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THE OBSTACLE PROBLEM AND DIRECT PRODUCTS OF UNFURLED SWALLOWTAILS

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1. Introduction. The obstacle problem has been studied in papers by Arnold, Givental, Shcherbak and other authors. This is the problem of investigating Lagrangian varieties naturally arising in variational problems with one-sided constraints.

EXAMPLE. Let us consider a Euclidean space M and an obstacle in it bounded by a smooth hypersurface Γ. The shortest path between two points $u, v \in M$ going around the obstacle consists of an interval of a ray (oriented straight line) l_1 going through the point u and tangent to Γ, of an interval of a geodesic l_2 on Γ tangent to l_1 and of an interval of a ray l_3 which is tangent to l_2 and goes through the end point v. We call l_1 *the inbound ray* (or the inbound geodesic), l_3 — *the outbound*.

Considering the shortest paths between u and points in some neighbourhood of v we get a family of inbound rays, a family of geodesics on Γ starting at points of tangency between the inbound rays and Γ and a family of outbound rays.

The family of geodesics on Γ starting at the points of tangency determines a Lagrangian variety $L_\Gamma \subset T^*\Gamma$: $L_\Gamma = \{(q,p) \in T^*\Gamma \mid$ *q belongs to a geodesic from the family*, *p is tangent to this geodesic*, $\|p\| = 1\}$.

DEFINITION. L_Γ is called the variety consisting of geodesics on Γ starting at points of tangency.

The variety L_Γ lies in the hypersurface H'' of all unit (co)vectors on Γ. Let l_Γ denote the image of L_Γ in the symplectic space of the characteristics of the hypersurface H''.

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In the same way the outbound rays determine a Lagrangian variety $L_M \subset H \subset T^*M$, where H is the hypersurface of all unit (co)vectors on M.

DEFINITION. L_M is called the variety consisting of outbound geodesics tangent to the family of geodesics on Γ given by l_Γ.

Let l_M denote the image of L_M in the symplectic space of the characteristics of the hypersurface H.

DEFINITION. l_M is called the variety of outbound geodesics tangent to the family of geodesics on Γ given by l_Γ.

Lagrangian varieties described above are really varieties but not manifolds. The simplest singularity is the unfurled (open) swallowtail.

DEFINITION ([3]). The *unfurled swallowtail* τ_n of dimension n is a Lagrangian subvariety in the symplectic space P_{2n} of polynomials of the form

$$x^{2n+1}/(2n+1)! + Q_1 x^{2n-1}/(2n-1)! + \ldots + Q_n x^n/n! - P_n x^{n-1}/(n-1)! + \ldots + (-1)^n P_1$$

(Q, P — Darboux coordinates) formed by polynomials with a root of multiplicity exceeding n.

The k-dimensional suspension over τ_n is $\tau_n \times \mathbf{R}^k \subset P_{2n} \times T^*\mathbf{R}^k$. It is denoted by $\tau_{n+k,k}$.

The following was proved in [3].

THEOREM 1. *If l_Γ is smooth then generically l_M is symplectomorphic to $\tau_{n+k-1,n}$ $(k+n = \dim M)$ in some neighbourhood of an outbound geodesic tangent to Γ of order k.*

It turns out that the assumption "l_Γ is smooth" generically is violated. Namely ([5]):

THEOREM 2. *Generically l_Γ is symplectomorphic to $\tau_{m+k-1,m}$ $(m+k = \dim \Gamma)$ in some neighbourhood of a geodesic on Γ which starts at a point of order k tangency between Γ and an inbound geodesic.*

So, we have a very natural question: what is l_M if l_Γ is singular? The following gives partial answer to this question.

1. *Generically l_M is formally* (*at least*) *diffeomorphic to $\tau_1 \times l_\Gamma$ in some neighbourhood of an outbound geodesic tangent to Γ of order* 2.

