s : M \to \mathbf{N}$ is constant and set $k = s(M)$. Then a curve γ describing $\mathcal{F}_1$, at a point, locally lies on a well defined affine k-plane*; *i.e. if we identify to each other an open set of M and one of $\mathbf{R}^n$ by means of an affine system of coordinates, then γ is locally contained just in an affine k-plane of $\mathbf{R}^n$.*

Continue to suppose $k = s(M)$. Let $\mathcal{G}$ be the vector subbundle of TM whose fibre at p is the subspace spanned by $\gamma^{(1)}(t_0)$, $\gamma^{(2)}(t_0)$, …, $\gamma^{(k)}(t_0)$ when γ describes $\mathcal{F}_1$ at $p = \gamma(t_0)$. By Lemma 3, as $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F., around each point there exist affine coordinates $(x_1, \ldots, x_n)$ on which $\mathcal{G}$ is defined by $dx_{k+1} = \ldots = dx_n = 0$. Therefore $\mathcal{G}$ is parallel and the foliation associated to it contains $\mathcal{F}_1$. Besides $dx_1 \wedge \ldots \wedge dx_k$ locally defines a volume form ω, on $\mathcal{G}$, parallel as well. This form ω does not depend on the choice of the affine coordinate system up to a constant factor.

Set $f(t) = \omega(\gamma^{(1)}(t), \ldots, \gamma^{(k)}(t))$ and $\tau(u) = \gamma(\lambda(u))$. Then $\omega(\tau^{(1)}(u), \ldots, \tau^{(k)}(u)) = (\lambda'(u))^\ell f(\lambda(u))$ where $\ell = k(k+1)/2$. Therefore for each $p \in M$ there exists a curve that we call γ again, describing $\mathcal{F}_1$ at this point, such that $\omega(\gamma^{(1)}(t), \ldots, \gamma^{(k)}(t))$ is constant. Obviously this property does not depend on the choice of ω. Moreover if $\omega(\tau^{(1)}(u), \ldots, \tau^{(k)}(u))$ is constant as well, where $\tau(u) = \gamma(\lambda(u))$, then $(\lambda'(u))^\ell$ is constant and λ has to be an affine function of u. In other words the relation between two of these parametrizations is given by an affine transformation of $\mathbf{R}$. Therefore $\mathcal{F}_1$ is equipped with an affine structure.

By construction this affine structure is preserved by translation along the leaves of $\mathcal{F}_2$ because ω is parallel. Consequently, as $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F., we have:

THEOREM 2. *If $s : M \to \mathbf{N}$ is constant then $\mathcal{F}_2$ has a transverse affine structure.*

Now suppose that each leaf of $\mathcal{F}_2$ is dense and has trivial holonomy. Then $s(M) = k$ for some $k \in \mathbf{N} - \{0\}$ and $\mathcal{F}_2$ has a transverse affine structure. By Seke's result (see

Theorem 8 of [9]), $\mathcal{F}_2$ is defined by a non-singular closed form α. Let X be the vector field tangent to $\mathcal{F}_1$ such that $\alpha(X) = 1$. Obviously X is $\mathcal{F}_2$-foliate.

LEMMA 4. *Let X be an $\mathcal{F}_2$-foliate vector field tangent to $\mathcal{F}_1$; then X is parallel along $\mathcal{F}_2$. Moreover, if Y is parallel along $\mathcal{F}_2$ so is $\nabla_X Y$.*

By Lemma 4 each vector field $X_j = \nabla^j_X X$ is $\mathcal{F}_2$-parallel; so $\nabla^k_X X = \sum_{j=0}^{k-1} a_j X_j$, where $a_j \in \mathbf{R}$, $j = 0, \ldots, k-1$, since $s(M) = k$.

Set $TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2$ and let φ be the projection onto $T\mathcal{F}_2$. Given vector fields Z_1 and Z_2 tangent to $\mathcal{F}_2$ set, by definition, $\nabla'_{Z_1} Z_2 = \varphi(\nabla_{Z_1} Z_2)$. As $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F. it is easily seen that ∇' is a connection on the leaves of $\mathcal{F}_2$ whose torsion and curvature vanish.

