SYMPLECTIC SINGULARITIES AND GEOMETRY OF GAUGE FIELDS
BANACH CENTER PUBLICATIONS, VOLUME 39
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

MONGE-AMPÈRE EQUATIONS VIEWED FROM CONTACT GEOMETRY

TOHRU MORIMOTO

Department of Mathematics, Kyoto University of Education
1 Fujinomori-cho, Fukakusa, Fushimi-ku, Kyoto 612, Japan
E-mail: morimoto@wsml.kyokyo-u.ac.jp

Introduction. In this note I give an elementary survey on Monge-Ampère equations from the view point of contact geometry. The main sources are Goursat [Gou], Matsuda [Ma5], Morimoto [Mo1], some recent topics that I have talked on several occasions ([Mo3] etc.), and Ishikawa and Morimoto [I-M].

The Monge-Ampère equations, even if limited to the equations in two independent variables, are very rich in concrete examples arising from Analysis, Geometry, and Physics. On the other hand, the Monge-Ampère equations are stable under contact transformation and can be well described in contact geometry.

One of the main purposes of this survey is to bring into relief various geometric problems through geometrization of the Monge-Ampère equations.

Contents.

1. Monge-Ampère exterior differential systems. Let us first recall the notion of an exterior differential system. Let M be a differential manifold and let $\mathcal{A}$ denote the sheaf of germs of differential forms on M. An *exterior differential system* on M is a subsheaf Σ of $\mathcal{A}$ such that

(1) Each stalk Σ_x, $x \in M$, is an ideal of $\mathcal{A}_x$,
(2) Σ is closed under exterior differentiation, i.e., $d\Sigma \subset \Sigma$,

1991 *Mathematics Subject Classification*: 35A30, 58G99.
The paper is in final form and no version of it will be published elsewhere.

(3) Σ is locally finitely generated.

An integral manifold of the exterior differential system Σ is an immersed submanifold $\iota : N \to M$ such that $\iota^*\phi = 0$ for any section ϕ of Σ.

For detailed treatises on exterior differential systems refer to Cartan [Ca2], Kuranishi [Ku], Bryant et.al [B-C-3G] etc.

Now consider a contact manifold (M, D) of dimension $2n+1$. By definition, a contact structure D is a subbundle of the tangent bundle TM of M of codimension 1 defined locally by a 1-form ω satisfying $\omega \wedge (d\omega)^n \neq 0$ everywhere. Such a 1-form ω is called a contact form of the contact structure D.

Definition. An exterior differential system Σ on a contact manifold (M, D) is called a *Monge-Ampère exterior differential system* (or simply *M-A system*) if Σ is locally generated by a contact form ω of D and an n-form θ.

By a solution of a M-A system Σ we mean an integral manifold of Σ of dimension n. Note that an integral manifold of Σ is a fortiori an integral manifold of D, namely an isotropic submanifold, and a Legendre submanifold if the dimension takes the maximum value n. Hence a solution of a M-A system is, in particular, a Legendre submanifold.

To justify our terminology, let us see that a solution of a M-A system turns out to be a solution of a so-called Monge-Ampère equation when expressed in terms of a suitable canonical coordinate system.

Let Σ be a M-A system on a contact manifold M of dimension $2n+1$ and let $\iota : S \to M$ be a Legendre submanifold. Take a point $a \in S$. By Darboux's theorem there is a local coordinate system (called a canonical coordinate system) $x^1, x^2, \dots, x^n, z, p_1, \dots, p_n$ of M around $\iota(a)$ such that the contact structure is locally defined by the 1-form $\omega = dz - \sum_{i=1}^n p_i dx^i$. Moreover we can choose a canonical coordinate system so that $\iota^* dx^1$, $\dots$, $\iota^* dx^n$ are linearly independent at a. Then the image $\iota(V)$ may be expressed in a neighbourhood V of a as a graph:

$$\begin{cases} z = \phi(x^1, \dots, x^n) \\ p_j = \psi_j(x^1, \dots, x^n) \end{cases}$$

Since S is a Legendre submanifold we have

$$\psi_j = \frac{\partial \phi}{\partial x^j}, \qquad j = 1, \dots, n.$$

Let θ be an n-form which, together with ω, generates the M-A system Σ. Write it down in the canonical coordinates as

$$(1.1) \qquad \theta \equiv \sum_{i_1 < \dots < i_l,\, j_1 < \dots < j_{n-l}} F^{j_1 \dots j_{n-l}}_{i_1 \dots i_l} dx^{i_1} \wedge \cdots \wedge dx^{i_l} \wedge dp_{j_1} \wedge \cdots \wedge dp_{j_{n-l}} \ (\mathrm{mod}\, \omega).$$

Then $\iota_{|_V} : V \to M$ is a solution of Σ if and only if

$$(1.2) \qquad \sum F^{j_1 \dots j_{n-l}}_{i_1 \dots i_l}(x^1, \dots, x^n, \phi, \frac{\partial \phi}{\partial x^1}, \dots, \frac{\partial \phi}{\partial x^n}) \Delta^{i_1 \dots i_l}_{j_1 \dots j_{n-l}}(\phi) = 0,$$

where $\Delta^{i_1\dots i_l}_{j_1\dots j_{n-l}}(\phi)$ denotes the minor of the Hessian matrix of ϕ given by

$$\Delta^{i_1\dots i_l}_{j_1\dots j_{n-l}}(\phi) = \operatorname{sgn}\begin{pmatrix}1,2,\dots,l,l+1,\dots,n\\ i_1,\dots,i_l,k_1,\dots,k_{n-l}\end{pmatrix}\det\begin{pmatrix}\dfrac{\partial^2\phi}{\partial x^{j_1}\partial x^{k_1}} & \cdots & \dfrac{\partial^2\phi}{\partial x^{j_1}\partial x^{k_{n-l}}}\\ \cdots & \cdots & \cdots\\ \dfrac{\partial^2\phi}{\partial x^{j_{n-l}}\partial x^{k_1}} & \cdots & \dfrac{\partial^2\phi}{\partial x^{j_{n-l}}\partial x^{k_{n-l}}}\end{pmatrix}$$

with $\{1,2,\dots,n\} = \{i_1,\dots,i_l,k_1,\dots,k_{n-l}\}$ and $k_1 < \dots < k_{n-l}$.