2. *If l_Γ is locally diffeomorphic to $l \times \mathbf{R}$, where l is some analytic variety*, *then generically l_M is formally* (*at least*) *diffeomorphic to $\tau_2 \times l$ in some neighbourhood of an outbound geodesic tangent to Γ of order* 3.

Remark. It follows from Theorem 2 that generically the condition "l_Γ is locally diffeomorphic to $l \times \mathbf{R}$" is fulfilled except when $\dim M = 4$ and l_Γ is considered in some neighbourhood of an inbound ray tangent to Γ of order 3. Indeed, l_Γ is locally symplectomorphic to $\tau_{m+k-1,m}$ and $m = 0$ along isolated geodesics on Γ. Along such "the most degenerate" single geodesic the order of tangency between Γ and the outbound ray is equal to 1, at isolated points the order of tangency is equal to 2, and nowhere except of the tangency point between Γ and the incoming geodesic the order of tangency exceeds 2.

Combining these results with Theorem 2 we get the following:

Generically in the obstacle problem

1. l_M *is formally diffeomorphic to* $\tau_1 \times \tau_{n,k}$, $n = \dim M - 2$, *in some neighbourhood of an outbound geodesic tangent to* Γ *of order* 2.

2. l_M *is formally diffeomorphic to* $\tau_2 \times \tau_{n-1,k}$, $n = \dim M - 2$, *in some neighbourhood of an outbound geodesic tangent to* Γ *of order* 3 *provided either* $\dim M \neq 4$ *or the outbound ray does not coincide with the corresponding inbound ray.*

2. The obstacle problem in terms of symplectic geometry. Let M be the configuration manifold of a Hamiltonian system with a Hamiltonian h, $\Gamma = \{q \in M \mid \widetilde{F}(q) = 0\}$ be a smooth hypersurface in M restricting an obstacle, T^*M be the total space of the cotangent bundle over M, ω be the standard symplectic form on T^*M, $\pi : T^*M \to M$ be the cotangent bundle projection, $F = \pi^*\widetilde{F}$. Let us consider the hypersurfaces $H = \{x \in T^*M \mid h(x) = 0\}$ and $T^*_\Gamma M = \{x \in T^*M \mid F(x) = 0\}$. Let $\rho : T^*_\Gamma M \to T^*\Gamma$ be the natural projection along the characteristics of the hypersurface $T^*_\Gamma M$, $\kappa : H \to N$ be the natural projection along the characteristics of the hypersurface H (locally N is a symplectic manifold).

Lagrangian varieties formed by extremals of the action functional of a Hamiltonian system with one-sided constraint can be described as follows ([1], [2]):

Let $B \subset M$ be a submanifold (initial front), $L_B \subset T^*M$ be a Lagrangian submanifold of covectors at points of B vanishing on tangent to B spaces, L be the union of the characteristics of the hypersurface H going through the points of $L_B \cap H$. For B in general position L is a Lagrangian manifold (at least locally). For B and Γ in general position the manifold L transversally intersects $T^*_\Gamma M$, hence $l = T^*_\Gamma M \cap L$ is smooth and not tangent to the characteristics of the hypersurface $T^*_\Gamma M$, hence $\rho(l) \subset T^*\Gamma$ is a Lagrangian submanifold. Let us consider H' and H'' — the sets of critical points and critical values of the restriction $\rho|_{H\cap T^*_\Gamma M}$. Denote the projection along the characteristics of the hypersurface $H'' \subset T^*\Gamma$ by $\kappa'' : H'' \to N''$, the union of the characteristics of H'' going through the points of $\rho(l) \cap H''$ by L_Γ, its image in N'' by l_Γ. Generically L_Γ is singular ([5]).

