MONGE-AMPÈRE EQUATIONS AND SURFACES WITH NEGATIVE GAUSSIAN CURVATURE

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Abstract. In [24], we studied the singularities of solutions of Monge-Ampère equations of hyperbolic type. Then we saw that the singularities of solutions do not coincide with the singularities of solution surfaces. In this note we first study the singularities of solution surfaces. Next, as the applications, we consider the singularities of surfaces with negative Gaussian curvature. Our problems are as follows: 1) What kinds of singularities may appear?, and 2) How can we extend the surfaces beyond the singularities?

1. Introduction. In our previous work [24], we studied the singularities of solutions of real Monge-Ampère equations of hyperbolic type as follows: Let $z = z(x, y)$ be an unknown function defined for $(x, y) \in \mathbf{R}^2$, then the equation is written as

$$(1.1) \qquad F(x, y, z, p, q, r, s, t) = Ar + Bs + Ct + D(rt - s^2) - E = 0$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, and $t = \partial^2 z/\partial y^2$. Here we assume that A, B, C, D and E are real smooth functions of (x, y, z, p, q). As we can see that the singularities of solution $z = z(x, y)$ of (1.1) do not coincide with the singularities of solution surface $\{(x, y, z); z = z(x, y)\}$, we will first study the singularities of solution surfaces of (1.1). Next, as the application, we consider the singularities of surfaces with negative Gaussian curvature. Concerning the second subject, we will give some comments. It is well known that a surface in $\mathbf{R}^3$ with constant negative Gaussian curvature has singular points. (For example, see D. Hilbert [8] and E. Holmgren [9].) But we do not know what kinds of singularities may appear. Moreover we would like to extend the solution surface beyond the singularities. Our problems are to answer to the above two questions. As it seems to us that we do not have any result on these problems, we think that, though we assume a little strong conditions, this is one step to construct the global theory on hyperbolic surfaces. We cannot treat, by the method developed in this note,

the surface whose Gaussian curvature is negative constant, though we will give some comments on this subject in the last section. In this note we will give only the sketch of our results. The detailed proofs will be given in the forthcoming paper.

2. Characteristic method and intermediate integrals. The aim of this talk is to construct the singularities of the solution surfaces in the case where equation (1.1) is hyperbolic. For our aim, we have to represent the solutions explicitly. To do so, we apply the characteristic method developed principally by D. Darboux and E. Goursat ([3], [5], [6]). As it seems to us that the method is not familiar today, we will briefly explain it in Section 2 "from our point of view". But this method depends on a property that the dimension of space is two. The main idea of Darboux and Goursat is how to reduce the solvability of (1.1) to the integration of first order partial differential equations, though their theory is local. But, as their method is constructive, it is very useful for our purpose. Let

$$\Gamma : (x, y, z, p, q) = (x(\alpha), y(\alpha), z(\alpha), p(\alpha), q(\alpha)), \quad \alpha \in \mathbf{R}^1,$$

be a smooth curve in $\mathbf{R}^5$, and suppose that it satisfies the following "strip condition":

$$(2.1) \qquad \frac{dz}{d\alpha}(\alpha) = p(\alpha)\frac{dx}{d\alpha}(\alpha) + q(\alpha)\frac{dy}{d\alpha}(\alpha).$$

As a "characteristic strip" means that one cannot determine the values of the second order derivatives of solution along the strip Γ, we have the following

DEFINITION 2.1. A curve Γ in $\mathbf{R}^5 = \{(x, y, z, p, q)\}$ is a "characteristic strip" if it satisfies (2.1) and

$$(2.2) \qquad \det \begin{bmatrix} F_r & F_s & F_t \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{bmatrix} = F_t \dot{x}^2 - F_s \dot{x}\dot{y} + F_r \dot{y}^2 = 0$$

where $F_t = \partial F/\partial t$, $F_s = \partial F/\partial s$, $F_r = \partial F/\partial r$, $\dot{x} = dx/d\alpha$ and $\dot{y} = dy/d\alpha$.

