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A NOTE ON SINGULARITIES AT INFINITY OF COMPLEX POLYNOMIALS

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Abstract. Let f be a complex polynomial. We relate the behaviour of f "at infinity" to the sheaf of vanishing cycles of the family $\overline{f}$ of projective closures of fibres of f. We show that the absence of such cycles: (i) is equivalent to a condition on the asymptotic behaviour of gradient of f known as Malgrange's Condition, (ii) implies the C^∞-triviality of f. If the support of sheaf of vanishing cycles of $\overline{f}$ is a finite set, then it detects precisely the change of the topology of the fibres of f. Moreover, in this case, the generic fibre of f has the homotopy type of a bouquet of spheres.

Let $f : \mathbf{C}^n \to \mathbf{C}$ be a polynomial function. A value $t_0 \in \mathbf{C}$ of f is called *typical* if f is a C^∞-trivial fibration over a neighbourhood of t_0 and *atypical* otherwise. The set of atypical values, called *the bifurcation set of* f, consists of the critical values of f and, maybe, some other values coming from the "singularities of f at infinity". What is "the singularity at infinity" is understood rather heuristically and, in general, no precise definition exists. Under various assumptions this notion can be given a precise meaning, for instance if "the singularities at infinity" are in some sense isolated as in [Pa], [S-T], or [Z].

Consider the family $\overline{f} : X \to \mathbf{C}$ of projective closures of fibres of f, X being the closure of the graph of f in $\mathbf{P}^n \times \mathbf{C}$. In this paper we study the singularities of f from the point of view of vanishing cycles of $\overline{f}$. In particular, as we show below (Theorem 1.2), the absence of vanishing cycles of $\overline{f} - t_0$ guarantees that t_0 is typical. Thus we

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may understand "the singularities at infinity" as those points at infinity at which $\overline{f}$ has nontrivial vanishing cycles. In the particular case when such points are isolated, they detect precisely the change of topology of the fibres of f (Corollary 1.7 below).

Our approach to the problem is similar to that of [Pa] and [S-T]. In order to trivialize f we study the levels of y_0 on X, where y_0 is a function defining, locally, the hyperplane at infinity. Using a topological version of a theorem of Ginsburg, Theorem 1.4 below, we show that the absence of vanishing cycles of $\overline{f}$ is equivalent to a condition on the asymptotic behaviour of the gradient of f known as Malgrange's Condition. Shortly speaking, Malgrange's Condition detects exactly the same special fibres of f as the vanishing cycles of $\overline{f}$. It is plausible that our method can be applied also to study other compactifications of f, see Remark 1.8.

Let $p \in X$ be a point at infinity. Then the condition $(p, dt) \notin W$ of Corollary 1.5, which as we prove is equivalent to both the absence of vanishing cycles of $\overline{f}$ at p and to Malgrange's Condition as $x \to p$, is exactly the condition of t-regularity of f at p of [S-T].

In [S-T] the authors show that if f has only isolated singularities, possibly also at infinity, then the generic fibre of f has a homotopy type of a bouquet of spheres of dimension $n-1$. In Section 2 we propose a proof of this theorem, based on the Morse Theory, which works also in our more general set-up.

In particular, the results of this paper generalize those of [Pa] and [S-T]. For an extensive bibliography related to the subject the reader is referred to [Di].

1. Vanishing cycles and Malgrange's Condition. Let $f(x_1, \ldots, x_n)$ be a complex polynomial of degree d and let $\widetilde{f}(x_0, x_1, \ldots, x_n)$ be the homogenization of f. Consider the family of the projective closures of the fibres of f given by $\overline{f} : X \to \mathbf{C}$, where

$$X = \{(x,t) \in \mathbf{P}^n \times \mathbf{C} \mid F(x,t) = \widetilde{f}(x) - t x_0^d = 0\}$$

and $\overline{f}$ is induced by the projection on the second factor. Let $H_\infty = \{x_0 = 0\} \subset \mathbf{P}^n$ be the hyperplane at infinity and let $X_\infty = X \cap (H_\infty \times \mathbf{C})$. The cone at infinity C_∞ of the fibres $f^{-1}(t)$ of f does not depend on t and hence $X_\infty = C_\infty \times \mathbf{C}$.

Let $i : X \hookrightarrow \mathbf{P}^n \times \mathbf{C}$ denote the inclusion and let $i_*\mathbf{C}_X$ denote the constant sheaf on X extended by zero onto $\mathbf{P}^n \times \mathbf{C}$. Let $\mathrm{Car}\,(X) \subset T^*(\mathbf{P}^n \times \mathbf{C})$ denote the characteristic cycle of $i_*\mathbf{C}_X$, see for instance [Br] or [Sa]. As a Lagrangian cycle in $T^*(\mathbf{P}^n \times \mathbf{C})$, $\mathrm{Car}\,(X)$ admits a presentation as a finite sum

$$\mathrm{Car}\,(X) = \sum m_j T^*_{X_j}(\mathbf{P}^n \times \mathbf{C}), \tag{1.1}$$