A second order nonlinear partial differential equation for one unknown function ϕ with n independent variables of the form (1.2) is known as Monge-Ampère equation. In particular, when $n = 2$, it has the following form familiar in the classical literature (see e.g., [Gou]):

$$Hr + 2Ks + Lt + M + N(rt - s^2) = 0,$$

where $p = \frac{\partial\phi}{\partial x}$, $q = \frac{\partial\phi}{\partial y}$, $r = \frac{\partial^2\phi}{\partial x^2}$, $s = \frac{\partial^2\phi}{\partial x\partial y}$, $t = \frac{\partial^2\phi}{\partial y^2}$ and $H, K, \dots, N$ are functions of x, y, z, p, q.

Thus a Monge-Ampère equation may be considered as a coordinate representation of a more intrinsic object of a Monge-Ampère exterior differential system.

Example 1. Consider $\mathbb{R}^5(x,y,z,p,q)$ as a contact manifold equipped with a contact form

$$\omega = dz - p\,dx - q\,dy.$$

Let Σ be a M-A system generated by the following 2-form (and ω):

$$\theta = dp \wedge dq.$$

If a solution of Σ is represented in the form

$$z = \phi(x,y), \quad p = \psi_1(x,y), \quad q = \psi_2(x,y),$$

then the function ϕ is a solution of the Monge-Ampère equation

$$rt - s^2 = 0.$$

If we introduce new coordinates $(\bar{x},\bar{y},\bar{z},\bar{p},\bar{q})$ defined by

$$\begin{cases} x = \bar{p}, \quad p = -\bar{x}\\ y = \bar{y}, \quad q = \bar{q}\\ z = \bar{z} - \bar{p}\bar{x}, \end{cases}$$

then we have

$$\omega = d\bar{z} - \bar{p}\,d\bar{x} - \bar{q}\,d\bar{y}$$
$$\theta = -d\bar{x} \wedge d\bar{q}.$$

Hence if a solution of Σ is represented in the form

$$\bar{z} = \bar{\phi}(\bar{x},\bar{y}), \quad \bar{p} = \bar{\psi}_1(\bar{x},\bar{y}), \quad \bar{q} = \bar{\psi}_2(\bar{x},\bar{y}),$$

the function $\bar{\phi}$ satisfies the Monge-Ampère equation

$$\bar{t} = 0.$$

Example 2. With the same notation as above, consider a M-A system generated by

$$\theta = dp \wedge dx.$$

Then the solutions with independent variables x, y satisfy the Monge-Ampère equation $s = 0$.

If we introduce another local coordinate system defined by

$$x = \bar{z}, \quad y = \bar{x}, \quad z = \bar{y}, \quad p = \frac{1}{\bar{q}}, \quad q = \frac{\bar{p}}{\bar{q}},$$

then we have

$$\omega = -p(d\bar{z} - \bar{p}\, d\bar{x} - \bar{q}\, d\bar{y}), \quad \theta = -\frac{1}{\bar{q}^2}(\bar{p}\, d\bar{q} \wedge d\bar{x} + \bar{q}\, d\bar{q} \wedge d\bar{y}).$$

Thus the solutions with independent variables $\bar{x}, \bar{y}$ satisfy

$$\bar{q}\bar{s} - \bar{p}\bar{t} = 0.$$

Let (M, D) and (M', D') be contact manifolds. A diffeomorphism $f : M \to M'$ is called a contact transformation if $f_* D = D'$. Let Σ, Σ' be M-A systems on M, M' respectively, we say that Σ and Σ' are contact equivalent (or equivalent, or isomorphic) if there exists a contact transformation f such that $f^* \Sigma' = \Sigma$. The notion of "locally contact equivalent" is defined in the obvious manner.

Let a Monge-Ampère equation of the form (1.2) be given. Associating to it a M-A system on the standard contact manifold $\mathbb{R}^{2n+1}$ generated by the n-form θ given by (1.1), we also say that two Monge-Ampère equations are contact equivalent if so are the associated M-A systems.

The above examples show that the M-A equations $rt - s^2 = 0$ and $t = 0$ are locally contact equivalent, and so are $s = 0$ and $qs - pt = 0$.

2. Characteristic systems. For further geometrization of the Monge-Ampère equations, we will introduce the characteristic systems of a M-A system. It is for the M-A systems on 5-dimensional complex contact manifolds that the characteristic systems are well defined and have nice geometric properties. For this reason from now on we will work on 5-dimensional complex contact manifolds unless otherwise stated. However, most of the following discussion will remain valid also in the real category under some additional assumptions.

In general, given an exterior differential system Σ on a manifold M, a subspace L of the tangent space $T_x M$ at $x \in M$ is called an integral element of Σ at x if for any germ of differential form $\alpha \in \Sigma_x$, the restriction of α to L vanishes.

Now let Σ be a M-A system on a contact manifold M of dimension 5 generated by ω and θ, where ω is a contact form of the contact manifold and θ is a 2-form.

For a non-zero vector $v \in T_x M$, it is clear that the line $L(v)$ generated by v is an integral element of Σ if and only if $\langle v, \omega \rangle = 0$, that is, $v \in D_x$.

Now supposing that we have chosen a 1-dimensional integral element $L(v)$ with $v \in D_x$, we are looking for a 2-dimensional integral element containing it.

For $v' \in T_xM$ the plane $L(v, v')$ is an integral element of Σ if and only if

$$\begin{cases} \langle v', \omega \rangle = 0 \\ \langle v \wedge v', \theta \rangle = 0 \\ \langle v \wedge v', d\omega \rangle = 0, \end{cases}$$

in other words, v' is a solution of the linear equation (polar equation)

$$\begin{cases} \omega = 0 \\ v \rfloor d\omega = 0 \\ v \rfloor \theta = 0, \end{cases}$$

where the left hook $\rfloor$ denotes the interior product. The rank of this equation is 3 or 2, according to which we call v regular or singular, respectively. If v is regular there exists a unique 2-dimensional integral element containing v. If v is singular the 2-dimensional integral elements containing v form a 1-dimensional manifold.

