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DIRECT IMAGE OF THE de RHAM SYSTEM ASSOCIATED WITH A RATIONAL DOUBLE POINT —A FIVE FINGERS EXERCISE

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1. Introduction. In 1976, M. Kashiwara [6] introduced the notion of direct image of $\mathcal{D}$-modules in his study of b-functions. The notion of direct image enjoys nice functorial properties, and the structure of direct image of $\mathcal{D}$-modules arouses great interest in various problems. In this paper we study the direct image of the de Rham system associated with a resolution of a rational double point singularity. In Section 2, we briefly recall some basic notions which are used later. In Section 3, we consider the surface with a rational double point of the type A_m. We give some explicit integral representation formulae for the Dirac delta function.

2. The de Rham system and the direct image functor.

de Rham system. Let X be a complex manifold of dimension n, $\mathcal{O}_X$ the sheaf of holomorphic functions. Let $\mathcal{D}_X$ be the sheaf on X of rings of partial differential operators with holomorphic coefficients. The sheaf $\mathcal{O}_X$ is naturally endowed with a structure of left $\mathcal{D}_X$-Module by differentiation. For instance, let $(x_1, x_2, \dots, x_n)$ be a system of local coordinates of X. For any germ h of holomorphic function, we have $\frac{\partial}{\partial x_j} h = \frac{\partial h}{\partial x_j}$. But if we regard h as a section of $\mathcal{D}_X$, i.e. as a linear partial differential operator of order zero, we have

$$\frac{\partial}{\partial x_j} h = \frac{\partial h}{\partial x_j} + h \frac{\partial}{\partial x_j}, \quad j = 1, 2, \dots, n.$$

Hence we have

$$\mathcal{O}_X \cong \mathcal{D}_X / \mathcal{D}_X \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \big).$$

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In fact, the sheaf $\mathcal{O}_X$ is generated by the constant function 1 over the sheaf of rings $\mathcal{D}_X$ and the annihilating ideal of the function 1 is locally equal to the following ideal:

$$\mathcal{D}_X\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right).$$

The coherent left $\mathcal{D}_X$-Module $\mathcal{O}_X$ is called the *de Rham system*.

Algebraic local cohomology. Let Y be a closed analytic subset of X, $\mathcal{J}_Y$ the defining ideal of Y. For each positive integer k, we set

$$\mathcal{H}^k_{[Y]}(\mathcal{O}_X) = \lim_{m\to\infty} \mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^m_Y, \mathcal{O}_X).$$

Since the sheaf $\mathcal{O}_X$ is a left $\mathcal{D}_X$-Module, the algebraic local cohomology group $\mathcal{H}^k_{[Y]}(\mathcal{O}_X)$ is endowed with the structure of left $\mathcal{D}_X$-Module. Moreover, Z. Mebkhout [8] and M. Kashiwara [6] proved the following facts:

(i) $\mathcal{H}^k_{[Y]}(\mathcal{O}_X)$ is a coherent $\mathcal{D}_X$-Module,
(ii) $\mathcal{H}^k_{[Y]}(\mathcal{O}_X)$ is a regular holonomic system.

When Y is a complex submanifold, we have the following result.

Proposition (Kashiwara [4].) *If Y is defined by $x_1 = \ldots = x_d = 0$ for a local coordinate system $(x_1, \ldots, x_n)$ of X, then:*

(i) $\mathcal{H}^k_{[Y]}(\mathcal{O}_X) = 0$ *for* $k \neq d$,
(ii) $\mathcal{H}^d_{[Y]}(\mathcal{O}_X) \cong \mathcal{D}_X/\mathcal{D}_X(x_1, \ldots, x_d, \frac{\partial}{\partial x_{d+1}}, \ldots, \frac{\partial}{\partial x_n})$.

Direct image. Let us recall briefly the notion of the direct image of $\mathcal{D}$-Modules.

Let X, Z be complex manifolds, $f : Z \to X$ a proper holomorphic map. We set

$$\mathcal{D}_{X\leftarrow Z} = f^{-1}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_X} \Omega_Z,$$

where Ω_Z and Ω_X are the sheaves of the highest degree holomorphic forms on Z and X respectively. Note that $\mathcal{D}_{X\leftarrow Z}$ is a $(f^{-1}\mathcal{D}_X, \mathcal{D}_Z)$-bi-Module.

For any coherent left $\mathcal{D}_Z$-Module $\mathcal{M}$, we set

$$\int_f \mathcal{M} = \mathbf{R}f_*(\mathcal{D}_{X\leftarrow Z} \otimes^{\mathbf{L}}_{\mathcal{D}_Z} \mathcal{M})$$

in the derived category $D^b(\mathcal{D}_X)$ of $\mathcal{D}_X$-Modules (we refer to [3], [6] and [9]).

We have the following fundamental result.

Proposition (Kashiwara, cf. [3]) *Let Y be a complex d-codimensional submanifold of X. Let i be the natural embedding map. Then we have*

$$\int_i \mathcal{O}_Y = \mathcal{H}^d_{[Y]}(\mathcal{O}_X).$$

Example ([10], [11]). As an illustration of the direct image, let us examine the de Rham system associated with the resolution of a plane curve singularity.