Denote the union of the characteristics of the hypersurface H going through the points of $H \cap \rho^{-1}(L_\Gamma)$ by L_M. Finally we introduce a filtration of $H \cap T^*_\Gamma M = H^{(0)}$ by the order of tangency between the characteristics of H and the hypersurface $T^*_\Gamma M$: $H^{(0)} \supset H^{(1)} \supset \ldots$, where $H^{(i)} = \{x \in T^*M \mid F(x) = h(x) = \langle F, h\rangle(x) = \ldots = \big\langle \ldots \langle F, h\rangle, \ldots, h\big\rangle(x) = 0\}$ (i times the Poisson bracket $\langle\cdot,\cdot\rangle$). It is not difficult to see that $H' = H^{(1)}$.

Example. In the previous example concerning extremals on a Riemannian manifold with a boundary we have: $H = \{(q,p) \in T^*M \mid \|p\| = 1\}$, $H^{(0)}$ is the set of all unit covectors at points of Γ, $H^{(1)}$ is the set of all unit covectors tangent to Γ, $H^{(2)} \setminus H^{(3)}$ — tangent to Γ of order 2 etc. The initial front B is the initial point u, the manifold L is given by a system of rays (geodesics) going through u: $L = \{(q,p) \in T^*M \mid q$ *belongs to a geodesic going through* u, p *is tangent to this geodesic*, $\|p\| = 1\}$. The varieties L_Γ, l_Γ, L_M, l_M are exactly the same as described above.

Keeping in mind this example, for any Hamiltonian, we will call L_Γ the variety consisting of geodesics on Γ starting at points of tangency; l_Γ — the variety of geodesics on Γ starting at points of tangency; L_M — the variety consisting of outbound geodesics tangent to the family of geodesics on Γ given by l_Γ and l_M — the variety of outbound geodesics tangent to the family of geodesics on Γ given by l_Γ.

3. Results. In what follows we assume that the Hamiltonian h is quadratic and convex in momenta.

THEOREM 3. *Generically l_M is formally diffeomorphic to $\tau_1 \times l_\Gamma$ in some neighbourhood of a characteristic of the hypersurface H tangent to $T^*_\Gamma M$ of order* 2 (*i.e. going through a point of* $H^{(2)} \setminus H^{(3)}$).

Remark. The genericity conditions are the following:

1. The restriction of κ (the projection along the characteristics of the hypersurface H) to $H^{(0)}$ is locally equivalent to the A_2-singularity.
2. l_Γ is locally diffeomorphic to an analytic variety.

THEOREM 4. *If l_Γ is locally diffeomorphic to $l \times \mathbf{R}$ for some analytic l then generically l_M is formally diffeomorphic to $\tau_2 \times l$ in a neighbourhood of a characteristic of the hypersurface H tangent to $T^*_\Gamma M$ of order* 3 (*i.e. going through a point of* $H^{(3)} \setminus H^{(4)}$).

Remarks.

1. The genericity conditions are the following:

1.1. The restriction of κ to $H^{(0)}$ is locally equivalent to the A_3-singularity.

1.2. Let $x \in \rho^{-1}(L_\Gamma) \cap (H^{(3)} \setminus H^{(4)})$ be the point under consideration (more precisely, the characteristic considered in the theorem goes through this point). We need the following: at the point $\rho(x) = (y_1, y_2) \in l \times \mathbf{R}^2 (\cong L_\Gamma)$ the edge $y_1 \times \mathbf{R}^2$ is transversal to $\rho(H^{(3)})$ in H''.

2. It is not difficult to see that the decomposition of l_Γ into $l \times \mathbf{R}$ (or L_Γ into $l \times \mathbf{R}^2$) generically is possible except when $\dim M = 4$ and the considered geodesics of the hypersurface H belongs to L (issues from the initial front). The reasons are exactly the same as in the case of geometrical optics (see the introduction).

4. Proofs. The main result we use in the proofs is the following theorem (proved in [1]) which gives the symplectic classification of pairs $(H, H^{(0)})$, where H is a hypersurface in a symplectic manifold, $H^{(0)}$ is a hypersurface in H.

