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MULTIPLE-SCALE ANALYSIS FOR PAINLEVÉ TRANSCENDENTS WITH A LARGE PARAMETER

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1. Introduction. The purpose of this note is to give a survey of a part of the article [AKT3] which is concerned with the exact WKB analysis of Painlevé transcendents with a large parameter. The exact WKB analysis is an analysis based on the systematic use of WKB solutions and Borel resummed WKB solutions of differential equations. A WKB solution is a kind of formal solution that is expanded in the (negative) power series of a large parameter. Such a series is divergent in general, but is easily constructed. By taking Borel resummation of a WKB solution, we get a holomorphic solution to the original equation. The correspondence between WKB solutions to holomorphic solutions obtained by Borel resummation is, however, not so simple (connection problems). If one knows the correspondence completely, then one can obtain large amount of global information about the solutions. In fact, we know the correspondence, at least generically, in the case of second order linear ordinary differential equations of Fuchsian type (with a large parameter) and we can calculate the monodromy groups of the equations (cf. [AKT2]). In [AKT3], we investigate the Painlevé equations from such a point of view (cf. [KT] also). Thus we are interested in

(i) constructing formal solutions of Painlevé equations,
(ii) solving the connection problems for these formal solutions.

In this note, we focus on the former problem and we give an outline of the construction of formal solutions of the Painlevé equations.

2. Formal solutions of the first Painlevé equation.

2.1 *Formal solution without free parameter.*

Let us consider the first Painlevé equation with a large parameter η:

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$$P_I: \quad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

To begin with, we look for a formal solution that has an expansion in the negative powers of η. Put the expression

$$\lambda = \lambda(t, \eta) = \sum_{j=0}^{\infty} \eta^{-j} \lambda_j(t) \tag{1}$$

into P_I and compare the coefficients of the powers of η of the both sides:

$$\sum_{j=0}^{\infty} \eta^{-j} \frac{d^2\lambda_j}{dt^2} = 6\eta^2 \sum \eta^{-j} \sum_{k=0}^{j} \lambda_k \lambda_{j-k} + 6\eta^2 t.$$

Then we have the following recursive relation:

$$\lambda_0 = \sqrt{-\frac{t}{6}} \tag{2}$$

$$\lambda_j = \frac{1}{12\lambda_0} \left(\frac{d^2\lambda_{j-2}}{dt^2} - 6 \sum_{k=1}^{j-1} \lambda_k \lambda_{j-k} \right) \quad (j \geq 2). \tag{3}$$

Thus we have

THEOREM 2.1 ([KT]). *There exists a formal solution λ of the form* (1) *of P_I. Such a solution is unique up to the choice of the branch of the square root in λ_0. Moreover, λ_{2j-1} vanishes identically for every $j = 1, 2, 3, \ldots$.*

It is easy to see that λ_{2j} $(j = 1, 2, 3, \ldots)$ has the form

$$\lambda_{2j} = -\frac{c_j}{\left(\sqrt{-\frac{t}{6}}\right)^{5j-1}},$$

where c_j is a constant that has the order of $(2j)! c_0^j$ (for some c_0). Hence the series λ does not converge. But it is pre-Borel summable in the sense of [AKT1].

2.2 *Formal solutions with two free parameters.* The formal solution constructed in Section 2.1 is a counterpart of a WKB solution in the exact WKB analysis of Schrödinger equations (cf. [AKT2], [V]). It is a basic object in the exact WKB analysis of Painlevé equations (cf. [KT]) and plays an important role. Our next step is to construct a family of formal solutions that have two free parameters. To find such a family is a natural problem because the Painlevé equations are of second order. We will employ the method of multi-scale analysis. We note that it is [JK] that first used this method in the analysis of the first and the second Painlevé equations. Our analysis is not limited in the first and the second cases (cf. [AKT3]) and we can treat not only the leading parts but also all the terms of formal solutions.

We take the following change of unknown function in P_I:

$$\lambda = \sqrt{s} + \eta^{-\frac{1}{2}} \Lambda,$$

where $s = -\dfrac{t}{6}$. Then we have an equation for Λ:

$$\frac{d^2\Lambda}{dt^2} = \eta^2 12\sqrt{s}\ \Lambda + \eta^{\frac{3}{2}}\, 6\Lambda^2 + \eta^{\frac{1}{2}} \frac{1}{144 s^{\frac{3}{2}}}. \tag{4}$$