This being remarked, now we define the characteristic variety $\mathcal{V}(\Sigma)$ of Σ as the union of the 1-dimensional singular integral elements:

$$\mathcal{V}(\Sigma)_x = \{v \in D_x \ ; \ v \rfloor \theta \equiv 0 \ (\mathrm{mod}\, \omega, v \rfloor d\omega)\}$$
$$\mathcal{V}(\Sigma) = \bigcup_{x \in M} \mathcal{V}(\Sigma)_x \ .$$

Then we have

PROPOSITION 2.1. *For each $x \in M$, there are following three cases to distinguish*:

i) *There exist 2-dimensional subspaces E_x, F_x of D_x such that*

$$\mathcal{V}(\Sigma)_x = E_x \cup F_x, \quad D_x = E_x \oplus F_x.$$

Moreover, E_x and F_x are perpendicular with respect to $d\omega$, i.e., $d\omega(v, v') = 0$ for $v \in E_x, v' \in F_x$.

ii) *There exists a 2-dimensional subspace E_x of D_x such that $\mathcal{V}(\Sigma) = E_x$ and E_x is isotropic, i.e., $d\omega(v, v') = 0$ for $v, v' \in E_x$.*

iii) $\mathcal{V}(\Sigma)_x = D_x$.

Proof. We denote by Ω and Θ respectively the restrictions of $d\omega$ and θ to D_x. If $\Theta + \lambda\Omega = 0$ for some $\lambda \in \mathbb{C}$, then we have the case iii), where the Monge-Ampère equation degenerates to be trivial at x.

If $\Theta \not\equiv 0 \ (\mathrm{mod}\, \Omega)$, let λ_1, λ_2 be the roots of the quadratic equation for λ: $(\Theta + \lambda\Omega)^2 = 0$. Then $\Theta + \lambda_i\Omega$ is decomposable and we can write:

$$\Theta + \lambda_1\Omega = \alpha_1 \wedge \alpha_2$$
$$\Theta + \lambda_2\Omega = \beta_1 \wedge \beta_2$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in D_x^*$. Let E_x and F_x be the null space of $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$, respectively. Then we see immediately that $\mathcal{V}(\Sigma)_x = E_x \cup F_x$.

If $\lambda_1 \neq \lambda_2$, the formula

$$(\lambda_1 - \lambda_2)\Omega = \alpha_1 \wedge \alpha_2 - \beta_1 \wedge \beta_2$$

shows that E_x and F_x are perpendicular. Moreover since $\Omega \wedge \Omega \neq 0$, $\alpha_1, \alpha_2, \beta_1, \beta_2$ are linearly independent and hence $D_x = E_x \oplus F_x$.

If $\lambda_1 = \lambda_2$ then $E_x = F_x$. Moreover since the equation $(\alpha_1 \wedge \alpha_2 + \mu\Omega)^2 = 0$ has only one solution $\mu = 0$, we have $\Omega \wedge \alpha_1 \wedge \alpha_2 = 0$. Then by Cartan's lemma we have

$$\Omega = \alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2$$

for some $\gamma_1, \gamma_2 \in D_x^*$, which shows that E_x is isotropic. ∎

Remark. In the real category $\mathcal{V}(\Sigma)$ does not necessarily decompose into two subspaces. If n is greater than 2 the characteristic variety is in general much more complicated.

We have thus associated to a M-A system Σ the characteristic variety $\mathcal{V}(\Sigma)$ which, under some generic condition, decomposes into vector bundles:

$$\mathcal{V}(\Sigma) = E \cup F$$

with E, F subbundles of D of rank 2 satisfying $E^\perp = F$.

Conversely, given a subbundle E of D of rank 2, it is immediate to see that there exists a unique M-A system Σ such that $\mathcal{V}(\Sigma) = E \cup E^\perp$.

Each bundle E or F is called the characteristic system associated with Σ.

Example 3 (cf. Example 1). Let Σ be the M-A system on $\mathbb{R}^5$ defined by $\theta = dp \wedge dq$. Clearly one of the characteristic systems, say E, is given by

$$\omega = dp = dq = 0.$$

Since $dp \wedge dq \wedge d\omega = 0$, we see that $E^\perp = E$. Therefore the other characteristic system coincides with E.

Example 4. Let Σ be the M-A system on $\mathbb{R}^5$ defined by

$$\theta = dq \wedge dy + f\, dx \wedge dy,$$

where f is a function on R^5. Since

$$\theta = (dq + f\, dx) \wedge dy \equiv -dp \wedge dx + f\, dx \wedge dy \ (\mathrm{mod}\, d\omega) = dx \wedge (dp + f\, dy),$$

the characteristic systems E, F are given by

$$E : \omega = dq + f\, dx = dy = 0$$
$$F : \omega = dx = dp + f\, dy = 0.$$

3. Monge's method of integration. By the method of Hamilton-Jacobi one can solve any first order partial differential equation for one unknown function by integrating an ordinary differential equation, a Hamiltonian vector field. From around the turn of the 18th century much efforts have been paid to solve higher order partial differential equations only by integrating ordinary differential equations. In particular, Monge and Darboux therein found interesting methods. It is to a certain class of Monge-Ampère equations that the Monge's method applies. (For a classical exposition see [Gou].) We remark also that some improvement of this method was given by M. Matsuda [Ma1,2]. In this section we will give an exposition of Monge's method in our formulation of contact geometry to make clear its geometric essence.

Let Σ be a M-A system on a 5-dimensional contact manifold M. Let $\mathcal{V}(\Sigma)$ be the characteristic variety. According to Proposition 2.1, we write $\mathcal{V}(\Sigma)_x = E_x \cup F_x$. To study the solutions of Σ the following observation is fundamental:

PROPOSITION 3.1. *A 2-dimensional submanifold $\iota : S \to M$ is a solution of Σ if and only if the following conditions are satisfied*:

(1) $\iota_* T_x S \subset D_{\iota(x)}$ *for all $x \in S$.*
(2) $\iota_* T_x S \cap E_{\iota(x)} \neq 0$ *for all $x \in S$.*
(3) $\iota_* T_x S \cap F_{\iota(x)} \neq 0$ *for all $x \in S$.*

Let us remark that conditions (2) and (3) above are equivalent under condition (1).