where X_j are algebraic subsets of $\mathbf{P}^n \times \mathbf{C}$ and $m_j \in \mathbf{Z}$. By [BDK] or [LM], $\mathrm{Car}\,(X)$, or rather the singular support $SS(i_*\mathbf{C}_X) \subset T^*(\mathbf{P}^n \times \mathbf{C})$, can be understood as the closure of the set of points $(x, \xi) \in T^*(\mathbf{P}^n \times \mathbf{C})$, $\xi \neq 0$, such that there exists $g : (\mathbf{P}^n \times \mathbf{C}, x) \to (\mathbf{C}^n, 0)$ with $\xi = dg(x)$ and nontrivial vanishing cycles $\Phi_g(i_*\mathbf{C}_X)$. In our case simply

$$SS(i_*\mathbf{C}_X) = |\mathrm{Car}\,(X)| = \bigcup T^*_{X_j}(\mathbf{P}^n \times \mathbf{C}), \tag{1.2}$$

where the union is taken over all X_j for which $m_j \neq 0$. Indeed, this is a consequence of perversity of (complex of) sheaves $i_*\mathbf{C}_X[-1]$ (see the proof of Definition-Proposition 1.1 below for a more detailed argument).

Given $t_0 \in \mathbf{C}$ we denote by $\Phi_{\overline{f}-t_0}(i_*\mathbf{C}_X)$ the sheaf of vanishing cycles of $\overline{f}-t_0$. Since $\Phi_{\overline{f}-t_0}(i_*\mathbf{C}_X)$ is zero for all by finitely many $t_0 \in \mathbf{C}$, there is no confusion if we denote by $\Phi(\overline{f})$ the direct sum of all nonzero $\Phi_{\overline{f}-t_0}(i_*\mathbf{C}_X)$. Given $p \in \mathbf{P}^n \times \mathbf{C}$, we denote by $(p, dt) \in T^*(\mathbf{P}^n \times \mathbf{C})$ the covector at p given by the projection on $\mathbf{C}$.

Definition-Proposition 1.1. We say that $\overline{f}$ is *non-characteristic at* $p \in X$ if one of the following equivalent conditions hold:

(i) $(p, dt) \notin |\mathrm{Car}\,(X)|$;

(ii) $p \notin \mathrm{supp}\,(\Phi(\overline{f}))$.

Similarly we say that $\overline{f}$ is *non-characteristic over* t_0 (or *over* t_0 *at* ∞) if $\overline{f}$ is non-characteristic at every $p \in \overline{f}^{-1}(t_0)$ (resp. at every $p \in \overline{f}^{-1}(t_0) \cap X_\infty$).

Proof. Since X is a hypersurface of a nonsingular variety $\mathbf{P}^n \times \mathbf{C}$ the sheaf $i_*\mathbf{C}_X[-1]$ is perverse. Thus the proposition is virtually an immediate consequence of [BDK] or [LM]. Nevertheless, for the reader convenience, we sketch below the standard argument.

Fix a Thom stratification $(\mathcal{S}, \mathcal{S}')$ of $\overline{f}$, that is Whitney stratifications $\mathcal{S}$ of X and $\mathcal{S}'$ of $\mathbf{C}$ respectively, satisfying the Thom condition $a_{\overline{f}}$. $\mathcal{S}'$ consists of a finite set Δ and an open stratum $\mathbf{C} \setminus \Delta$. Then $Z = \{p \in X \mid (p, dt) \in |\mathrm{Car}\,(X)|\}$ is contained in $\overline{f}^{-1}(\Delta)$. Also $\widetilde{Z} = \mathrm{supp}\,(\Phi(\overline{f})) \subset \overline{f}^{-1}(\Delta)$. In what follows we restrict ourselves to a neighbourhood of one of such fibres $\overline{f}^{-1}(t_0)$. Then $\overline{f}^{-1}(t_0)$ is a union of strata and so is $\widetilde{Z}$. Subdividing $\mathcal{S}$, if necessary, we assume that Z is also a union of strata.

Suppose $Z \neq \widetilde{Z}$. Note that by definition both Z and $\widetilde{Z}$ are closed in $\mathbf{P}^n \times \mathbf{C}$. Consider the stratum S_0 of the largest dimension in $(Z \setminus \widetilde{Z}) \cup (\widetilde{Z} \setminus Z)$. Then S_0 is open in one of Z or $\widetilde{Z}$ (depending on the case) and empty in the other. Let p_0 be a generic point of S_0. Our argument is purely local so we may assume that locally at p_0, $(S_0, p_0) = (\mathbf{C}^d \times \{0\}, 0) \subset (\mathbf{C}^{n+1}, 0)$, and that $t - t_0$ is the $(d+1)$-st coordinate. Let $\pi : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^d \times \{0\}, 0)$ denote the projection. Then $N = \pi^{-1}(0)$ is a transverse slice to S_0 at p_0. Let $X' = X \cap N$ and denote $f' = (\overline{f} - t_0)|_{X'}$, both defined in a sufficiently small neighbourhood of p_0. By assumption on $\mathcal{S}$, the Milnor fibre of $\overline{f}$ at p_0 is product-like along S_0 and $\widetilde{Z} \cap N = \mathrm{supp}\,(\Phi_{f'})$ near p_0. We show that similarly $Z \cap N = \{p \in X \cap N \mid (p, dt) \in |\mathrm{Car}\,(X \cap N)|\}$. Since $\mathcal{S}$ is a Whitney stratification of X, we may take as X_j in (1.1) the closures of some, not necessarily all, strata of $\mathcal{S}$. Then near p_0, a stratum S contributes nontrivially to $|\mathrm{Car}\,(X)|$ if and only if $S \cap N$ makes a nontrivial contribution to $|\mathrm{Car}\,(N \cap X)|$, see for instance [BDK]. Moreover, by Whitney (a) condition $(p_0, dt) \in T^*_S(\mathbf{P}^n \times \mathbf{C})$ if and only if $(p_0, dt) \in T^*_{S'}N$, where $S' = S \cap N$. This shows $Z \cap N = \{p \in X \cap N \mid (p, dt) \in |\mathrm{Car}\,(X \cap N)|\}$ as required.