Denote the discriminant of (2.2) by Δ, then

$$\Delta = F_s^2 - 4F_r F_t = B^2 - 4(AC + DE).$$

If $\Delta < 0$, equation (1.1) is called to be elliptic. If $\Delta > 0$, it is hyperbolic. In this note, we will treat the equations of hyperbolic type. More precisely, we assume $\Delta \geq 0$ and also $D \neq 0$. Let λ_1 and λ_2 be the solutions of $\lambda^2 + B\lambda + (AC + DE) = 0$, then the characteristic strip satisfies the following equations:

$$(2.3) \qquad \begin{cases} dz - p\,dx - q\,dy = 0 \\ D\,dp + C\,dx + \lambda_1\,dy = 0 \\ D\,dq + \lambda_2\,dx + A\,dy = 0 \end{cases}$$

or

$$(2.4) \qquad \begin{cases} dz - p\,dx - q\,dy = 0 \\ D\,dp + C\,dx + \lambda_2\,dy = 0 \\ D\,dq + \lambda_1\,dx + A\,dy = 0. \end{cases}$$

Let $\omega_0 = dz - p\,dx - q\,dy$, $\omega_1 = D\,dp + C\,dx + \lambda_1\,dy$ and $\omega_2 = D\,dq + \lambda_2\,dx + A\,dy$. Take an exterior product of ω_1 and ω_2, and substitute into their product the contact relations $\omega_0 = 0$, $dp = r\,dx + s\,dy$ and $dq = s\,dx + r\,dy$. Then we get

$$(2.5) \qquad \omega_1 \wedge \omega_2 = D\left\{Ar + Bs + Ct + D(rt - s^2) - E\right\} dx \wedge dy.$$

In a space whose dimension is greater than two, the decomposition as above is not possible in general. Here we introduce the notion of "first integral".

DEFINITION 2.2. A function $V = V(x,y,z,p,q)$ is called the *first integral* of $\{\omega_0, \omega_1, \omega_2\}$ if $dV \equiv 0 \mod \{\omega_0, \omega_1, \omega_2\}$.

R e m a r k. We can easily see that a function $V = V(x,y,z,p,q)$ is the first integral of (2.3) (or (2.4)) if it is constant on any solution of (2.3) (or of (2.4), respectively).

G. Darboux [3] and E. Goursat [5], especially in [5], had considered equations (1.1) under the assumption that (2.3), or (2.4), has at least two independent first integrals. We denote them by u and v. Then we get the following

PROPOSITION 2.3. *Assume that $\lambda_1 \neq \lambda_2$, and that (2.3), or (2.4), has two independent first integrals $\{u, v\}$. Then we can prove that there exists a function $k = k(x,y,z,p,q) \neq 0$ satisfying*

$$(2.6) \qquad du \wedge dv = k\,\omega_1 \wedge \omega_2 = kD\{Ar + Bs + Ct + D(rt - s^2) - E\}dx \wedge dy.$$

If equation (1.1) is written as (2.6), it would be obvious that (2.3), or (2.4), has two independent first integrals $\{u, v\}$. Therefore the representation (2.6) means the characterization of "Monge-Ampère equations of Darboux-Goursat type".

Let $\{u, v\}$ be two independent first integrals of (2.3). For any function g of two variables whose gradient does not vanish, $g(u, v) = 0$ is called an *intermediate integral* of (1.1). Let C_0 be an initial strip defined in $\mathbf{R}^5 = \{(x,y,z,p,q)\}$. If the strip C_0 is not characteristic, we can find an intermediate integral $g(u, v)$ which vanishes on C_0. Here we put $g(u, v) = f(x,y,z,p,q)$. The *Cauchy problem* for (1.1) satisfying the initial condition C_0 is to look for a solution $z = z(x, y)$ of (1.1) which contains the strip C_0, i.e., the two-dimensional surface $\{(x, y, z(x,y), \partial z/\partial x(x,y), \partial z/\partial y(x,y))\}$ in $\mathbf{R}^5$ containing C_0. The representation (2.6) assures that, as $du \wedge dv = 0$ on a surface $g(u, v) = 0$, a smooth solution of $f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0$ satisfies equation (1.1). Therefore we get the following

THEOREM 2.4 ([3], [5]). *Assume that the initial strip C_0 is not characteristic. Then a function $z = z(x, y)$ is a solution of the Cauchy problem for (1.1) with the initial condition C_0 if and only if it is a solution of $f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0$ satisfying the same initial condition C_0.*

3. How to construct examples. When a partial differential equation of first order

$$(3.1) \qquad\qquad\qquad f(x,y,z,p,q) = 0$$

is given, we will construct Monge-Ampère equation which accepts equation (3.1) as its intermediate integral. Assume $(\mathrm{grad}\, f) \neq 0$. Then we can locally find a function $g =$