On the other hand, as each X_j is $\mathcal{F}_2$-parallel and $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F., it yields $\nabla'\varphi(X_j) = 0$; so $[\varphi(X_j), \varphi(X_\ell)] = 0$. But $\varphi(X_1), \ldots, \varphi(X_{k-1})$ are linearly independent because $s(M) = k$. Therefore $k \leq \operatorname{rank} M + 1$ (we recall that *the rank of a compact manifold* is the maximum number of commuting vector fields linearly independent everywhere). For example if $\mathcal{F}_1$ is as much twisted as possible, i.e. $s(M) = n$, then $\operatorname{rank} M \geq n-1$ and M is a bundle over S^1 with fibre $\mathbf{T}^{n-1}$ (see [2]).

In short:

THEOREM 3. *If every leaf of $\mathcal{F}_2$ is dense and has trivial holonomy then there exist $k \in \mathbf{N} - \{0\}$, $a_0, \ldots, a_{k-1} \in \mathbf{R}$ and a non-singular vector field X such that:*

(a) $s(M) = k \leq \operatorname{rank} M + 1$.

(b) *X is tangent to $\mathcal{F}_1$, $\mathcal{F}_2$-parallel and $\mathcal{F}_2$-foliate; therefore $\mathcal{F}_2$ is given by a non-singular closed 1-form.*

(c) $\nabla^k_X X = \sum_{j=0}^{k-1} a_j \nabla^j_X X$.

Moreover X is unique up to a constant factor.

Let us remark that in affine coordinates $\mathcal{F}_1$ is described by a solution of the equation

$$\gamma^{(k+1)} = \sum_{j=1}^{k} a_{j-1} \gamma^{(j)},$$

therefore its shape is completely known.

EXAMPLE. Consider the torus $\mathbf{T}^n$ equipped with the canonical affine structure. Then $\mathcal{F}_2$ has trivial holonomy (see the remark following Proposition 1).

As the slope of $\mathcal{F}_1$ along each leaf of $\mathcal{F}_2$ is constant, if all the leaves of this last foliation are dense then the vector field given by Theorem 3 is geodesic, i.e. $X = \sum_{j=1}^n b_j \frac{\partial}{\partial \theta_j}$ where $b_1, \ldots, b_n \in \mathbf{R}$. On the other hand $\mathcal{F}_2$ is defined by a closed 1-form $\alpha = \sum_{j=1}^n b'_j d\theta_j + \alpha'$, where $b'_1, \ldots, b'_n \in \mathbf{R}$ with $\sum_{j=1}^n b_j b'_j = 1$ and α' is the pull-back of a closed 1-form defined on the quotient of $\mathbf{T}^n$ by the closures of the orbits of X.

If all the leaves of $\mathcal{F}_2$ are compact, consider an embedded curve $\tau : S^1 \to M$ transverse to $\mathcal{F}_2$ and cutting each of its leaves once. Then $\mathcal{F}_1$ is describes by its value on τ. Indeed given a vector field X' along τ, transverse to $\mathcal{F}_2$, the parallel translation along the leaves of this foliation gives rise to a vector field X on $\mathbf{T}^n$, which defines a foliation $\mathcal{F}_1$. Suppose that X is never tangent to $\mathcal{F}_2$; then $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F. if and only if X is $\mathcal{F}_2$-foliate.

4. The non-trivial holonomy case. In this section we assume that $\mathcal{F}_2$ has at least a leaf with non-trivial holonomy. Given natural numbers $j_1 = 1 < j_2 < \ldots < j_\ell$ let $M_{1j_2\ldots j_\ell}$ be the set of all the points $p \in M$ for which the following property holds: there exist $t_0 \in \mathbf{R}$ and an affine coordinate system, defined around p, such that the curve $\gamma(t) = (t, t^{j_2}, \ldots, t^{j_\ell}, 0, \ldots, 0)$ describes $\mathcal{F}_1$ at $\gamma(t_0) = p$. By construction $M_{1j_2\ldots j_\ell}$ is an $\mathcal{F}_2$-saturated open set. Besides $\partial M_{1j_2\ldots j_\ell} = \emptyset$, i.e. either $M_{1j_2\ldots j_\ell} = \emptyset$ or $M_{1j_2\ldots j_\ell} = M$.