Consequently we may assume that one of the sets $\widetilde{Z} \cap N, Z \cap N$ equals $\{p_0\}$ and the other is empty.

Suppose S_0 is a one point set. Then the statement follows from the following formula, see e.g. [Sa, (4.6)], on the intersection index at (p_0, dt) of $\mathrm{Car}\,(X)$ and $\{dt\} = \{(p, dt) \mid p \in \mathbf{P}^n \times \mathbf{C}\} \subset T^*(\mathbf{P}^n \times \mathbf{C})$,

$$(\mathrm{Car}\,(X).\{dt\})_{(p_0, dt)} = (-1)^n \chi(\Phi(\overline{f}))(p_0). \tag{1.3}$$

Indeed, $p_0 \notin Z$ if and only if the left-hand side of (1.3) vanishes. On the other hand since the support of the sheaf of vanishing cycles is zero-dimensional (or empty) and $\Phi(\overline{f})$ is perverse (after a shift in degrees), the vanishing of the Euler characteristic is equivalent to the vanishing of the cohomologies of $\Phi(\overline{f})$. Hence the right-hand side of (1.3) vanishes if and only if $p \notin \widetilde{Z}$. This ends the proof. ■

Suppose that we would like to trivialize topogically $\overline{f}$ near $p \in X$. If such trivialization exists, then $\overline{f}$ has no vanishing cycles near p. We do not know whether the converse is true. The theorem below shows that this is the case if we require only the triviality of f.

THEOREM 1.2.

(i) *If $\overline{f}$ is non-characteristic over t_0 then f is C^∞-trivial over a neighbourhood of t_0, that is t_0 is typical. In particular, if* $\operatorname{supp}(\Phi(\overline{f}))$ *is empty in X then f is C^∞-trivial.*

(ii) *Similarly if $\overline{f}$ is non-characteristic over t_0 at infinity then f is C^∞-trivial over a neighbourhood of t_0 and near infinity (i.e. in the complement of a sufficiently big ball in $\mathbf{C}^n$).*

Note that the existence of a stratification trivializing $\overline{f}$ is, at least a priori, a much stronger assumption than the absence of vanishing cycles. Thus, in the proof of Theorem 1.2, we prefer to avoid the use of stratifications. In particular, we do not trivialize $\overline{f}$ but only f. To prove Theorem 1.2 we use affine trivializations and the following condition on the asymptotic behaviour of gradient of $\overline{f}$.

We say, after [Ph], that f satisfies Malgrange's Condition for $t_0 \in \mathbf{C}$ if, for $|x|$ large enough and for $f(x)$ close to t_0,

$$\exists \delta > 0 \quad |x|\,|\operatorname{grad} f(x)| \geq \delta\,. \tag{M}$$

It is well-known that Malgrange's Condition gives C^∞-triviality of f near infinity, for the proof see e.g. [Pa]. Hence Theorem 1.2 follows from the following theorem.

THEOREM 1.3. *$\overline{f}$ is non-characteristic over t_0 at infinity if and only if Malgrange's Condition holds for t_0.*

Proof. First we outline the idea of proof. Given $p \in X_\infty$. The hyperplane at infinity is defined near p by a single function, say y_0. Let $g : (X, p) \to (\mathbf{C}, 0)$ denote $y_0|_X$. Consider $W = T^*_g|_{X_\infty}$ — the space of limits of hyperplanes tangent to the levels $\{g = c\}$, as $c \to 0$. The key point of the proof is to show that $\overline{f}$ is non-characteristic at p if and only if $(p, dt) \notin W$. This will follow from a theorem of Ginsburg. Then suppose that $(p, dt) \notin W$ which in some local coordinates around p is equivalent to condition (1.5) below. Here we note that (1.5) shows that, locally near p, we may use the levels of $g = y_0|_X$ to trivialize f. We show later in Section 2 how to "glue" such local trivializations to a global one but we do not need this to prove Theorem 1.3. To complete the proof we just show that (1.5) translates exactly to Malgrange's Condition if we return to the original affine coordinates.

We begin the proof with setting up the notation. Fix $p \in X_\infty$. We assume that $p = ((0:0:\ldots:0:1), 0) \in \mathbf{P}^n \times \mathbf{C}$, so that $y_0 = x_n^{-1}$, $y_i = x_i/x_n$ for $i = 1, \ldots, n-1$, and t, form a local system of coordinates at p. In this new coordinate system X is defined by

$$F(y_0, y_1, \ldots, y_{n-1}, t) = \widetilde{f}(y_0, y_1, \ldots, y_{n-1}, 1) - t y_0^d = 0.$$

Let Ω be a small neighbourhood of p. Consider the relative conormal $T_g^*(\Omega) \subset T^*(\Omega)$ to g. Then the divisor W in $T_g^*(\Omega)$ defined by $g = 0$ is a Lagrangian conical subvariety of $T^*(\Omega)$, [HMS], and hence is of the form $\sum p_j T_{Z_j}^*(\Omega)$, for some $Z_j \subset X_\infty \cap \Omega$.