THEOREM 5. *In some neighbourhood of a point where the restriction to $H^{(0)}$ of the natural projection along the characteristics of the hypersurface H is equivalent to the A_k-singularity* (*i.e. in some coordinates may be written in the form* $\{(x, t_1, \ldots, t_n) \mid x^{k+1} + x^{k-1}t_1 + \ldots + t_k = 0\} \mapsto (t_1, \ldots, t_n)$) *the pair* $(H, H^{(0)})$ *is reducible to the form* $(\{q_0 = 0\}, \{q_0 = F = 0\})$ *by a formal* (*at least*) *symplectomorphism. Here q, p are Darboux coordinates and*

(a) $F = p_0^2 + p_1$ *if* $k = 1$,
(b) $F = p_0^3 + p_1 p_0 + q_1$ *if* $k = 2$,
(c) $F = p_0^4 + p_1 p_0^2 + q_2 p_0 + p_2$ *if* $k = 3$.

R e m a r k. The case (a) was firstly proved in [4] on the C^∞–level.

4.1. *Proof of Theorem* 3. For a quadratic and convex in momenta Hamiltonian h we have $\rho^{-1}(H'') \subset H'(= H^{(1)})$ and the restriction of ρ to $H \cap T^*_\Gamma M$ is equivalent to the A_1-singularity. Hence the restriction of ρ to H' is a diffeomorphism and the hypersurface $H'' \subset T^*\Gamma$ is smooth. Denote by l_H the preimage of L_Γ in H: $l_H = \rho^{-1}(L_\Gamma) \cap H$. The variety l_H is isotropic (because $\rho^{-1}(L_\Gamma)$ is Lagrangian); $L_\Gamma \subset H''$, hence $l_H = (\rho|_{H^{(1)}})^{-1}(L_\Gamma)$, hence l_H is diffeomorphic to L_Γ.

Let $x \in l_H \cap (H^{(2)} \setminus H^{(3)})$, from part (b) of Theorem 5 it follows that the pair $(H, H^{(0)}) = (\{h = 0\}, \{h = 0\} \cap T^*_\Gamma M)$ can be reduced to the form $(\{q_0 = 0\}, \{q_0 = p_0^3 + p_1p_0 + q_1 = 0\})$ by a formal symplectomorphism. Hence in some neighbourhood of x we have:

$$\begin{aligned} &H\ (= \{q_0 = 0\}) \supset H^{(0)}\ (= \{q_0 = p_0^3 + p_1p_0 + q_1 = 0\}) \\ &\supset H^{(1)}\ (= \{q_0 = 0, p_1 = -3p_0^2,\ q_1 = 2p_0^3\}) \supset H^{(2)}\ (= \{q_0 = q_1 = p_0 = p_1 = 0\}), \end{aligned}$$

the functions $(p_0, q_2, \dots, q_n, p_2, \dots, p_n) = (p_0, q', p')$ are coordinate functions on $H^{(1)}$, $\omega|_{H^{(1)}} = \mathrm{d}p_2 \wedge \mathrm{d}q_2 + \ldots + \mathrm{d}p_n \wedge \mathrm{d}q_n$.

Now we are going to prove that $l_H \in H^{(1)}$ can be given by a system of equations not dependent on p_0. The main ingredient of the proof is the fact that l_H is isotropic.

PROPOSITION 1. *At points $y \in l_H$ where l_H is smooth $\partial/\partial p_0 \in T_y l_H$.*

P r o o f. Let us consider an auxiliary fibration of coordinate spaces: $\alpha : \mathbf{R}^{2n-1} \to \mathbf{R}^{2n-2}$, $\alpha : (p_0, q', p') \mapsto (q', p')$, and the symplectic structure $\omega' = \mathrm{d}p' \wedge \mathrm{d}q'$ on $\mathbf{R}^{2n-2}$. Our $l_H \subset \mathbf{R}^{2n-1}$ and $(\alpha^*\omega')|_{l_H} = (\omega|_{H^{(1)}})|_{l_H} = \omega|_{l_H} = 0$ because l_H is isotropic. Assume that $\partial/\partial p_0 \notin T_y l_H$ and l_H is smooth at y. Projecting $T_y l_H$: $\alpha_{*,y} : T_y l_H \to T_{\alpha(y)}\mathbf{R}^{2n-2}$ we get an isotropic subspace of dimension n in the symplectic space $T_{\alpha(y)}\mathbf{R}^{2n-2}$. This contradiction proves the proposition. ∎