Let $X = \mathbf{C}^2$ with coordinates (x, y). Let $Y = \{(x, y) \mid x^5 - y^3 = 0\}$. Let $T = \mathbf{C}$ with coordinate t, $\pi : T \to X$ with $\pi(t) = (t^3, t^5)$. Let $i : T \to Z$ be the natural embedding map, where $Z = X \times T$. We have the following commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{\quad i \quad} & Z \\ \big\downarrow & & \big\downarrow \text{proj} \\ Y & \xrightarrow{\qquad} & X \end{array}$$

here proj is the natural projection map proj : $X \times T \to X$.

Now we set

$$u = \int_{\pi} 1$$

where 1 stands for the constant function, which is a generator over $\mathcal{D}_T$ of the de Rham system $\mathcal{O}_T$. We have

$$u = \int_{\text{proj}} \int_i 1 = \int_{\text{proj}} \delta(x - t^3)\delta(y - t^5).$$

Then u satisfies the following system of linear partial differential equations:

$$P_1 u = P_2 u = P_3 u = 0,$$

where

$$\begin{aligned} P_1 &= x^5 - y^3, \\ P_2 &= 3x\frac{\partial}{\partial x} + 5y\frac{\partial}{\partial y} + 7, \\ P_3 &= 3y^2\frac{\partial^3}{\partial x^2 \partial y} + 5x^4\frac{\partial^3}{\partial x \partial y^2} + 25x^3\frac{\partial^2}{\partial y^2} + 9y\frac{\partial^2}{\partial x^2}. \end{aligned}$$

Furthermore we have

$$\mathcal{D}_X u = \mathcal{D}_X / \mathcal{D}_X(P_1, P_2, P_3)$$

and u is equal to $xy\delta(x^5 - y^3)$ up to non-zero constant.

3. Calculation and a result. In this section we take a resolution of a surface with a rational double point and consider the de Rham system on the resolution. One of our aims is to calculate the $\mathcal{D}_X$-Module structure of the direct image of the de Rham system. We present here the key point of our calculation.

Resolution. Let $X = \mathbf{C}^3$ with coordinates (x, y, z). Let S be the surface with a rational double point at the origin defined by

$$S = \{(x, y, z) \in X \mid z^{m+1} = xy\}.$$

We resolve the singularity of the surface S as follows. Let $W_0, W_1, \ldots, W_m$ be copies of $\mathbf{C}^2$ with coordinates $(u_0, v_0), (u_1, v_1), \ldots, (u_m, v_m)$ respectively. Following a standard argument, we patch them up and construct a non-singular surface M by using the following transition functions:

$$u_{k+1} = 1/v_k, \quad v_{k+1} = u_k v_k^2, \quad \text{for} \quad k = 0, 1, 2, \ldots, m - 1.$$

We introduce a holomorphic map $\pi : M \to X$ by

$$\begin{cases} x = u_k^{k+1} v_k^k \\ y = u_k^{m-k} v_k^{m-k+1} \\ z = u_k v_k & \text{on } W_k,\ k = 0, \dots, m. \end{cases}$$

It is easy to see that $\pi : M \to X$ is well defined and π is a resolution of the singularity of the surface S. The exceptional set of the resolution consists of curves $C_1, \dots, C_m$, where $C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}$.

Set $Z = X \times P^1 \times P^1 \times \ldots \times P^1$. Let $([\xi_1, \eta_1], [\xi_2, \eta_2], \dots, [\xi_m, \eta_m])$ be the standard homogeneous coordinates in the product $P^1 \times P^1 \times \ldots \times P^1$. Set

$$p_k = \xi_k / \eta_k, \quad q_k = \eta_k / \xi_k, \qquad k = 1, 2, \dots, m$$

and

$$p_1 = u_k^{m-k-1} v_k^{m-k},\ p_2 = u_k^{m-k-2} v_k^{m-k-1},\ \dots,\ p_{m-k} = v_k,$$
$$q_{m-k+1} = u_k,\ q_{m-k+2} = u_k^2 v_k,\ \dots,\ q_m = u_k^k v_k^{m-k-1} \quad \text{for} \quad k = 0, \dots, m-1.$$

This defines a holomorphic embedding map $i : M \to Z$. Note that we have $i(C_k) = [0,1] \times \ldots \times [0,1] \times P^1 \times [1,0] \times \ldots \times [1,0]$. We have the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\ \ i\ \ } & Z \\ \downarrow & & \downarrow \text{proj} \\ S & \longrightarrow & X \end{array}$$

here proj is the natural projection map $\text{proj} : X \times P^1 \times P^1 \times \ldots \times P^1 \to X$.

Calculation. Let us examine the integrals along π of the de Rham system $\mathcal{O}_M$.