Let $U = X \setminus X_\infty$ and let $j : U \to X$ denote the inclusion. Then

$$\mathrm{Car}\,(X) = \mathrm{Car}\,(j_! \mathbf{C}_U) + \mathrm{Car}\,(\mathbf{C}_{X_\infty}). \tag{1.4}$$

Since U is nonsingular, the Lagrangian conical cycle $\mathrm{Car}\,(j_!\mathbf{C}_U)$ has to be of the form

$$\mathrm{Car}\,(j_!\mathbf{C}_U) = T_X^*(\mathbf{P}^n \times \mathbf{C}) + \sum m_j T_{Y_j}^*(\mathbf{P}^n \times \mathbf{C}).$$

Now we state a theorem which relates W and $\mathrm{Car}\,(j_!\mathbf{C}_U)$. This theorem follows immediately from [BMM, Théorème 3.4.2], which is a consequence of results of Ginsburg [G, Theorems 3.3 and 5.5]. We are allowed to use this result since $i_* \mathbf{C}_X[-1]$ is perverse.

THEOREM 1.4. *Let* $\mathrm{Car}\,(j_!\mathbf{C}_U) = T_X^*(\mathbf{P}^n \times \mathbf{C}) + \sum m_j T_{Y_j}^*(\mathbf{P}^n \times \mathbf{C})$ *as above, where* $Y_j \subset X_\infty$ *and* $m_j \neq 0$. *Then in a neighbourhood* Ω *of* $p \in X_\infty$

$$W = \sum p_j T_{Z_j}^*(\Omega),$$

where $Z_j = Y_j \cap \Omega$ *and* $p_j \neq 0$ *if* $Y_j \neq \emptyset$.

COROLLARY 1.5. $(p, dt) \notin |\mathrm{Car}\,(X)|$ *if and only if* $(p, dt) \notin |W|$.

Proof. First note that $(p, dt) \notin |\mathrm{Car}\,(\mathbf{C}_{X_\infty})|$ since $X_\infty = C_\infty \times \mathbf{C}$. Thus clearly $(p, dt) \in |\mathrm{Car}\,(X)|$ iff $(p, dt) \in |\mathrm{Car}\,(j_!\mathbf{C}_U)|$.

Over $\mathrm{Sing}\, X$ we have $T_X^*(\mathbf{P}^n \times \mathbf{C}) \subset \bigcup T_{X_j}^*(\mathbf{P}^n \times \mathbf{C})$, where we take all $X_j \neq X$ given by (1.1), and hence the corollary follows from Theorem 1.4.

Suppose that $p \notin \mathrm{Sing}\, X$. Let $f = f_0 + f_1 + \cdots + f_d$ be the decomposition of f into homogeneous components. Recall after [Di, Ch. 1 §4] or [Pa] that the singular part of X is precisely $A \times \mathbf{C}$, where

$$A = \{x \in H_\infty \mid \partial f_d/\partial x_1 = \cdots = \partial f_d/\partial x_n = f_{d-1} = 0\}.$$

The singular part of X_∞ can be bigger and equals precisely $B \times \mathbf{C}$, where

$$B = \{x \in H_\infty \mid \partial f_d/\partial x_1 = \cdots = \partial f_d/\partial x_n = 0\}.$$

On $X_\infty \setminus B \times \mathbf{C}$ the application $(y_0, t)|_X$ is submersive so neither $(p, dt) \in |\mathrm{Car}\,(X)|$ nor $(p, dt) \in |W|$. If $p \in (B \setminus A) \times \mathbf{C}$, then $\partial F/\partial y_0(p) \neq 0$ and the other partial derivatives vanish at p. Consequently, $(p, dt) \notin |\mathrm{Car}\,(X)|$, and hence, by Theorem 1.4, $(p, dt) \notin |W|$. This ends the proof of the corollary. ∎

Proof of Theorem 1.3 (continued). By definition $W = T_g^*|_{X_\infty}$ is the limit, as $y_0 \to 0$, of conormal spaces to levels of g. Hence $(p, dt) \notin |W|$ if and only if

$$|\partial F/\partial t(y, t)| \le C |(\partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1})(y, t)|, \tag{1.5}$$

for all $(y, t) \in X$ and close to p. Since $\partial F/\partial t = y_0^d$, (1.5) translated to the old affine coordinates $(x_1, \ldots, x_n)$ reads (see e.g. [S-T])

$$|x|\,|(\partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})| \ge \delta,$$

where $\delta > 0$ depends on C. This of course implies Malgrange's Condition (M). To see that actually (1.5) is equivalent to (M) we show the following lemma. (For the proof of

Theorem 1.3 we need only the first part of the lemma. The second part will be used in Section 2 below.)

LEMMA 1.6. *Condition* (1.5) *is equivalent to*

$$|\partial F/\partial t(y,t)| \le C|(y_0\partial F/\partial y_0, \partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1})(y,t)|, \tag{1.6}$$

for all $(y,t)\in X$ *and close to* p. *Moreover*, (1.5) *implies*

$$|y_0\partial F/\partial y_0(y,t)| \ll |(\partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1})(y,t)|, \tag{1.7}$$

locally near p *and as* $X \ni (y,t) \to X_\infty$.