It follows from Proposition 1 that for any smooth function g on $H^{(1)}$, vanishing on l_H, $\partial g/\partial p_0 = 0$ at points where l_H is smooth. But the singular locus is a proper and closed subset of l_H and $\partial g/\partial p_0$ is continuous hence $\partial g/\partial p_0|_{l_H} = 0$. Let us denote by $C^\infty(H^{(1)})$ the ring of germs at x of infinitely smooth functions on $H^{(1)}$, by $J(l_H)$ the ideal of germs of vanishing on l_H functions. Let $f = (f_1, \dots, f_r)^T$, where f_i are some smooth representatives of the ideal $J(l_H)$ generators. We get:

$$\partial f/\partial p_0(p_0, q', p') = \xi(p_0, q', p')f(p_0, q', p')$$

where $\xi = \|\xi_{i,j}\|$ is an $r \times r$-matrix, $\xi_{i,j}$ are some smooth functions. Let $\Psi(p_0, q', p')$ be a principal matrix solution of the system of ordinary differential equations dependent on the parameters q', p'

$$dy/dp_0 = \xi(p_0, q', p')y.$$

Then $f(p_0, q', p') = \Psi(p_0, q', p')\psi(q', p')$, where $\psi = (\psi_1, \dots, \psi_r)^T$, $\psi_i(q', p')$ are some smooth functions. Hence $l_H = \{(p_0, q', p') \mid \psi_1(q', p') = \ldots = \psi_r(q', p') = 0\}$. From this and the fact that l_H ($\cong L_\Gamma$) is locally diffeomorphic to $l_\Gamma \times \mathbf{R}$ it follows that the variety $\{(q', p') \mid \psi(q', p') = 0\}$ is locally diffeomorphic to l_Γ. Projecting l_H along the characteristics of the hypersurface H we get:

$$l_M = \{(q_1, q', p_1, p') \mid 4p_1^3 + 27q_1^2 = 0, \psi(q', p') = 0\}.$$

Hence l_M is locally diffeomorphic to $\tau_1 \times l_\Gamma$.

4.2. *Proof of Theorem* 4. Let $x \in l_H \cap (H^{(3)} \setminus H^{(4)})$, from part (c) of Theorem 5 it follows that the pair $(H, H^{(0)}) = (\{h = 0\}, \{h = 0\} \cap T^*_\Gamma M)$ can be reduced to the form $(\{q_0 = 0\},\ \{q_0 = p_0^4 + p_1p_0^2 + q_2p_0 + p_2 = 0\})$ by a formal symplectomorphism. Hence in some neighbourhood of x we have

$$\begin{aligned} H\ (= \{q_0 = 0\}) &\supset H^{(0)}\ (= \{q_0 = p_0^4 + p_1p_0^2 + q_2p_0 + p_2 = 0\}) \\ &\supset H^{(1)}\ (= \{q_0 = 0,\ q_2 = -4p_0^3 - 2p_1p_0,\ p_2 = 3p_0^4 + p_1p_0^2\}) \\ &\supset H^{(2)}\ (= \{q_0 = 0,\ p_1 = -6p_0^2,\ q_2 = 8p_0^3,\ p_2 = -3p_0^4\}), \end{aligned}$$

the functions $(p_0, q_1, p_1, q_3, \ldots, q_n, p_3, \ldots, p_n) = (p_0, q_1, p_1, q'', p'')$ are coordinate functions on $H^{(1)}$, the restriction of the symplectic structure to $H^{(1)}$:

$$\omega' = \omega|_{H^{(1)}} = \mathrm{d}p_1 \wedge \mathrm{d}(q_1 + 12p_0^5/5 + 2p_1p_0^3/3) + \mathrm{d}p'' \wedge \mathrm{d}q''$$