Proof. We use the same notation as in the preceding section. Let θ, ω be local generators of Σ. If $E_{\iota(x)}$ is defined by

$$\omega_{\iota(x)} = \alpha_1 = \alpha_2 = 0$$

with $\alpha_1, \alpha_2 \in T_x^* M$, then

$$\theta_{\iota(x)} \equiv \alpha_1 \wedge \alpha_2 \ (\operatorname{mod} \omega_{\iota(x)}, d\omega_{\iota(x)}).$$

It follows from this that a Legendre subspace $L \subset D_{\iota(x)}$ is an integral element of Σ if and only if $L \cap E_{\iota(x)} \neq 0$. ■

Now we are going to consider a Cauchy problem for Σ. Let $c : I \to M$ be a one-dimensional integral curve, that is, $c^*\omega = 0$ or $c_* T_t I \subset D_{c(t)}$ for all $t \in I$. We say the curve c is non-characteristic if

$$c_* T_t I \cap \mathcal{V}(\Sigma)_{c(t)} = 0 \quad \text{for all } t \in I.$$

Given a non-characteristic curve c, we want to find a solution of Σ by extending the initial curve c.

In view of Proposition 3.1, every solution is generated by two families of characteristic curves (integral curves of E or F). It is, therefore, natural to expect that the surface generated by the integral curves starting from the initial curve c of characteristic vector field (i.e., a section of $\mathcal{V}(\Sigma)$) would be a solution.

More precisely, let X be an everywhere non-zero characteristic vector field, say, $X \in \Gamma(E)$. Let $u(t,s)$ be a family of integral curves of X defined by

$$\begin{cases} u(t,0) = c(t), & t \in I \\ \dfrac{\partial u}{\partial s}(t,s) = X_{u(t,s)}. \end{cases}$$

Then the map $u(t,s)$ gives an immersion $u : U \to M$, where U is some neighbourhood of $I \times \{0\}$ in $I \times \mathbb{R}$.

It is clear that u satisfies condition (2) of Proposition 3.1. However, it does not necessarily satisfy the condition (1) and is not a Legendre submanifold. The following proposition gives a sufficient condition for u to be a solution of Σ (cf. [Ga2], [Ma1]).

PROPOSITION 3.2. *The notation being as above, if X satisfies the following two conditions then $u : U \to M$ is a solution of Σ*:

(1) $\langle d\omega, \frac{dc}{dt} \wedge X_{c(t)} \rangle = 0, \quad t \in I,$

(2) $L_X^2\omega \equiv 0 \ (\mathrm{mod}\,\omega, L_X\omega)$.

Proof. Denote by u_s the local 1-parameter transformation group generated by X. Since the tangent space $u_*T_{(s,t)}U$ is spanned by $\{X_{(u(t,s)}, (u_s)_*\frac{dc}{dt}(t)\}$, it suffices to prove

$$(*)\qquad \begin{cases} \langle (u_s)_*c'(t), \omega\rangle = 0 \\ \langle (u_s)_*c'(t)\wedge X_{u(t,s)}, d\omega\rangle = 0. \end{cases}$$

By assumption (2), we can write

$$L_X^2\omega = a\omega + bL_X\omega$$

with some functions a, b on M. For a fixed $t \in I$, we define functions $y_1(s), y_2(s)$ by

$$\begin{cases} y_1(s) = \langle (u_s)^*\omega, c'(t)\rangle \\ y_2(s) = \langle (u_s)^*L_X\omega, c'(t)\rangle. \end{cases}$$

Then y_1, y_2 satisfy the following ordinary differential equations:

$$\begin{cases} \dfrac{dy_1}{ds} = y_2 \\ \dfrac{dy_2}{ds} = ay_1 + by_2. \end{cases}$$

But $y_1(0) = y_2(0) = 0$ by assumption (1). It then follows from the uniqueness of solution that $y_1 = y_2 = 0$, which proves (*). ∎

A non-zero vector field $X \in \Gamma(E)$ is called an *integral characteristic vector field* of E if condition (2) of Proposition 3.2 is satisfied, and further it is called *adapted to the initial curve* c if condition (1) is satisfied.

Now let us consider how to find an adapted integrable characteristic vector field. We assume the characteristic variety decomposes into vector bundles: $\mathcal{V}(\Sigma) = E \cup F$, so that $E^\perp = F$.

Now suppose that there exists a first integral h of F, that is, $Yh = 0$ for any $Y \in \Gamma(F)$. Let X_h be the vector field defined by

$$(3.1)\qquad \begin{cases} \langle X_h, \omega\rangle = 0 \\ L_{X_h}\omega \equiv dh \ (\mathrm{mod}\,\omega). \end{cases}$$

Then we see firstly that $X_h \in \Gamma(E)$, because $d\omega(X_h, Y) = \langle dh, Y\rangle = 0$ for all $Y \in \Gamma(F)$. Secondly we see that X_h is integrable, because

$$\begin{aligned} L_{X_h}^2\omega &= L_{X_h}(dh + \lambda\omega) \equiv L_{X_h}(dh) \ (\mathrm{mod}\,\omega, L_X\omega) \\ &= d(X_h \cdot h) = d(X_h \wedge X_h, d\omega) = 0. \end{aligned}$$

Thus a first integral h of F gives rise to an integral characteristic vector field X_h of E.

Next suppose that there exist two independent first integrals h_1, h_2 of F. We assume moreover $(dh_1)_x, (dh_2)_x, \omega_x$ are independent everywhere. (Since any contact form cannot be written as $\omega = \lambda_1 h_1 + \lambda_2 h_2$, this condition is satisfied for generic points.)

Then for any non-characteristic integral curve c of Σ, we can locally find an integral characteristic vector field X_h adapted to c as follows: For $t_0 \in I$, one of $\langle dh_i, \frac{dc}{dt}(t_0)\rangle$ $(i = 1, 2)$ is not zero (otherwise $c_*T_{t_0)}I \subset F_{c(t_0)}$). Hence there exists a local function $h(h_1, h_2)$

such that $h(c(t)) = 0$ and $\{dh, \omega\}$ are independent. Then, since h is constant on the curve c we have

$$\langle c'(t) \wedge (X_h)_{c(t)}, d\omega \rangle = \langle c'(t), dh \rangle = 0,$$

which shows that X_h is adapted to c. By integrating X_h, we obtain a solution $u : U \to M$ of Σ with initial curve c.