Proof. (1.6) follows trivially (1.5). We show that (1.6) implies (1.5) and (1.7). This suffices to be shown on each real analytic curve $(y(s),t(s))$, where $s \in [0,\varepsilon)$, $X \ni (y(s),t(s) \to p' \in X_\infty$. Since $F(y(s),t(s)) \equiv 0$

$$0 = \frac{d}{ds}F(y(s),t(s)) = \frac{dt}{ds}\frac{\partial F}{\partial t} + \frac{dy_0}{ds}\frac{\partial F}{\partial y_0} + \sum_{i=1}^{n-1}\frac{dy_i}{ds}\frac{\partial F}{\partial y_i},$$

which gives

$$\begin{aligned}\left|y_0\frac{\partial F}{\partial y_0}\right| \sim s\left|\frac{dy_0}{ds}\frac{\partial F}{\partial y_0}\right| &\le C|(y,t)-p'|\,|(\partial F/\partial t, \partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1})| \\ &\ll \|(\partial F/\partial t, \partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1})\|.\end{aligned}$$

This together with (1.6) gives (1.5) and (1.7). This ends the proof of Lemma 1.6. ∎

To complete the proof of Theorem 1.3 it suffices to note that (1.6) translates exactly to Malgrange's Condition when we return to the original affine coordinates $(x_1,\ldots,x_n)$. This ends the proof of Theorems 1.2 and 1.3. ∎

Let $\mathcal{F} = i_*\mathbf{C}_X[-1]$. Then $\mathrm{Car}(\mathcal{F})$ is a positive cycle and the sheaf $\Phi_{\overline{f}-t_0}\mathcal{F}$ is perverse on $\mathbf{P}^n \times \{t_0\}$. Suppose now that p is an isolated point of $\Phi(\overline{f})$, $t_0 = \overline{f}(p)$. Then, by perversity of $\Phi_{\overline{f}-t_0}\mathcal{F}$, the stalk cohomologies of the complex $\Phi_{\overline{f}-t_0}\mathcal{F}(p)$ are nonzero only in the middle dimension n. Of course, in this case, the "geometric" vanishing cycles of $\overline{f}$ appear only in dimension $n-1$. For such p denote

$$\mu(p) = (-1)^n\chi(\Phi_{\overline{f}-t_0}\mathcal{F}(p)).$$

Note that $\mu(p) > 0$ and equals the Milnor number of f at p if p is affine (i.e. $p \in X\setminus X_\infty$).

Given $t\in\mathbf{C}$, we denote by F_t the fibre of f at t. By F_g we denote the generic fibre of f. The following corollary of Theorem 1.2 and the index formula (1.3) generalizes the Hà-Lê Theorem [Hà-Lê] and its previous higher dimensional generalizations [Pa, Theorem 1], [S-T, 5.6].

COROLLARY 1.7. *Given* $t_0 \in \mathbf{C}$. *Suppose that* $\mathrm{supp}\,(\Phi_{\overline{f}-t_0}\mathcal{F})$ *is finite or empty. Then* t_0 *is atypical if and only if* $\chi(F_{t_0}) \ne \chi(F_g)$.

Proof. Clearly if t_0 is typical the Euler characteristic does not change. To show the converse it is convenient (though not necessary) to use the formalism of perverse sheaves. First note that since for all t, $\chi(\overline{f}^{-1}(t)) - \chi(F_t) = \chi(C_\infty)$ does not depend on t, one may replace in the statement $\chi(F_{t_0}) \neq \chi(F_g)$ by $\chi(\overline{f}^{-1}(t_0)) \neq \chi(\overline{f}^{-1}(t_g))$, where t_g is chosen generic.

Let $\Sigma_{t_0} = \operatorname{supp}(\Phi_{\overline{f}-t_0}\mathcal{F})$ be finite. Then, we rewrite (1.3) as

$$(1.8) \qquad (\operatorname{Car}(\mathcal{F}).\{dt\})_{\Sigma_{t_0}} = \sum_{p\in\Sigma_{t_0}} \mu(p) = (-1)^{n-1}(\chi(\overline{f}^{-1}(t_g)) - \chi(\overline{f}^{-1}(t_0))),$$

where t_g is chosen generic. Since $\operatorname{Car}(\mathcal{F})$ is a positive cycle and Σ_{t_0} finite, the left-hand side of the above formula is nonnegative. It is zero exactly if $\operatorname{Car}(\mathcal{F})$ and $\{dt\}$ do not intersect over t_0, that is if $\overline{f}$ is non-characteristic over t_0. But this implies, by Theorem 1.2, that t_0 is typical. This ends the proof. ∎

Remark 1.8. Note that for our crucial argument (that is Theorem 1.4) it is $\operatorname{Car}(j_!\mathbf{C}_U)$ and not $\operatorname{Car}(X)$ that really matters. In particular, our method can be applied to other compactifications of f. Here is one of possible statements.

Let X be a compactification of $\mathbf{C}^n$ such that f extends onto X and let $\overline{f}$ denote this extension. Let U denote the image of $\mathbf{C}^n$ in X and $j : U \to X$ the embedding. Let $X_\infty = X \setminus U$ be the divisor at infinity.

We also assume that X is a hypersurface in a nonsingular variety which implies that $j_!\mathbf{C}_U$ is perverse. Take $p \in X_\infty$, $\overline{f}(p) = t_0$. Then our argument shows:

(i) If $p \notin \Phi_{\overline{f}-t_0}(j_!\mathbf{C}_U)$ then inequality (1.5) holds near p;

(ii) If $\operatorname{supp}(\Phi_{\overline{f}-t_0}(j_!\mathbf{C}_U)) = \emptyset$, then t_0 is typical.