or, after the change $Q_1 = q_1 + 12p_0^5/5 + 2p_1p_0^3/3$, $\omega' = \mathrm{d}p_1 \wedge \mathrm{d}Q_1 + \mathrm{d}p'' \wedge \mathrm{d}q''$. As in the proof of Theorem 3 one proves that l_H may be given by a system of equations independent of p_0:

$$l_H = \{(p_0, Q_1, p_1, q'', p'') \in H^{(1)} \mid \psi_1(Q_1, p_1, q'', p'') = \ldots = \psi_r(Q_1, p_1, q'', p'') = 0\}.$$

From the condition of the theorem $\big(L_\Gamma \cong l \times \mathbf{R}^2$, the edge $y_1 \times \mathbf{R}^2$ is transversal to $\rho(H^{(3)})$ at the point $\rho(x) = (y_1, y_2) \in l \times \mathbf{R}^2\big)$ it follows that the variety l_H is locally diffeomorphic to $l \times \mathbf{R}^2$, the edge $x_1 \times \mathbf{R}^2$ is transversal to the submanifold $H^{(3)} \subset H^{(1)}$, $H^{(3)} = \{p_0 = p_1 = 0\}$ at the point $x = (x_1, x_2) \in l \times \mathbf{R}^2$.

PROPOSITION 2.

$$\begin{aligned} T_x(x_1 \times \mathbf{R}^2) \subset \operatorname{span}\{&\partial/\partial p_0, (\partial\psi_i/\partial p_1)(x)\partial/\partial Q_1 + (\partial\psi_i/\partial p'')(x)\partial/\partial q'' \\ &- (\partial\psi_i/\partial Q_1)(x)\partial/\partial p_1 - (\partial\psi_i/\partial q'')(x)\partial/\partial p'',\ i = 1, \ldots, r\} \end{aligned}$$

P r o o f. We can locally decompose l_H into $\tilde{l} \times \mathbf{R}^m$, where m is maximal. Obviously $T_x(x_1 \times \mathbf{R}^2) \subset T_x(\tilde{x}_1 \times \mathbf{R}^m)$, where $x = (\tilde{x}_1, \tilde{x}_2) \in \tilde{l} \times \mathbf{R}^m$. We claim that

$$\begin{aligned} T_x(\tilde{x}_1 \times \mathbf{R}^m) = \operatorname{span}\{&\partial/\partial p_0, (\partial\psi_i/\partial p_1)(x)\partial/\partial Q_1 + (\partial\psi_i/\partial p'')(x)\partial/\partial q'' \\ &- (\partial\psi_i/\partial Q_1)(x)\partial/\partial p_1 - (\partial\psi_i/\partial q'')(x)\partial/\partial p'',\ i = 1, \ldots, r\}. \end{aligned}$$

This follows from the facts:

1. The image of l_H under the projection $\alpha : \mathbf{R}^{2n-1} \to \mathbf{R}^{2n-2}$, $\alpha : (p_0, Q_1, p_1, q'', p'') \mapsto (Q_1, p_1, q'', p'')$, is Lagrangian (the symplectic structure is $\mathrm{d}p_1 \wedge \mathrm{d}Q_1 + \mathrm{d}p'' \wedge \mathrm{d}q''$).
2. $T_x(\tilde{x}_1 \times \mathbf{R}^m) \cong \mathbf{R}\partial/\partial p_0 \oplus T_{\alpha(x)}\alpha(\tilde{x}_1 \times \mathbf{R}^m)$.
3. The tangent space to $\alpha(\tilde{x}_1 \times \mathbf{R}^m)$ is spanned by the Hamiltonian vector fields with Hamiltonians ψ_i (functions ψ_i are representatives of $J(l_H)$ and, simultaneously, representatives of $J(\alpha(l_H))$). ■