It should be noted that $h \equiv 0$ on any solution S of Σ with initial curve c. In fact, since F defines on S a one-dimensional foliation transversal to c and $h = 0$ on c and h is constant along the foliation, h must be identically zero on S. This implies that any solution with initial curve c is also a solution of the first order partial differential equation $h = 0$. (Recall that a function on a contact manifold may be regarded as a first order PDE.)

It also immediately follows that a solution of Σ with initial curve c is unique as a germ of submanifold.

Summarizing the above discussion, we have

THEOREM 3.3. *If one of the characteristic systems of a M-A system admits independent two first integrals, then any non-characteristic integral curve can be extended to a solution of the M-A system. This solution is locally unique and can be obtained by integrating an adapted integrable characteristic vector field.*

This is the integrating method of Monge. The function (or the first order PDE) h is called *intermediate integral*.

4. Classification. In this section we treat the problem of classifying the M-A systems vis-à-vis the local contact equivalence. Since there are infinitely many different classes and the complete classification is far from expected, here we will be content to give an outline of the classification for a rough grasp of the variety of M-A systems and add some indication for more detailed classification.

First of all note that two M-A systems Σ and Σ' on contact manifolds M and M' respectively are contact equivalent by a contact transformation $\phi : M \to M'$ if and only if $\phi_* \mathcal{V}(\Sigma) = \mathcal{V}(\Sigma')$, which is, in turn, equivalent to saying that $\phi_* E = E'$ (or $\phi_* E = F'$) when the characteristic varieties are decomposed into vector bundles: $\mathcal{V}(\Sigma) = E \cup F$, $\mathcal{V}(\Sigma') = E' \cup F'$. Thus the classification of the M-A systems generically reduces to classifying the pairs (D, E), where D is a contact structure of rank 4 and E is a subbundle of rank 2 of D.

Here we recall the work of É. Cartan [Ca1], in which he investigated the classification of the Pfaff equations

$$\omega_1 = \omega_2 = \omega_3 = 0$$

on a five-dimensional space, in other words, the subbundles of rank 2 of the tangent bundle of a five-dimensional manifold.

According to N. Tanaka [Ta], given a subbundle E of the tangent bundle TM of a manifold M, we define the derived systems of E as follows: Define inductively the subsheaves $\mathcal{E}^k (k = 1, 2, \ldots)$ of the sheaf $\underline{TM}$ of the germs of vector fields on M by setting

$\mathcal{E}^1 = \underline{E}$, the sheaf of the germs of sections of E and

$$\mathcal{E}^{k+1} = \mathcal{E}^k + [\mathcal{E}^1, \mathcal{E}^k].$$

Then in a neighbourhood of a generic point all $\mathcal{E}^k$ are vector bundles, that is, there exist subbundles E^k such that $\mathcal{E}^k = \underline{E^k}$ $(k = 1, 2, \dots)$.

Now in the case $\dim M = 5$ and $\operatorname{rank} E = 2$, if the derived systems are all vector bundles, there are the following five cases to distinguish:

(0) $\operatorname{rank} E^2 = 2$
(1) $\operatorname{rank} E^2 = \operatorname{rank} E^3 = 3$
(2) $\operatorname{rank} E^2 = 3,\ \operatorname{rank} E^3 = \operatorname{rank} E^4 = 4$
(3) $\operatorname{rank} E^2 = 3,\ \operatorname{rank} E^3 = 4,\ \operatorname{rank} E^4 = 5$
(4) $\operatorname{rank} E^2 = 3,\ \operatorname{rank} E^3 = 5.$

For a M-A system Σ with characteristic variety $\mathcal{V}(\Sigma) = E \cup F$, we say that Σ is *hyperbolic* if $E_x \neq F_x$ for all x, and *parabolic* if $E = F$.

Furthermore we say that a hyperbolic M-A system Σ is in the class H_{ij} if E is of type (i) and F of type (j) in the list above. Since there is no canonical way to distinguish E and F we may assume $i \leq j$. We say that a parabolic M-A system Σ is in the class P_j if E is of type (j).

It should be noted that the classes H_{ij} and P_j are invariant under contact equivalence.

PROPOSITION 4.1. *If one of the characteristic systems of a M-A system is completely integrable then the two characteristic systems coincide. Moreover such M-A systems are all locally contact equivalent.*

P r o o f. Let Σ be a M-A system and assume that one of the characteristic systems, say E, is completely integrable. For $u, v \in E_x$, take a local sections X, Y of E such that $X_x = u, Y_x = v$. Then we have

$$\begin{aligned} d\omega(u,v) &= d\omega(X,Y)_x = [X\omega(X) - Y\omega(X) - \omega([X,Y])]_x \\ &= -\omega([X,Y])_x = 0, \end{aligned}$$

which shows that E_x is isotropic and therefore $E_x^\perp = E_x$. Hence the two characteristic systems coincide.

To prove the last half of the assertion, we first note that if E is completely integrable then we have locally a fibring $\pi : M \to X$ with the fibres being leaves of E and therefore Legendre submanifolds of M. Then our assertion follows from the fact that Legendre fibrings are all locally contact equivalent, and this fact can be shown as follows: Let $\pi : M \to X$ be a Legendre fibring with M being $(2n+1)$-dimensional manifold equipped with a contact structure D, so that X is $n+1$-dimensional and each fibre is Legendre submanifold. Let $\mathrm{Gr}(X, n) \to X$ be the Grassmann bundle whose fibre at $x \in X$ consists of all n-dimensional subspaces of $T_x X$. Then there is a canonical map $f : M \to \mathrm{Gr}(X, n)$ defined by $f(p) = (\pi(p), \pi_* D_p)$ for $p \in M$. It is easy to see that $\mathrm{Gr}(X, n)$ has a canonical contact structure and that f is a fibre preserving contact immersion. It then follows that two Legendre fibrings of same dimension are locally contact equivalent. ∎

An example is fulfilled by a Monge-Ampère equation $rt - s^2 = 0$. The corresponding M-A system is generated by a 2-form $dp \wedge dq$. One of its characteristic system is given by

$$\omega = dp = dq = 0,$$

which is clearly completely integrable. The above proposition shows that there is no M-A system in the class H_{0j} and that there is only one (up to local contact equivalence) in the class P_0.