Indeed, in this case, we may trivialize f using the technique developed in Section 2 below.

Remark 1.9. The key step of our proof, the theorem of Ginsburg, was originally stated and proven in the D-module set-up. Its topological counterpart, which we are using, can be proven by purely topological means as in [MM] (see also [LM]).

Remark 1.10. In some cases Malgrange's Condition can be strengthened. For instance, if $n = 2$ then Malgrange's Condition at t_0 is equivalent to the "tameness" at t_0, that is

$$(1.9) \qquad \exists\delta > 0 \quad |\operatorname{grad} f(x)| \geq \delta\,,$$

as $|x| \to \infty$ and $f(x) \to t_0$, see [Hà]. Also if f is convenient with nondegenerate principal part at infinity, then $|\operatorname{grad} f(x)| \geq \delta > 0$ for $|x|$ sufficiently large, see [Ph].

Suppose that all Y_j appearing in the decomposition $\operatorname{Car}(j_!\mathbf{C}_U) = T^*_X(\mathbf{P}^n \times \mathbf{C}) + \sum m_j T^*_{Y_j}(\mathbf{P}^n \times \mathbf{C})$ are either trivial $Y_j = C_j \times \mathbf{C}$ or contained in the special fibres of $\overline{f}$. This holds for instance if $\operatorname{Sing} X = A \times \mathbf{C}$ is one-dimensional, the case considered in details in [Pa]. Then, by Theorem 1.4, the same triviality holds for the cycles Z_j appearing in the decomposition of $W = T^*_g|_{X_\infty}$. Thus near $p \in X_\infty$, provided $(p, dt) \notin \operatorname{Car}(X)$, the local inequality characterizing this condition is

$$|\partial F/\partial t(y)| \ll |(\partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1})(y)|\,,$$

as $y \to X_\infty$. Now we may argue as in [Pa, §3.1]. First, by ojasiewicz Inequality [], the above condition is equivalent to

$$\exists N > 0 \qquad |\partial F/\partial t| \leq |y_0|^{1/N} |(\partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1})|\,,$$

which is equivalent to the following affine condition

$$(1.10) \qquad \exists N \geq 0\, \exists \delta > 0 \qquad |x|^{(N-1)/N} |\mathrm{grad}\, f(x)| \geq \delta\,.$$

If the above condition is satisfied, then we may actually trivialize $\overline{f}$ so that the induced trivialization on X_∞ is by the product $X_\infty = C_\infty \times \mathbf{C}$. The reader may consult [Pa] for the details.

EXAMPLE 1.11. In general, for $n > 2$, one cannot expect Malgrange's Condition to be equivalent to (1.10). The following example shows that condition (1.10) is not generic that is may fail over all values of f.

Let $f(x,y,z) = x + x^2y + x^4yz$. Given $t_0 \neq 0$. Consider the curve

$$\gamma(s) = (x(s), y(s), z(s)) = (s, 2t_0 s^{-2}, -\frac{1}{2} s^{-2}(1 + (4t_0)^{-1}s)),$$

as $s \to 0$. Then $f(\gamma(s)) \to t_0$, $\partial f/\partial x(\gamma(s)) \equiv 0$ and it is easy to check that

$$|\gamma(s)|\, |\mathrm{grad}\, f(\gamma(s))| \to \text{const.} \neq 0,$$

so (M) cannot be improved to (1.10).

2. A bouquet theorem.

THEOREM 2.1. *Suppose that* $\mathrm{supp}\,(\Phi(\overline{f}))$ *is finite. Then the generic fibre of* f *is homotopy equivalent to a bouquet of spheres of dimension* $n-1$.

Proof. First we give an outline of the proof. Fix for each $p \in X_\infty$ a local coordinate system $y_0, y_1, \ldots, y_{n-1}, t$ at p such that the hyperplane at infinity is given by $y_0 = 0$. We construct a global nonnegative "control function" $\varphi : X \to \mathbf{R}$, which locally behaves as $|y_0|^2$. Actually we do assume that φ is of such form $\varphi = |y_0|^2$ near all points of $\Sigma_\infty = \mathrm{supp}\,(\Phi(\overline{f})) \cap X_\infty$. We will also require that the fibres $\varphi^{-1}(\varepsilon)$, for sufficiently small $\varepsilon > 0$, are transverse to the fibres of $\overline{f}$ everywhere except the points on the polar curves Γ_i defined at $p_i \in \Sigma_\infty$. Take any $t_0 \in \mathbf{C}$, such that $\overline{f}^{-1}(t_0) \cap \Sigma_\infty = \emptyset$ and a disc $D_N = \{t \in \mathbf{C} \mid |t - t_0| < N\}$, with N large enough so it contains all atypical values of f. Then we apply the Morse Theory to $\psi = |f - t_0|^2$ restricted to $M_\varepsilon = \varphi^{-1}([\varepsilon, \infty)) \cap \overline{f}^{-1}(D_N)$, where $\varepsilon > 0$ is sufficiently small. We show that all critical points of ψ, not on the zero fibre $\psi^{-1}(0)$, are isolated, and they either are the critical points of f or lie on $\partial M_\varepsilon \cap \Gamma_i$. It is well known that each point of the first type can be perturbed to finitely many nondegenerate critical points of index n. By a local calculation we show that the points of the second type make a similar contribution to the homotopy type of M_ϵ. Consequently we show that, up to homotopy, M_ε can be obtained from $F_{t_0} = f^{-1}(t_0)$ by attaching a finite number of cells of dimension n. Since M_ε is contractible and F_{t_0} has a homotopy type of an $n-1$-dimensional CW-complex, F_{t_0} has a homotopy type of a bouquet of spheres by Whitehead Theorem.