It follows from Proposition 2 that there exists ψ_i (say ψ_1) such that the vector

$$(\partial\psi_1/\partial p_1)(x)\partial/\partial Q_1 + (\partial\psi_1/\partial p'')(x)\partial/\partial q'' - (\partial\psi_1/\partial Q_1)(x)\partial/\partial p_1 - (\partial\psi_1/\partial q'')(x)\partial/\partial p''$$

is transversal to the hypersurface $\{p_1 = 0\}$, that is, $\partial\psi_1/\partial Q_1(x) \neq 0$. Using the implicit function theorem we get:

$$l_H = \{(p_0, Q_1, p_1, q'', p'') \in H^{(1)} \mid Q_1 = g_1(p_1, q'', p''),$$
$$g_i(p_1, q'', p'') = \psi_i(g_1(p_1, q'', p''), p_1, q'', p'') = 0,\ i = 2, \ldots, r\}.$$

In the coordinate (p_1, q'', p'')-space $\mathbf{R}^{2n-3}$ we consider the variety $l' = \{(p_1, q'', p'') \mid g_2(p_1, q'', p'') = \ldots = g_r(p_1, q'', p'') = 0\}$. The variety l_H can be constructed from l' by the embedding into the hypersurface $\{Q_1 = g_1(p_1, q'', p'')\}$ and the multiplication by p_0-axis. Hence l' is diffeomorphic to $l \times \mathbf{R}$ in some neighbourhood of the considered point $x' = (x_1, x_2') \in l \times \mathbf{R}$ $(= l \times \mathbf{R}^2 \cap \{p_0 = 0\})$. From the conditions of the theorem it follows that the edge $x_1 \times \mathbf{R}$ is transversal to the hypersurface $\{p_1 = 0\}$, hence it is transversal to $\{p_1 = \varepsilon\}$ for any sufficiently small ε. Thus the intersections $l' \cap \{p_1 = 0\}$ $(\cong l)$ and $l' \cap \{p_1 = \varepsilon\}$ $(\cong l)$ are locally diffeomorphic: there exists a diffeomorphism $G_\varepsilon : \{p_1 = 0\} \to \{p_1 = \varepsilon\}$ sending $\{(q'', p'') \mid g_i(0, q'', p'') = 0,\ i = 2, \ldots, r\}$ onto $\{(q'', p'') \mid g_i(\varepsilon, q'', p'') = 0,\ i = 2, \ldots, r\}$. The diffeomorphism

$$G : H^{(1)} \to H^{(1)},\ G : (p_0, Q_1, p_1, q'', p'') \mapsto (p_0, Q_1, p_1, G_{p_1}^{-1}(q'', p''))$$

brings the variety l_H to the form

$$l_H = \{(p_0, Q_1, p_1, q'', p'') \in H^{(1)} \mid Q_1 = g_1(p_1, G_{p_1}(q'', p'')) = \phi(p_1, q'', p''),$$
$$g_i(0, q'', p'') = 0,\ i = 2, \ldots, r\}.$$

Hence $l_H = \{(q, p) \in T^*M \mid q_0 = 0,\ q_2 = -4p_0^3 - 2p_1p_0,\ p_2 = 3p_0^4 + p_1p_0^2,\ q_1 = -12p_0^5/5 - 2p_1p_0^3/3 + \phi(p_1, q'', p''),\ g_i(0, q'', p'') = 0,\ i = 2, \ldots, r\}$. Changing $Q = q_1 - \phi(p_1, q'', p'')$ and projecting l_H along the characteristics of the hypersurface H (forgetting p_0) we finally get:

$$l_M = \{(Q, q'', p_1, p'') \mid g_2(0, q'', p'') = \ldots = g_r(0, q'', p'') = 0, \textit{the polynomial in } p_0 :$$
$$p_0^5/5 + p_1p_0^3/3 + q_2p_0^2/2 + p_2p_0 + Q/2 \textit{ has a root of multiplicity } \geq 3\}.$$

This proves the theorem.

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