$g(x, y, z, p, q)$ satisfying

$$(3.2) \qquad \operatorname{rank} \begin{pmatrix} \operatorname{grad} f \\ \operatorname{grad} g \end{pmatrix} = 2.$$

Here we take a product $df \wedge dg$ and substitute there the contact relations $\omega_0 = 0$, $dp = r\,dx + s\,dy$ and $dq = s\,dx + r\,dy$. Then we get

$$(3.3) \qquad df \wedge dg = F(x, y, z, p, q, r, s, t)dx \wedge dy.$$

This representation teaches us that the equation $F = 0$ has two independent first integrals f and g. At today's point, we do not have the proof of global existence of the function $g = g(x, y, z, p, q)$ satisfying the property (3.2). Therefore the equation $F = F(x, y, z, p, q, r, s, t)$ has the meaning only in a domain where the function $g(x, y, z, p, q)$ is defined. But we can construct the equation $F(x, y, z, p, q, r, s, t) = 0$ in the large for equations of certain types as follows.

EXAMPLE 1. Assume that f is of Hamilton-Jacobi type, i.e., $f = p + h(x, y, z, q)$. Choosing the function g as $g = q + k(x, y, z)$, we construct the equation $F = 0$ by

$$(3.4) \qquad \begin{aligned} df \wedge dg &= \{(rt - s^2) + Ar + Bs + Ct - E\}dx \wedge dy \\ &= F(x, y, z, p, q, r, s, t)dx \wedge dy \end{aligned}$$

where A, B, C and E are certain functions of (x, y, z, p, q).

EXAMPLE 2. Assume that f is quasi-linear, i.e., $f = ap + bq + c$ where a, b and c are real smooth functions of (x, y, z) and $(a, b) \neq (0, 0)$. Choose $g = -bp + aq + c'$ where c' is an arbitrary function of (x, y, z). Then the equation $F = 0$ is obtained by

$$(3.5) \qquad \begin{aligned} df \wedge dg &= \{(a^2 + b^2)(rt - s^2) + Ar + Bs + Ct - E\}dx \wedge dy \\ &= F(x, y, z, p, q, r, s, t)dx \wedge dy \end{aligned}$$

where A, B, C and E are functions of (x, y, z, p, q).

EXAMPLE 3. Any non-characteristic Cauchy problem for first order partial differential equations can be locally reduced to the following form:

$$(3.6) \qquad \frac{\partial u}{\partial x} + f(x, y, u, \frac{\partial u}{\partial y}) = 0 \quad \text{in} \quad \{(x, y); x > 0, y \in \mathbf{R}^1\},$$

$$(3.7) \qquad u(0, y) = \varphi(y) \quad \text{on} \quad \{(0, y); x = 0, y \in \mathbf{R}^1\}.$$

By Example 1, we can construct Monge-Ampère equation which accepts equation (3.6) as the intermediate integral. Following the Cauchy problem $(3.6) - (3.7)$, we define the initial strip C_0 by

$$C_0 : (x, y, z, p, q) = (0, \alpha, \varphi(\alpha), -f(0, \alpha, \varphi(\alpha), \varphi'(\alpha)), \varphi'(\alpha)), \ \alpha \in \mathbf{R}^1.$$

Then we can easily see that the Cauchy problem for Monge-Ampère equation with the initial strip C_0 has the Cauchy problem $(3.6) - (3.7)$ as its intermediate integral.

4. Solution surfaces of Monge-Ampère equations. In this section we will study the singularities of solution surfaces of Monge-Ampère equations. Here it would be better to make clear the meaning of "singularity" of surfaces, though we may write very elementary facts.

DEFINITION 4.1. A point (x^0, y^0, z^0) is called to be a *singularity* of a solution $z = z(x, y)$ of (1.1) if and only if $z^0 = z(x^0, y^0)$ and $z(x, y) \notin C^2$ in a neighbourhood of (x^0, y^0).