Now we present the details of our construction. Since the role of (affine) isolated critical points of f is classical and well-understood we concentrate on the points of $\Sigma_\infty = \mathrm{supp}\,(\Phi(\overline{f})) \cap X_\infty$. Let L be the line bundle on $\mathbf{P}^n \times \mathbf{C}$ associated to the divisor $H_\infty \times \mathbf{C}$. Then $H_\infty \times \mathbf{C}$ is the zero set of a holomorphic section $\mathbf{v}$ of L. Choose a Hermitian metric on L and let $\varphi : X \to \mathbf{R}$ be given by

$$\varphi(p) = \|\mathbf{v}(p)\|^2.$$

For each $p_i \in \Sigma_\infty$ fix a local coordinate system $y_0, y_1, \dots, y_{n-1}, t$ at p_i like the one used in the proof of Theorem 1.3 and assume that in a neighbourhood U_i of each $p_i \in \Sigma_\infty$ our metric is standard with respect to this system.

LEMMA 2.2. *There is a neighbourhood U of X_∞ in X such that φ is regular on $U \setminus X_\infty$. Moreover, on $U \setminus \bigcup U_i$ the levels of φ and $\overline{f}$ are transverse.*

Proof. On $U_i \setminus X_\infty$, $\varphi = |y_0|^2$ is regular.

Suppose $p \in X_\infty \setminus \operatorname{supp}(\Phi(\overline{f}))$. Let $y_0, y_1, \dots, y_{n-1}, t$ be a local system of coordinates at p such that $y_0 = 0$ defines $H_\infty \times \mathbf{C}$ locally at p. Then $\mathbf{v} = y_0\mathbf{e}$, where $\mathbf{e}$ is a holomorphic section of L defined and nowhere vanishing in a neighbourhood of p. To prove the lemma it suffices to show that near p and on $X \setminus X_\infty$, dF, dt, and the holomorphic part of $d|\mathbf{v}|^2$ are linearly independent.

Denote by D the holomorphic connection associated to the metric on L. Let θ denote the local connection form associated to $\mathbf{e}$ as a local frame at p. Then

$$d|\mathbf{v}|^2 = (D\mathbf{v}, \mathbf{v}) + (\mathbf{v}, D\mathbf{v}) = (\bar{y}_0 dy_0 + |y_0|^2\theta)|\mathbf{e}|^2 + \overline{(\bar{y}_0 dy_0 + |y_0|^2\theta)}|\mathbf{e}|^2.$$

Hence the holomorphic part of $d|\mathbf{v}|^2$ is parallel to $dy_0 + y_0\theta$. But it follows from (1.5) and (1.7) of Lemma 1.6 that

$$dF = \frac{\partial F}{\partial y_0} dy_0 + \sum_{i=1}^{n-1} \frac{\partial F}{\partial y_i} dy_i + \frac{\partial F}{\partial t} dt, \qquad dy_0 + y_0\theta, \quad \text{and} \quad dt$$

are linearly independent. This shows the lemma. ∎

Fix $t_0 \in \mathbf{C}$, such that $f^{-1}(t_0)$ has no singularities at infinity that is $\overline{f}^{-1}(t_0) \cap \Sigma_\infty = \emptyset$. Choose N large enough so that $D_N = \{t \in \mathbf{C} \mid |t - t_0| < N\}$ contains all atypical values of f. Let $M_\varepsilon = \varphi^{-1}([\varepsilon, \infty)) \cap \overline{f}^{-1}(D_N)$, where $\varepsilon > 0$ is sufficiently small. Then, Lemma 2.2 implies that both $M_\varepsilon \hookrightarrow X \setminus X_\infty \simeq \mathbf{C}^n$ and $M_\varepsilon \cap F_{t_0} \hookrightarrow F_{t_0}$ are homotopy equivalences.

Consider $\psi : M_\varepsilon \to \mathbf{R}$ defined by

$$\psi(x) = |f(x) - t_0|^2.$$

The critical points of ψ which are not in the zero fibre $\psi^{-1}(0)$, are either critical points of f or the points on the boundary ∂M_ε and these lie in the neighbourhoods U_i. By assumption, the points of the first type are isolated and they are in the interior $\operatorname{Int}(M_\varepsilon)$. Near such a point p, ψ can be perturbed to a Morse function with exactly $\mu(p)$ nondegenerate critical points of index n.