THEOREM 4.2. *There is only one M-A system up to local contact equivalence in the class H_{11}. In other words, let Σ be a hyperbolic M-A system with characteristic systems E, F. If the derived systems E^2, F^2 respectively of E, F are both completely integrable, then Σ is locally contact equivalent to the M-A system corresponding to the equation $s = 0$.*

This theorem goes back to S. Lie. For a classical proof see Goursat [Gou], or Matsuda [Ma5]. Another proof based on the general method for the equivalence problems of geometric structures can be found in Morimoto [Mo4].

Let us just see that the equation $s = 0$ is in fact in H_{11}. The corresponding M-A system is defined by the 2-form

$$dq \wedge dy \equiv -dp \wedge dx \ (\operatorname{mod} \omega, d\omega).$$

Hence the characteristic systems are given by

$$E : \omega = dq = dy = 0$$
$$F : \omega = dp = dx = 0$$

Their derived systems E^2, F^2 are given by

$$E^2 : dq = dy = 0$$
$$F^2 : dp = dx = 0,$$

which are clearly completely integrable.

We can also find in the book of Goursat [Gou] the following propositions:

PROPOSITION 4.3. *If a M-A system is hyperbolic and each of its characteristic system admits a first integral, then the M-A system is locally equivalent to a Monge-Ampère equation of the following form:*

$$s + f(x, y, z, p, q) = 0.$$

Proof. Let $E,\ F$ be the characteristic systems of a hyperbolic M-A system Σ and assume that they have first integrals $x,\ y$, respectively. Since $E \cap F = 0$, dx, dy are linearly independent. In general on a contact manifold with a fixed contact form ω, we define the bracket $[f,\ g]$ for functions $f,\ g$ by

$$[f,\ g] = d\omega(X_f,\ X_g),$$

where the vector field X_f is given by (3.1). Then we see that $[x,\ y] = 0$, since $X_x,\ X_y$ are sections of $F,\ E$ respectively and therefore perpendicular. It then follows from a fundamental theorem of contact geometry that the functions $x,\ y$ can be extended to a normal coordinate system x, y, z, p, q. Let E be defined by the Pfaff equation:

$$dx = \alpha = \omega = 0,$$

where we may suppose that α is written as

$$\alpha = \alpha_2 dy + \alpha_3 dp + \alpha_4 dq.$$

But since $d\omega(X_\alpha,\ X_y) = 0$, we see $\alpha_4 = 0$. Moreover we can easily see that $\alpha_3 \neq 0$. Hence we may choose α as

$$\alpha = f\,dy + dp.$$

Thus our M-A system is generated by the 2-form

$$dx \wedge (dp + f\,dy),$$

which is equivalent to the Monge-Ampère equation

$$s + f = 0. \ \blacksquare$$

Proposition 4.4. *If a M-A system is parabolic and its characteristic system admits a first integral, then the M-A system is locally equivalent to a Monge-Ampère equation of the following form:*

$$t + f(x, y, z, p, q) = 0.$$

Proof. Similar to the proof of Proposition 4.3. $\blacksquare$

The discussions above provide us with a rough idea of classifying the M-A systems. Now we give some remarks and indications for further detailed classification.

1) H_{ij} $(1 \leq i \leq j \leq 4)$ is not empty: In fact each H_{ij} contains infinite number of different equivalence classes except that H_{11} does only one.

2) P_0 contains only one equivalence class, P_1 is empty, and P_j $(j = 2, 3, 4)$ contains infinite number of classes.

3) The M-A systems in H_{1j} are Monge integrable.

4) A M-A system may be treated as a G-structure on a contact manifold (or generalized G-structure on a filtered manifold as developed in Morimoto [Mo4]) and we can apply the general methods of equivalence for (generalized) G-structures to obtain further invariants of M-A systems. However, since the classification spreads into many branches, it is difficult to carry out all the calculation. One of the goals which may be attainable is to classify the homogeneous M-A systems in each H_{ij} or P_j. (We say a M-A system Σ is homogeneous if the automorphism group $\mathrm{Aut}(\Sigma)$ of contact transformations is transitive.) (see [Mo1]).

5) Most generic is a M-A system Σ belonging to H_{44}, which satisfies

$$\dim \mathrm{Aut}\,(\Sigma) \leq 8,$$

and if the equality holds then it is locally equivalent to a M-A system defined on the homogeneous space $SL(3, \mathbb{C})/SL(2, \mathbb{C})$ and invariant by the actions of $SL(3, \mathbb{C})$. In a suitable local expression it can be written in the following form:

$$rt - s^2 = (z - xp - yq)^4.$$

See [Mo1], [Mo2], and for particular solutions of this equation see A. Kushner and B. Doubrov [D-K].

6) Another approach to a classification of Monge-Ampère equations is developed by Lychagin, Rubtsov and Chekalov ([L], [L-R-C]).

5. Global solutions, singularities. So far we have been mainly concerned with local problems. But since our formulation of Monge-Ampère equations is quite free from coordinate expressions, we are ready to consider various global problems as well as singularities of solutions. Here we will touch upon some of such problems.

5.1. *A global model of the equation $rt - s^2 = 0$.* Let V be a 4-dimensional vector space and V^* its dual space. Denote by $P(V)$ and $P(V^*)$ be the projective spaces of the 1-dimensional subspaces of V and V^*, respectively. Set

$$Q = \{([x], [\xi]) \in P(V) \times P(V^*) : \langle x, \xi \rangle = 0\}.$$

With the canonical projections $\rho : Q \to P(V)$ and $\rho' : Q \to P(V^*)$, we can identify Q with the projective cotangent bundle of $P(V)$ and that of $P(V^*)$:

$$Q \cong PT^*P(V) \cong PT^*P(V^*).$$

Moreover there is a canonical contact structure D on Q such that the above isomorphisms are contact isomorphisms. We then have

$$D = \operatorname{Ker} \rho_* \oplus \operatorname{Ker} \rho'_*.$$

The subbundle $\operatorname{Ker} \rho'_*$ of D of rank 2 then defines a M-A system that has $\operatorname{Ker} \rho'_*$ as a characteristic system; we denote by Σ_0 the M-A system on Q thus defined. Since $\operatorname{Ker} \rho'_*$ is completely integrable, the M-A system Σ_0 is in the class P_0 and locally isomorphic to $rt - s^2 = 0$ by Proposition 4.1. It should be remarked that this M-A system is canonically associated with projective geometry.