DEFINITION 4.2. Let S be a surface in $\mathbf{R}^3$. S is *regular* at a point (x^0, y^0, z^0) if we can choose parameters $(\alpha, \beta) \in \mathbf{R}^2$ as follows: $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$ and $z = z(\alpha, \beta)$ satisfy the two conditions:

(i) $(x(\alpha^0, \beta^0), y(\alpha^0, \beta^0), z(\alpha^0, \beta^0)) = (x^0, y^0, z^0)$, and $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$ and $z = z(\alpha, \beta)$ are of class C^1 in a neighbourhood of (α^0, β^0) ,

(ii) $$\mathrm{rank} \begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial z}{\partial \alpha} \\ \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial z}{\partial \beta} \end{pmatrix} = 2.$$

DEFINITION 4.3. A point (x^0, y^0, z^0) is called to be a *singularity* of a solution surface S if and only if S is not regular at the point (x^0, y^0, z^0).

First we will show that the singularities of solutions do not generally coincide with those of solution surfaces.

THEOREM 4.4. *Assume the following two conditions*:

I) (2.3), *or* (2.4), *has two independent first integrals, and*

II) *The intermediate integral is quasi-linear with respect to* (p, q).

Then the smooth solution surface of (1.1) *exists in the large, though the solution* $z = z(x, y)$ *of* (1.1) *may have singularities.*

S k e t c h   o f   t h e   p r o o f. By the assumption, we can write the Cauchy problem for the intermediate integral $f = f(x, y, z, p, q)$ as

(4.1) $$\begin{cases} f = f(x, y, z, p, q) = ap + bq - c = 0 \\ C_0 : (x, y, z, p, q) = (x_0(\alpha), y_0(\alpha), z_0(\alpha), p_0(\alpha), q_0(\alpha)), \end{cases}$$

where a, b and c are smooth functions of (x, y, z). The curve C_0 is the initial condition. Then the characteristic equations for (4.1) are

(4.2) $$\frac{\partial x}{\partial \beta} = a, \ \frac{\partial y}{\partial \beta} = b, \ \frac{\partial z}{\partial \beta} = c; \quad x(0) = x_0(\alpha), \ y(0) = y_0(\alpha), \ z(0) = z_0(\alpha).$$

We denote the solutions of (4.2) by $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$ and $z = z(\alpha, \beta)$. If the initial condition C_0 is not characteristic, we can prove that

$$\mathrm{rank} \begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial z}{\partial \alpha} \\ \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial z}{\partial \beta} \end{pmatrix} = 2 \quad \text{for any } (\alpha, \beta) \in \mathbf{R}^2.$$

Therefore, though the solution $z = z(x, y)$ has singularities at the points where the Jacobian $D(x, y)/D(\alpha, \beta) = 0$, the solution surface is regular even at these points. $\blacksquare$

Next we consider the case where the singularities may appear on the solution surfaces of (1.1).

THEOREM 4.5. *Assume the following two conditions*:

   I) (2.3), *or* (2.4), *has two independent first integrals, and*

   II) *The intermediate integral is of Hamilton-Jacobi type.*

Then $z = z(x, y)$ *is singular at a point* (x^0, y^0) *if and only if the solution surface* $\{(x, y, z); z = z(x, y)\}$ *is not regular at a point* (x^0, y^0, z^0) *where* $z^0 = z(x^0, y^0)$.

S k e t c h   o f   t h e   p r o o f. By the assumption on the intermediate integral $f = f(x, y, z, p, q)$, we can write the Cauchy problem as follows:

(4.3)
$$\begin{cases} f(x, y, z, \partial z/\partial x, \partial z/\partial y) = \partial z/\partial x + h(x, y, z, \partial z/\partial y) = 0 \\ z(x(\alpha), y(\alpha)) = z(\alpha), \quad (\partial z/\partial x)(x(\alpha), y(\alpha)) = p(\alpha), \\ \qquad\qquad (\partial z/\partial y)(x(\alpha), y(\alpha)) = q(\alpha). \end{cases}$$

The characteristic differential equations for (4.3) are written as follows:

(4.4)
$$\begin{cases} \dfrac{dx}{d\beta} = \dfrac{\partial f}{\partial p}(x, y, z, p, q), \quad \dfrac{dy}{d\beta} = \dfrac{\partial f}{\partial q}(x, y, z, p, q) \\[2mm] \dfrac{dz}{d\beta} = p\dfrac{\partial f}{\partial p}(x, y, z, p, q) + q\dfrac{\partial f}{\partial q}(x, y, z, p, q) \\[2mm] \dfrac{dp}{d\beta} = -\dfrac{\partial f}{\partial x}(x, y, z, p, q) - p\dfrac{\partial f}{\partial z}(x, y, z, p, q) \\[2mm] \dfrac{dq}{d\beta} = -\dfrac{\partial f}{\partial y}(x, y, z, p, q) - q\dfrac{\partial f}{\partial z}(x, y, z, p, q), \end{cases}$$