Indeed, if $\partial M_{1j_2\ldots j_\ell} \neq \emptyset$ it contains a minimal set. Therefore, by reasoning as before, the boundary of $M_{1j_2\ldots j_\ell}$ contains a leaf H of $\mathcal{F}_2$ with non-trivial holonomy. By Proposition 1 there exist natural numbers $r_1 = 1 < r_2 < \ldots < r_k$ such that $H \subset M_{1r_2\ldots r_k}$.

Lemma 5. *Consider, on $\mathbf{R}^n$, the curves $\gamma(t) = (t, t^{j_2}, \ldots, t^{j_\ell}, 0, \ldots, 0)$ and $\lambda(t) = (t, t^{r_2}, \ldots, t^{r_k}, 0, \ldots, 0)$ where $1 < j_2 < \ldots < j_\ell$ and $1 < r_2 < \ldots < r_k$. Suppose that there exists an affine transformation $A : \mathbf{R}^n \to \mathbf{R}^n$ and a non-empty open interval I such that $A(\gamma(I)) \subset \lambda(\mathbf{R})$. Then $\ell = k$ and $j_i = k_i$, $i = 2, \ldots, \ell$.*

Now Lemma 5 says us that $H \subset M_{1j_2\ldots j_\ell}$ so $H \cap \partial M_{1j_2\ldots j_\ell} = \emptyset$, *contradiction.*

We have assumed that $\mathcal{F}_2$ has non-trivial holonomy; therefore by Proposition 1 there exist natural numbers $r_1 = 1 < r_2 < \ldots < r_k$, a point $p_0 \in M_{1r_2\ldots r_k}$ and an affine coordinate system, defined around this one, such that the curve $\gamma(t) = (t, t^{r_2}, \ldots, t^{r_k}, 0, \ldots, 0)$ describes $\mathcal{F}_1$ at $p_0 = \gamma(0)$. Moreover $M = M_{1r_2\ldots r_k}$.

First case: Function $s : M \to \mathbf{N}$ is constant. Then $s(M) = k$ and $r_i = i$, $i = 2, \ldots, k$ since $p_0 = \gamma(0)$ for some p_0. It is easily seen that given $a \in \mathbf{R}^+$ and $b \in \mathbf{R}$ there exists an affine transformation A of $\mathbf{R}^n$, preserving the orientation such that $A(\gamma(t)) = \gamma(at + b)$, i.e. all the points of γ are affinely equivalent on $\mathbf{R}^n$. Therefore for each $p \in M$ we can find affine coordinates, defined around this point, on which the curve $\gamma(t) = (t, t^2, \ldots, t^k, 0, \ldots, 0)$ describes $\mathcal{F}_1$ at $\gamma(0) = p$.

In Example (d) and the second part of Example (f) (set $r_j = j$), $\mathcal{F}_2$ has one minimal set if $n \geq 3$ and two minimal sets if $n = 2$; they are compact leaves. In Example (g) the only minimal set of $\mathcal{F}_2$ is M itself.

On the other hand, as $\mathcal{F}_2$ has a transverse affine structure (Theorem 2), if the fundamental group of M is Abelian then the minimal sets of $\mathcal{F}_2$ are just the compact leaves (see [4] and [9]).

Second case: Function $s : M \to \mathbf{N}$ is not constant. Consider the curve $\gamma(t) = (t, t^{r_2}, \ldots, t^{r_k}, 0, \ldots, 0)$ and for each $a \in \mathbf{R} - \{0\}$ the affine transformation of $\mathbf{R}^n$: $A_a(x) = (ax_1, a^{r_2}x_2, \ldots, a^{r_k}x_k, x_{k+1}, \ldots, x_n)$. Then $A_a(\gamma(t)) = \gamma(at)$; so all the points of γ, unless the origin, are affinely equivalent and they have the same number of linearly independent successive derivatives, which equals k. Set $M' = s^{-1}(k)$. Then:

1) $k < r_k$, and $s(p) < k$ if $p = \gamma(0)$. Moreover $s(M - M')$ is the first natural number i such that $i + 1 < r_{i+1}$.

2) $M - M'$ is transversely finite; therefore it is the union of a finite number of compact leaves of $\mathcal{F}_2$, each of them with non-trivial holonomy (Corollary 1.1).

3) The leaves of $\mathcal{F}_2$ contained in M' have trivial holonomy (because $s(p) < k$ if $p = \gamma(0)$) and M' does not contain any minimal set.