To study the critical points on the boundary ∂M_ε we follow closely [Hm-Lê]. Fix $p_i \in \Sigma_\infty$ and the associated local coordinate system $y_0, \dots, y_{n-1}, t$. Then the levels of $\overline{f}$ and $y_0|_X$ are not transverse exactly along the polar curve

(2.1) $\qquad \Gamma_i = \text{Closure}\,\{(y,t) \in X \setminus X_\infty \mid \partial F/\partial y_1 = \dots = \partial F/\partial y_{n-1} = F = 0\}.$

Hence the singularities of $\psi|_{\partial M_\varepsilon}$ must lie on Γ_i. Fix a branch Γ of Γ_i and choose a local parametrization $\gamma(s) = (y(s), t(s))$ of Γ such that $y_0 = s^m$, $m > 0$. On Γ, $\overline{f}$ cannot be constant since then $\partial F/\partial t$ would also vanish on Γ which together with (2.1) would imply $\Gamma \subset \operatorname{Sing} X$. Hence

$$(\overline{f} - t_0)(s) = a_0 + a_k s^k + \ldots,$$

where $a_0 \neq 0$, $a_k \neq 0$, and $k > 0$. Let $s = re^{i\alpha}$. Then

$$\psi(\gamma(s)) = |a_o|^2 + 2r^k \operatorname{Re}(a_0 \bar{a}_k e^{ik\alpha}) + \ldots.$$

Hence

$$\begin{aligned} \partial\psi/\partial\alpha &= 2kr^k \operatorname{Re}(ia_0\bar{a}_k e^{ik\alpha}) + \ldots \\ \partial^2\psi/\partial\alpha^2 &= -2k^2r^k \operatorname{Re}(a_0\bar{a}_k e^{ik\alpha}) + \ldots \\ \partial\psi/\partial r &= 2kr^{k-1} \operatorname{Re}(a_0\bar{a}_k e^{ik\alpha}) + \ldots. \end{aligned}$$

The above formulas show that, on $\{|y_0|^2 = \varepsilon\}$, $\varepsilon > 0$ and sufficiently small, ψ has exactly $2k$ critical points. At these points $\partial^2\psi/\partial\alpha^2 \neq 0$ and

$$\operatorname{sign}(\partial^2\psi/\partial\alpha^2) = -\operatorname{sign}(\partial\psi/\partial r). \tag{2.2}$$

Suppose, for simplicity, that the points on $\Gamma \setminus p_i$ are of multiplicity 1, that is $\overline{f}$ restricted to each nonzero level $\{y_0 = c\} \cap X$, has at the points of Γ nondegenerate critical points. Fix a critical point p of ψ. Now we use the parametrization γ of Γ which we described above. Let $p = \gamma(s_0)$ and we assume that $s_0 = \varepsilon^{2/m}$. Then $\alpha = \operatorname{Arg}(s)$ and $z_1 = y_1 - \gamma_1(s_0e^{i\alpha}), \ldots, z_{n-1} = y_{n-1} - \gamma_{n-1}(s_0e^{i\alpha})$ form a local coordinate system on ∂M_ε at p. Consequently $\partial\psi/\partial z_i = \partial\psi/\partial y_i = 0$ on $\partial M_\varepsilon \cap \Gamma$ and hence $\partial^2\psi/\partial z_i\partial\alpha(p) = 0$. In particular, the Hessian $H(\psi|_{\partial M_\varepsilon})(p)$ splits with respect to coordinates $z = (z_1, \ldots, z_{n-1})$ and α. The index coming from the first summand is $n-1$. By (2.2) the index coming from the second one $\partial^2\psi/\partial\alpha^2$ is 1 if the gradient $\operatorname{grad}\psi(p)$ is directed outward of M_ε and -1 if the gradient is directed inward. Consequently, by the Morse theory on manifolds with boundary, see e.g. [Hm-Lê], only the latter critical points contribute to the homotopy type of the sets $\{\psi \leq \text{const.}\}$, and each of them gives a cell of dimension n.

The same property holds even if the points on Γ are not of multiplicity 1. Indeed, then we perturb ψ near its critical points by taking for instance $\widetilde{\psi}(y,t) = |t - t_0 - \sum_{i=1}^{n-1} a_i y_i|^2$, where a_i are small and generic. Since such perturbation preserves the property (2.2), the critical points we get make a similar contribution to the homotopy type.

Consequently we show that, up to homotopy, M_ε can be obtained from F_{t_0} by attaching a finite number of cells of dimension n. Since M_ε is contractible and F_{t_0} has a homotopy type of an $n-1$-dimensional CW-complex, F_{t_0} has a homotopy type of a bouquet of spheres by Whitehead Theorem. This ends the proof of Theorem 2.1. ∎

Remark 2.3. Our proof shows that Theorem 2.1 still holds for such special fibres F_{t_0} of f which do not have singularities at infinity. On the other hand, Siersma and Tibăr [S-T] give the following example of a special fibre for which the statement of Theorem 2.1 does not hold: $f(x,y) = x^2y + x : \mathbf{C}^2 \to \mathbf{C}$ and $t_0 = 0$.

Note that our arguments work and the statement of Theorem 2.1 remains true, for all t_0, if we replace the homotopy type of the fibre $f^{-1}(t_0)$ by the one of $f^{-1}(D_{t_0})$, where D_{t_0} is a small disc around t_0. But if $f^{-1}(t_0)$ has singularities at infinity and we repeat the construction of the proof of Theorem 2.1, we find out that all the critical points of $\psi = |f - t_0|^2$ on the polar curves corresponding to $p_i \in \Sigma_\infty \cap \overline{f}^{-1}(t_0)$ are essential. In this case, all the gradients are directed inward and a half of critical points is of index n and the other half of index $n-1$.

In particular, the above example shows that for atypical t_0 the inclusion $f^{-1}(t_0) \hookrightarrow f^{-1}(D_{t_0})$ need not be a homotopy equivalence.

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