(4.5) $x(0) = x(\alpha), \ y(0) = y(\alpha), \ z(0) = z(\alpha), \ p(0) = p(\alpha), \ q(0) = q(\alpha), \quad \alpha \in \mathbf{R}^1.$

We denote the solutions of $(4.4) - (4.5)$ by $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$, $z = z(\alpha, \beta)$, $p = p(\alpha, \beta)$ and $q = q(\alpha, \beta)$. As $\omega_0 = 0$ on the solution surface, we have

(4.6)
$$\mathrm{rank}\begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial z}{\partial \alpha} \\[2mm] \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial z}{\partial \beta} \end{pmatrix} = \mathrm{rank}\begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} \\[2mm] \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} \end{pmatrix} \quad \text{for any } (\alpha, \beta) \in \mathbf{R}^2.$$

It is well known that the solution $z = z(x, y)$ has singularities at the points where the Jacobian $D(x, y)/D(\alpha, \beta) = 0$. Then equation (4.6) means that the solution surface is also not regular at the points where the Jacobian vanishes. From these ideas, we can arrive at the above conclusion. ∎

Concerning the solution surface $S = \{(x, y, z); z = z(x, y)\}$, the problems which we are interested in are as follows: I) What kinds of singularities may appear?, and II) Can we extend the solution surface beyond the singularities? For the problem II), we have two directions. After the appearance of singularities, the solution $z = z(x, y)$ takes in general several values. One way is to introduce a physical point of view. Then a solution must be single-valued. For this aim, we cut off some parts of solution so that it could become a single-valued weak or generalized solution satisfying the entropy condition for equations of conservation law or the semi-concavity condition for Hamilton-Jacobi equations. By this procedure, the singularities may appear in the solution. See [22], [23], [10], [11], [13], [19] and [20]. Another way is to consider the above problem from geometric point of view.

Then we must accept multi-valued solutions. As Monge-Ampère equations appear often in geometric problems, we should take here the second approach. This means that, without cutting off some part of solution surfaces, we should accept the whole part of solution surfaces and consider the singularities of surfaces in the meaning of Definition 4.3.

To state our results, we introduce a smooth mapping H from $\mathbf{R}^2$ to $\mathbf{R}^2$ defined by

$$H(\alpha, \beta) = (x(\alpha, \beta), y(\alpha, \beta)), \quad (\alpha, \beta) \in \mathbf{R}^2,$$

where $x(\alpha, \beta)$ and $y(\alpha, \beta)$ are the solutions of $(4.4) - (4.5)$. Let us write $\Sigma = \{(\alpha, \beta) \in \mathbf{R}^2; D(x, y)/D(\alpha, \beta) = 0\}$ and $H(\Sigma) = \Gamma$. Then we get the following

THEOREM 4.6. *Let the assumptions of Theorem 4.5 be satisfied and the singularities of the mapping H be fold and cusp points only. Then the curve Γ becomes piecewise smooth and the solution surface has the singularities along Γ. Moreover we can uniquely extend the solution surface beyond the singularities in the space of C^1-functions which are of class C^2 except on piecewise smooth curves.*

The canonical forms of cusp and fold points are obtained by H. Whitney [25]. The uniqueness of the extension of solution surfaces beyond the singularities follows from Theorem 4.6 in [23].

R e m a r k. Let us explain the meaning of "Hamilton-Jacobi type" used in Theorem 4.5 and 4.6. In [22], we have studied the differences between Hamilton-Jacobi equations and equations of conservation law, under the assumption that $f(x, y, z, p, q)$ is smooth. Our conclusion is that the most characteristic property of Hamilton-Jacobi equations is the global solvability of the Cauchy problem for (4.4). On the other hand, if $f = 0$ is quasi-linear, the solutions $p(\alpha, \beta)$ and $q(\alpha, \beta)$ tend to infinity when the Jacobian $D(x, y)/D(\alpha, \beta)$ vanishes. Therefore, in the above theorem, "Hamilton-Jacobi type" means the global solvability of the Cauchy problem for (4.4). Recently S. Izumiya [11] has given the geometric characterization of Hamilton-Jacobi equations and quasi-linear partial differential equations of first order.