Besides around every point $p \in M$ we can find affine coordinates on which the curve $\gamma(t) = (t, t^{r_2}, \dots, t^{r_k}, 0, \dots, 0)$ describes $\mathcal{F}_1$ at p, where $p = \gamma(1)$ if $p \in M'$ and $p = \gamma(0)$ if $p \notin M'$.

Finally remark that the affine transformations of $\mathbf{R}^n$ sending a non-empty open interval of the curve $\gamma(t) = (t, t^{r_2}, \dots, t^{r_k}, 0, \dots, 0)$ on a subset of $\gamma(\mathbf{R})$ are the transformations A_a defined before. As $A_a(\gamma(t)) = \gamma(at)$ the parameter t gives rise to an affine structure on each leaf of $\mathcal{F}_1$, and a transverse affine structure of $\mathcal{F}_2$ because $\mathcal{F}_1$ and $\mathcal{F}_2$ are T.F. Moreover the holonomy of this affine transverse structure is a group of homotheties with the same center. This implies that on each connected component of M' either all the leaves of $\mathcal{F}_2$ are locally dense or all of them are proper. Even more this proves, in another way, that $\mathcal{F}_2$ has *almost no holonomy*, i.e. only the compact leaves have non-trivial holonomy (Theorem 7 of [9], see [4] as well).

In short:

THEOREM 4. *If $\mathcal{F}_2$ has non-trivial holonomy and function $s : M \to \mathbf{N}$ is not constant then*:

(I) *$s(M) = \{\ell_1, \ell_2\}$ where $\ell_1 < \ell_2$.*

(II) *$\mathcal{F}_2$ has a transverse affine structure whose holonomy is group of homotheties of $\mathbf{R}$ with a common center.*

(III) *$\mathcal{F}_2$ has almost no holonomy*; *moreover $s^{-1}(\ell_1)$ is the union of all compact leaves of this foliation* (*a finite number*).

Although there exist codimension one foliations with a transverse affine structure and exceptional minimal sets (see [6]), I do not know any example of translation foliations where $\mathcal{F}_2$ has an exceptional minimal set. Obviously in such a case $s : M \to \mathbf{N}$ has to be constant.

EXAMPLE. Suppose that $M = S^1 \times S^m$, $m \geq 2$. Then $\mathcal{F}_2$ has non-trivial holonomy and its minimal sets are compact leaves.

Indeed, by Theorem 3 if $\mathcal{F}_2$ has trivial holonomy, as $H^1(S^1 \times S^m, \mathbf{R}) = \mathbf{R}$, then $\mathcal{F}_2$ is defined by a fibration $\pi : S^1 \times S^m \to S^1$. Therefore each leaf of $\mathcal{F}_2$ is simply connected because the homotopy sequence. On the other hand (see Section 3) every leaf has an affine structure and, by parallel displacement, we may construct a parallel non-singular 1-form α on it. Obviously $d\alpha = 0$. So $[\alpha] \neq 0$ and the leaf is not simply connected, *contradiction.*

A transverse affine structure $\mathcal{S}$ of $\mathcal{F}_2$ gives rise, through the local $\mathcal{F}_2$-foliate vector fields which are tangent to $\mathcal{F}_1$, to an affine structure on each leaf of $\mathcal{F}_1$. When all these structures are complete we will say that $\mathcal{S}$ *is complete* (with respect to $\mathcal{F}_1$).

Let us call $\nabla(\mathcal{S})$ the connection on $\mathcal{F}_1$ associated to $\mathcal{S}$.

For example in the case of Theorem 3 the transverse affine structure $\mathcal{S}_1$ associated to X is complete because $\nabla(\mathcal{S}_1)_X X = 0$. On the other hand the transverse affine structure $\mathcal{S}_2$ built up from the property that $s(M) = k$ is complete iff $a_{k-1} = 0$, i.e. if and only if $\mathcal{S}_1 = \mathcal{S}_2$.

THEOREM 5. *Assume that $\mathcal{F}_2$ has a complete transverse affine structure $\mathcal{S}$. If the holonomy of $\mathcal{F}_2$ is not trivial then all its leaves are dense* (*therefore the function $s : M \to \mathbf{N}$ is constant*).