We use the following fact:

$$\int_\pi \mathcal{O}_M = \int_{\text{proj}} \int_i \mathcal{O}_M = \int_{\text{proj}} \mathcal{N}.$$

where $N = \mathcal{H}^{m+1}_{[i(M)]}(\mathcal{O}_Z)$.

We set, for instance on $\eta_1 \neq 0$, $\eta_2 \neq 0$, $\dots$, $\eta_m \neq 0$

$$g_m = -p_m \delta(y - x^m p_m^{m+1}) \delta(z - x p_m) \delta(p_1 - x^{m-1} p_m^m) \cdot$$
$$\delta(p_2 - x^{m-2} p_m^{m-1}) \cdots \delta(p_{m-2} - x^2 p_m^3) \delta'(p_{m-1} - x p_m^2) dp_1 \wedge dp_2 \wedge \cdots \wedge dp_m.$$

It is easy to verify that the differential form g_m is globally well-defined on Z as a relative differential form supported on $i(M)$:

$$g_m \in \Gamma(Z, \mathcal{N} \otimes \Omega_{P^1 \times \ldots \times P^1}),$$

and that g_m is not exact, but the differential forms $x g_m$, $y g_m$ and $z g_m$ are relatively exact. In fact, if we set

$$f = \delta(y - x^m p_m^{m+1}) \delta(z - x p_m) \delta(p_1 - x^{m-1} p_m^m) \delta(p_2 - x^{m-2} p_m^{m-1}) \cdot$$
$$\cdots \delta(p_{m-2} - x^2 p_m^3) \delta(p_{m-1} - x p_m^2) dp_1 \wedge dp_2 \wedge \cdots \wedge dp_{m-2} \wedge dp_m,$$

then the differential forms f, $p_{m-1} f$ and $p_{m-1}^2 f$ are globally well-defined. Furthermore we have

$$d(zf) = x g_m, \quad d(p_{m-1}^2 z^{m-2} f) = y g_m \quad \text{and} \quad d(p_{m-1} f) = z g_m,$$

where d is the relative exterior differentiation. These equalities hold globally. This implies that $\int_{\text{proj}} g_m$ is equal to a constant multiple of the delta-function on X supported at the origin $(0,0,0)$. In particular, we have

$$\int_{\text{proj}} g_m \in \mathcal{H}^3_{[0,0,0]}(\mathcal{O}_X).$$

Similarly, on $\eta_1 \neq 0$, $\eta_2 \neq 0$, ..., $\eta_m \neq 0$, we set

$$\begin{aligned} g_k = &-p_k\delta(y - x^m p_m^{m+1})\delta(z - xp_m)\delta(p_1 - x^{m-1}p_m^m)\delta(p_2 - x^{m-2}p_m^{m-1}) \cdot \\ &\cdots \delta'(p_{k-1} - x^{m-k+1}p_m^{m-k+2}) \cdots \delta(p_{m-1} - xp_m^2) dp_1 \wedge dp_2 \wedge \cdots \wedge dp_m. \end{aligned}$$

for $k = 2, \dots, m$ and

$$\begin{aligned} g_1 = &[(m+1)p_1\delta'(y - x^m p_m^{m+1})\delta(z - xp_m)\delta(p_1 - x^{m-1}p_m^m) \cdots \delta(p_{m-1} - xp_m^2) \\ &+\delta(y - x^m p_m^{m+1})\delta'(z - xp_m)\delta(p_1 - x^{m-1}p_m^m) \cdots \delta(p_{m-1} - xp_m^2) \\ &+mp_2\delta(y - x^m p_m^{m+1})\delta(z - xp_m)\delta'(p_1 - x^{m-1}p_m^m) \cdots \delta(p_{m-1} - xp_m^2) \\ &+(m-1)p_3\delta(y - x^m p_m^{m+1})\delta(z - xp_m)\delta(p_1 - x^{m-1}p_m^m) \cdot \\ &\quad \delta'(p_2 - x^{m-2}p_m^{m-1}) \cdots \delta(p_{m-1} - xp_m^2) \\ &+ \dots\dots \\ &+2p_m\delta(y - x^m p_m^{m+1})\delta(z - xp_m)\delta(p_1 - x^{m-1}p_m^m) \cdots \delta'(p_{m-1} - xp_m^2)] \cdot \\ &\quad dp_1 \wedge dp_2 \wedge \cdots \wedge dp_m. \end{aligned}$$

The differential forms $g_1, \dots, g_m$ are globally well-defined on Z as relative differential form supported on $i(M)$ and the integrals along the fibers of these differential forms are equal to the Dirac delta function up to non-zero constant factors. We can summarize the results of our calculation in the following form:

THEOREM. *The integrals along the fibers of the map* proj : $X \times P^1 \times \dots \times P^1 \to X$ *of the relative differential forms* $g_1, g_2, \dots, g_m$ *are equal to the delta-function supported at the origin* $(0,0,0)$ *up to non-zero constant:*

$$\int_{\text{proj}} g_k = \text{const} \cdot \delta(x)\delta(y)\delta(z) \quad k = 1, \dots, m.$$

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