5. Surfaces with negative Gaussian curvature. Let κ be the Gaussian curvature of the surface $z = z(x, y)$, then $z = z(x, y)$ satisfies the following equation:

$$(5.1) \qquad\qquad rt - s^2 = \kappa(1 + p^2 + q^2)^2.$$

We use the same notations as in Section 1. Here we are interested in the hyperbolic case, i.e., we assume $\kappa < 0$. Let us recall the classical theorem due to D. Hilbert as follows:

THEOREM 5.1 ([8]). *A surface S in $\mathbf{R}^3$ with constant negative Gaussian curvature has singular points.*

Therefore, when we may extend the classical solution of (5.1), the singularities may appear in general. But, if the Gaussian curvature is not strictly negative, there exists a surface in the large whose Gaussian curvature is negative.

EXAMPLE. Assume the Gaussian curvature $\kappa = -1/(1 + p^2 + q^2)^2$. Then equation (5.1) is written as

$$(5.2) \qquad\qquad rt - s^2 = -1.$$

The first integrals of (5.2) are given by $\{x + q, y - p\}$. Let the initial strip C_0 be

$$(5.3) \qquad\qquad C_0 : (x, y, z, p, q) = (0, \alpha, 1/2\alpha^2, 0, \alpha), \quad \alpha \in \mathbf{R}^1.$$

Then the intermediate integral $g = g(x, y, z, p, q)$ for the Cauchy problem $(5.2) - (5.3)$ is given by $g = p + q + x - y$. Therefore the solution of $(5.2) - (5.3)$ is written as $z = -1/2x^2 + 1/2y^2$. This means that there exists a smooth surface in the large whose Gaussian curvature is equal to $-1/(1 + p^2 + q^2)^2$.

As the generalization of Hilbert's theorem [8], N. V. Efimov proved the following

THEOREM 5.2 (Efimov [4]). *No surface can be immersed in $\mathbf{R}^3$ so as to be complete in the induced Riemannian metric, with strictly negative Gaussian curvature.*

We write $\lambda_1 = (1+p^2+q^2)(-\kappa)^{1/2}$, $\lambda_2 = -\lambda_1$, $\omega_1 = dp + \lambda_1\, dy$ and $\omega_2 = dq + \lambda_2\, dx$. Then we have on $\{\omega_0 = dz - p\, dx - q\, dy = 0, \; dp = r\, dx + s\, dy, \; dq = s\, dx + r\, dy\}$

$$\omega_1 \wedge \omega_2 = \{(rt - s^2) - \kappa(1 + p^2 + q^2)^2\}dx \wedge dy.$$

Therefore equation (5.1) is obtained by the product of ω_1 and ω_2. Then we get

THEOREM 5.3. *Assume that the Gaussian curvature κ is a function of (p, q) and strictly negative, then the system of one forms $\{\omega_0, \omega_1, \omega_2\}$ does not have two independent first integrals.*

Therefore we cannot apply our preceding method to solve (5.1). Then our problem is how we can get a family of characteristic strips. We can obtain it by solving the following system of first order partial differential equations:

$$(5.4) \qquad \begin{cases} \dfrac{\partial p}{\partial \alpha} + \lambda_1 \dfrac{\partial y}{\partial \alpha} = 0 \\[2mm] \dfrac{\partial q}{\partial \alpha} - \lambda_1 \dfrac{\partial x}{\partial \alpha} = 0 \\[2mm] \dfrac{\partial p}{\partial \beta} - \lambda_1 \dfrac{\partial y}{\partial \beta} = 0 \\[2mm] \dfrac{\partial q}{\partial \beta} + \lambda_1 \dfrac{\partial x}{\partial \beta} = 0. \end{cases}$$

The local solvability of (5.4) is already proved by H. Lewy [14] and J. Hadamard [7]. But, to develop the global theory, we would have to consider the global behaviour of the solutions of (5.4).

Finally we would like to give some comments on M. Kossowski [13]. He constructed local solutions of (1.1) by the method which is almost similar to the characteristic method. Then the biggest problem is how to get the family of characteristic strips. In [23], he could obtain it by solving certain system of equations which is numbered as (7) in [13]. His equation (7) in [13] is corresponding to (5.4) in our case. As he assumed the analyticity on equations (1.1), he could solve it by Cauchy-Kowalewski theorem. As we consider (1.1) in C^∞-space, we need the condition of hyperbolicity on (1.1).

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