Proof. Let $\widetilde{M}$ be the universal covering of M. Then the structure $(M, \mathcal{F}_1, \mathcal{F}_2)$ can be seen as the quotient of a structure $(\widetilde{M}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ of the same kind (i.e. $\widetilde{M}$ is an affine manifold and $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ are T.F.) by the action of a group G, isomorphic to the fundamental group of M, which operates properly discontinuously. Denote by $\pi : \widetilde{M} \to M$ the canonical projection. Since $\mathcal{F}_2$ is transversely orientable and has a transverse affine structure, there exist an $\widetilde{\mathcal{F}}_2$-basic submersion $\varphi : \widetilde{M} \to \mathbf{R}$ and a morphism $\rho : G \to \mathrm{Aff}^+(\mathbf{R})$ such that $\varphi(g \cdot p) = \rho(g) \cdot \varphi(p)$ for any $g \in G$ and $p \in \widetilde{M}$ (see [4]).

Let X be the vector field on $\widetilde{M}$, tangent to $\mathcal{F}_1$, such that $\varphi_*(X) = \partial/\partial t$. If $\gamma(t)$ is an integral curve of X then $\pi(\gamma(t))$ is a geodesic of $\nabla(\mathcal{S})$. Therefore X is complete; consequently the fibration $\varphi : \widetilde{M} \to \mathbf{R}$ is a product and each $\varphi^{-1}(t)$ is a leaf of $\widetilde{\mathcal{F}}_2$.

We can suppose, without loss of generality, that the leaf $\pi(\varphi^{-1}(0))$ has non-trivial holonomy: so there exists $0 < a < 1$ such that the map $t \to at$ belongs to $\rho(G)$.

First assume that $\rho(G)$ contains some translation. Then for each $t_0 \in \mathbf{R}$ the set $\rho(G)(t_0)$ is dense. As $\varphi : \widetilde{M} \to \mathbf{R}$ is a product, $\pi(\varphi^{-1}(\rho(G)(t_0)))$ is a dense leaf of $\mathcal{F}_2$. But all the leaves of this foliation can be written in this way, so they are dense.

If $\rho(G)$ does not contain any translation then it is a group of homotheties with center $0 \in \mathbf{R}$. Moreover $\mathcal{F}_2$ has only a minimal set: the compact leaf $\pi(\varphi^{-1}(0))$, and every leaf of $\mathcal{F}_1$ intersects $\pi(\varphi^{-1}(0))$ just once. Let us choose the orientation of $\mathcal{F}_1$ whose pull-back by π equals that given by X on $\widetilde{\mathcal{F}}_1$. If L is a leaf of $\mathcal{F}_1$ then its α-limit is contained in the closed set $\pi(\varphi^{-1}([0, +\infty)))$. Therefore no leaf of this α-limit cuts $\pi(\varphi^{-1}(0))$, *contradiction.* ■

Remark. We recall that if the holonomy of $\mathcal{F}_2$ is not trivial then this foliation at most has one transverse affine structure (see [4] and [9]).

COROLLARY 5.1. *Suppose that the affine manifold (M, ∇) is complete. If $\mathcal{F}_2$ has non-trivial holonomy then each of its leaves is dense* (*therefore $s : M \to \mathbf{N}$ is constant*).

Proof. Now the affine manifold $\widetilde{M}$ can be regarded as $\mathbf{R}^n$ endowed with the canonical affine structure. Then every leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}_1$ may be written, in suitable affine coordinates, in the form $\{(t, t^{r_2}, \ldots, t^{r_k}, 0, \ldots, 0) \mid t \in \mathbf{R}\}$. Indeed, $\widetilde{L}$ locally is a pseudo-parabola and it has no ends on this curve because is a leaf.

Let $\mathcal{S}$ be the transverse affine structure of $\mathcal{F}_2$ constructed before (see Theorems 2 and 4). Set $\gamma(t) = (t, t^{r_2}, \ldots, t^{r_k}, 0, \ldots, 0)$, $t \in \mathbf{R}$. If $\{(t, t^{r_2}, \ldots, t^{r_k}, 0, \ldots, 0) \mid t \in \mathbf{R}\}$ is a leaf of $\widetilde{\mathcal{F}}_1$ then it is easily seen that $\pi(\gamma(t))$ is a geodesic of $\nabla(\mathcal{S})$. Therefore $\mathcal{S}$ is complete. ■

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