By Proposition 3.1, a surface S in Q is a solution of Σ_0 if and only if S is a Legendre submanifold and the rank of $\rho' : S \to P(V^*)$ is less or equal to 1 everywhere. We may consider Σ_0 as a Monge-Ampère equation for a surface of $P(V)$; a surface $Y \subset P(V)$ is called a solution if the Legendre lift $\tilde{Y}$ (in other words, the projective cotangent lift, or the projective conormal bundle of Y) is a solution of Σ_0. It then turns out that a solution $Y \subset P(V)$ is a ruled surface generated by projective lines and the tangent spaces of Y are constant along each generating line. For instance the cone C defined by

$$z^2 = x^2 + y^2$$

with affine coordinates x, y, z is a solution of Σ_0. We note that the projective cotangent lift $\tilde{C}$ of C is diffeomorphic to a torus and has no singularity, while C does. As to the non-singular global solutions, we have the following

Theorem 5.1. *A compact connected smooth surface of $P(V)$ is a solution of Σ_0 if and only if it is a projective plane.*

See [I-M] for a proof. We remark an interesting contrast between this theorem and that of Bernstein which asserts that a solution defined on the whole xy-plane of the equation $rt - s^2 = 1$ is a polynomial of degree 2. The latter arises from the ellipticity of the equation, and the former from some properties of projective geometry and holds both in the real and the complex category.

If $S \subset Q$ is a compact connected smooth solution of Σ_0 and not the Legendre lift of a projective plane, then, by the above theorem, the projection $\rho : S \to P(V)$ must have always singularities. Here we ask a question: Is there any law for the number of singularities with respect to the projection $\rho : S \to P(V)$?

5.2. *A global model of* $s = 0$. As we have seen in our classification, the Monge-Ampère equation $s = 0$ is also local trivial. Let us give a global model of this equation.

Let $\pi : S^3 \to S^2$ be the Hopf fibring, that is,

$$S^3 = \{(z_1, z_2) \in \mathbb{C};\ |z_1|^2 + |z_2|^2 = 1\},$$

and π is the quotient map to $S^2 = \mathbb{C}P^1$ by the S^1-action:

$$(z_1, z_2)e^{i\theta} = (e^{i\theta} z_1, e^{i\theta} z_2).$$

Then there is a natural contact form ω on S^3 given by the restriction of the form

$$\frac{1}{2}\mathcal{I}m(\bar{z}_1 dz_1 + \bar{z}_2 dz_2),$$

which is invariant by the S^1-action.

Now preparing two copies of the Hopf fibring, $\pi : S^3 \to S^2$ and $\bar{\pi} : \bar{S}^3 \to \bar{S}^2$, we set

$$M = S^3 \times \bar{S}^3 / \sim,$$

the quotient space by the action of the diagonal Δ of $S^1 \times \bar{S}^1$. Then we see that the 1-form $\omega - \bar{\omega}$ induces a contact structure D on M. We have also the natural projections $\rho : M \to S^2$ and $\bar{\rho} : M \to \bar{S}^2$. If we set

$$E = D \cap \operatorname{Ker} \rho_*, \quad \bar{E} = D \cap \operatorname{Ker} \bar{\rho}_*,$$

then E and $\bar{E}$ are of rank 2 and perpendicular and we have

$$D = E \oplus \bar{E}.$$

Hence there is a unique M-A system Σ_{11} on M that has E and $\bar{E}$ as characteristic systems. Since E and $\bar{E}$ have respectively two independent first integrals given by ρ and $\bar{\rho}$, the M-A system Σ_{11} is locally equivalent to $s = 0$ by Theorem 4.2.

It would be interesting to study the global solutions of this M-A system.

5.3. *Singularities of solutions.* When we study singularities of a solution S of a M-A system Σ on a contact manifold M, we should distinguish the following different sorts of singularities:

(a) Singularities of S itself.

(b) Singularities with respect to a Legendre fibring: If there is given a Legendre fibring $\rho : M \to N$, in particular, if M is the projective cotangent bundle PT^*N of a manifold N, it will be interesting to study the singularities of S with respect to the projection $\rho : S \to N$.

(c) Singularities arising with respect to a prescribed space of independent variables. If $M = J^1_X N$, that is M is the space of the 1-jets of cross-sections of a fibred manifold $N \to X$, the singularities with respect to the projection $S \to X$ are the singularities which arise when one wants to regard S as an ordinary solution of the Monge-Ampère equation expressed in coordinate with independent variables in X.

Singularities of Legendre varieties have been studied by many people. Now, in relation with Monge-Ampère equations, we pose the following general questions:

(1) What happens to the singularities of a Legendre variety if imposed to be a solution of a M-A system?

(2) How does it depend on the choice of a M-A system?

Here we mention a result obtained by G. Ishikawa concerning to the questions above. Consider a map-germ $f : \mathbb{R}^2, 0 \to \mathbb{R}^5, 0$ defined by

$$(x, y, z; p, q) \circ f = (u, v^2, uv^3; v^3, \frac{3}{2}uv),$$

which is isotropic with respect to the contact form $\omega = dz - pdx - qdy$, that is, $f^*\omega = 0$. An isotropic map-germ g from $\mathbb{R}^2, 0$ to a contact manifold M is called an *open umbrella* if it is contact equivalent to the above map-germ f up to parametrization.

PROPOSITION 5.2. *An open umbrella can be a solution of a M-A system of type $rt - s^2 = 0$ but cannot be a solution of a M-A system of type $s = 0$.*

For further information on the singularities of a solution of Σ_0, we refer to Ishikawa ([Is1], [Is2]). We cite also another approaches to singularities of Monge-Ampère equations by M. Kossowski [Ko] and M. Tsuji [Ts].

There being so much literature on Monge-Ampère equations, the following references are not intended to be complete. However, they are considerably ameliorated thanks to the referees who informed me of papers relevant to our subjects, and to whom I am very grateful.

References

[B-C-3G] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, *Exterior differential systems*, Springer-Verlag New York, 1991.

[B-G1] R. L. Bryant and P. A. Griffiths, *Characteristic cohomology of differential systems I: General theory*, J. Amer. Math. Soc. 8 (1995), 507–627.

[B-G2] R. L. Bryant and P. A. Griffiths, *Characteristic cohomology of differential systems II: Conservation laws for a class of parabolic equations*, Duke Math. J. 78 (1995), 531–676.

[Ca1] É. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. École Norm. Sup. 27 (1910), 109–192.

[Ca2] É. Cartan, *Les systèmes différentiels extérieurs et leurs applications géométriques*, Hermann, Paris, 1945.

[D-K] B. Doubrov and A. Kushner, *The Morimoto equation*, ISLC Mathematical College Works, Lie-Lobachevsky Colloquium, Tartu, October 26–30, 1992, International Sophus Lie Centre, 1992, 30–32.

[Fo] A. F. Forsyth, *Theory of differential equations, IV*, Cambridge Univ. Press, London, 1906.

[Ga1] R. B. Gardner, *Invariants of Pfaffian systems*, Trans. Amer. Math. Soc. 126 (1967), 514–533.

[Ga2] R. B. Gardner, *The Cauchy problem for Pfaffian systems*, Comm. Pure Appl. Math. 22 (1969), 587–596.

[Ga3] R. B. Gardner, *A differential geometric generalization of characteristics*, Comm. Pure Appl. Math. 22 (1969), 597–626.

[Ga4] R. B. Gardner, *Characteristics and the geometry of hyperbolic equations in the plane*, J. Differential Equations 104 (1993), 60–116.

[Gou] E. Goursat, *Leçon sur l'intégration des équations aux derivées partielles du second ordre à deux variables independentes, I, II*, Hermann, Paris, 1896, 1898.

[H-T-T] D. Hartley, R. W. Tucker, and P. A. Tuckey, *Equivalence of the Darboux and Gardner methods for integrating hyperbolic equations in the plane*, Duke Math. J. 77 (1995), 167–192.

[Is1] G. Ishikawa, *Determinacy of the envelope of the osculating hyperplanes to a curve*, Bull. London Math. Soc. 25 (1993), 603–610.

[Is2] G. Ishikawa, *Developable of a curve and determinacy relative to osculation-type*, Quart. J. Math. Oxford Ser. (2) 46 (1995), 437–451.

[I-M] G. Ishikawa and T. Morimoto, *On solution surfaces of Monge-Ampère equations*, in preparation.

[Ko] M. Kossowski, *Homotopy invariants for solutions to symplectic Monge-Ampère equations*, J. Differential Equations 106 (1993), 294–311.

[Ku] M. Kuranishi, *Lectures on exterior differential systems*, Tata Institute of Fundamental Research, Bombay, 1962.

[LeP1] Th. Lepage, *Sur certaine formes différentielles associées aux équations du type de Monge-Ampère provenant du calcul des variations*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, 15 (1929), 829–848.

[LeP2] Th. Lepage, *Sur une propriété caractéristique des équations des extrémales des intégrales multiples*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, 16 (1930), 103–123.

[LeP3] Th. Lepage, *Interprétation géométrique de certains résultats de la théorie des formes symboliques de différentielles. Première communication*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, 16 (1930), 1334–1352.

[LeP4] Th. Lepage, *Interprétation géométrique de certains résultats de la théorie des formes symboliques de différentielles. Deuxième communication*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, 17 (1931), 163–178.

[LeP5] Th. Lepage, *Sur certaines formes différentielles symboliques de degré et de rang n à $2n+1$ variables*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, 20 (1934), 527–537.

[Ly] V. V. Lychagin, *Contact geometry and non-linear second order differential equations*, Russian Math. Surveys 34 (1979), 149–180.

[L-R-C] V. V. Lychagin, V. N. Rubtsov and I. V. Chekalov, *A classification of Monge-Ampère equations*, Ann. Sc. École Norm. Sup. (4) 26 (1993), 281–308.

[Ma1] M. Matsuda, *Two methods of integrating Monge-Ampère equations*, Trans. Amer. Math. Soc. 150 (1970), 327–343.

[Ma2] M. Matsuda, *Two methods of integrating Monge-Ampère equations, II*, Trans. Amer. Math. Soc. 166 (1972), 371–386.

[Ma3] M. Matsuda, *On Monge-Ampère equations,* Sugaku 24-2 (1972), 100–118 (in Japanese).

[Ma4] M. Matsuda, *Reduction of Monge-Ampère's equations by Imschenetsky transformations*, J. Math. Soc. Japan 25 (1973), 43–70.

[Ma5] M. Matsuda, *Theory of exterior differential forms*, Iwanamishoten, Tokyo, 1976 (in Japanese).

[Mo1] T. Morimoto, *La géométrie des équations de Monge-Ampère*, C. R. Acad. Sci. Paris 289 (1979), A-25–A-28.

[Mo2] T. Morimoto, *Open problems suggested by T. Morimoto*, Open problems in structure theory of non-linear integrable differential and difference systems: Proceedings of the fifteenth international symposium, Division of Mathematics, The Taniguchi Foundation, 1984, 27–29.

[Mo3] T. Morimoto, *On Monge-Ampère equations*, Sûrikaisekikenkyûsho Kôkyûroku (Real Singularities and Real Algebraic Geometry) 815 (1992), 97–109 (in Japanese).

[Mo4] T. Morimoto, *Geometric structures on filtered manifolds*, Hokkaido Math. J. 22 (1993), 263–347.

[S-S] I. M. Singer and S. Sternberg, *On the infinite groups of Lie and Cartan, Part I* (*The transitive groups*), J. Anal. Math. 15 (1965), 1–114.

[Ta] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ. 10 (1970), 1–82.

[Ts] M. Tsuji, *Formation of singularities for Monge-Ampère equations*, Bull. Soc. Math. France 119 (1995), 433–457.

[Tu] D. V. Tunitsky, *Hyperbolicity and multivalued solutions of Monge-Ampère equations*, Amer. Math. Soc. Transl. Ser. 2 167 (1995), 245–260.