We introduce a new variable τ by

$$\tau = \eta \int^t \sqrt{12\sqrt{s}}\ dt = -\eta \frac{48}{5}\sqrt{3} s^{\frac{5}{4}}$$

and look for a solution Λ of (4) in the form

$$\Lambda = \Lambda(t,\tau)|_{\tau=\tau(t)}.$$

This is the first step in the multiple-scale analysis (cf. [BO], for example). Since $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \dfrac{d\tau}{dt}\dfrac{\partial}{\partial \tau}$, $\Lambda(t,\tau)$ should satisfy, as a function of two independent variables (t,τ), the following equation:

$$\begin{aligned} \frac{\partial^2\Lambda}{\partial\tau^2} - \Lambda = {} & \eta^{-\frac{1}{2}} \frac{\Lambda^2}{2\sqrt{s}} - \eta^{-1}\left(\frac{1}{\sqrt{3}s^{\frac{1}{4}}} \frac{\partial^2\Lambda}{\partial t\partial\tau} - \frac{\sqrt{3}}{12^2 s^{\frac{5}{4}}}\frac{\partial\Lambda}{\partial\tau}\right) \\ & + \eta^{-\frac{3}{2}} \frac{1}{12^3 s^2} - \eta^{-2}\frac{1}{12\sqrt{s}}\frac{\partial^2\Lambda}{\partial t^2}. \end{aligned} \tag{5}$$

Suppose that Λ has an expansion

$$\Lambda = \sum_{k=0}^{\infty} \eta^{-\frac{k}{2}} \Lambda_{\frac{k}{2}}.$$

Put this into (5) and compare the coefficients of the both sides. Then we get the following series of equations for $\{\Lambda_{\frac{k}{2}}\}$'s:

$$\frac{\partial^2\Lambda_0}{\partial\tau^2} - \Lambda_0 = 0, \tag{6}$$

$$\frac{\partial^2\Lambda_{\frac{1}{2}}}{\partial\tau^2} - \Lambda_{\frac{1}{2}} = \frac{\Lambda_0^2}{2\sqrt{s}}, \tag{7}$$

$$\frac{\partial^2\Lambda_1}{\partial\tau^2} - \Lambda_1 = \frac{\Lambda_0\Lambda_{\frac{1}{2}}}{\sqrt{s}} - \frac{1}{\sqrt{3}s^{\frac{1}{4}}}\frac{\partial^2\Lambda_0}{\partial t\partial\tau} + \frac{\sqrt{3}}{12^2 s^{\frac{5}{4}}}\frac{\partial\Lambda_0}{\partial\tau}, \tag{8}$$

$$\frac{\partial^2\Lambda_{\frac{3}{2}}}{\partial\tau^2} - \Lambda_{\frac{3}{2}} = \frac{1}{2\sqrt{s}}(\Lambda_0\Lambda_1 + \Lambda_{\frac{1}{2}}^2) + \frac{1}{12^3 s^2} - \frac{1}{\sqrt{3}s^{\frac{1}{4}}}\frac{\partial^2\Lambda_{\frac{1}{2}}}{\partial t\partial\tau} + \frac{\sqrt{3}}{12^2 s^{\frac{5}{4}}}\frac{\partial\Lambda_{\frac{1}{2}}}{\partial\tau}, \tag{9}$$

$$\begin{aligned} \frac{\partial^2\Lambda_{\frac{k}{2}}}{\partial\tau^2} - \Lambda_{\frac{k}{2}} = {} & \frac{1}{2\sqrt{s}}\sum_{l=0}^{k-1}\Lambda_{\frac{l}{2}}\Lambda_{\frac{k-l-1}{2}} - \frac{1}{\sqrt{3}s^{\frac{1}{4}}}\frac{\partial^2\Lambda_{\frac{k-2}{2}}}{\partial t\partial\tau} + \frac{\sqrt{3}}{12^2 s^{\frac{5}{4}}}\frac{\partial\Lambda_{\frac{k-2}{2}}}{\partial\tau} \\ & - \frac{1}{12\sqrt{s}}\frac{\partial^2\Lambda_{\frac{k-4}{2}}}{\partial t^2}\ (k \geq 4). \end{aligned} \tag{10}$$

If we find a sequence of functions $\{\Lambda_{\frac{k}{2}}(t,\tau)\}$ that satisfy (6)–(10), we obtain a formal solution

$$\Lambda = \sum_{k=0}^{\infty} \eta^{-\frac{k}{2}} \Lambda_{\frac{k}{2}}(t,\tau)|_{\tau=\tau(t)}$$

of (4). One can solve Eq. (6) easily:

$$\Lambda_0 = a_1^{(0)} e^{\tau} + a_{-1}^{(0)} e^{-\tau},$$

where $a_{\pm 1}^{(0)} = a_{\pm 1}^{(0)}(t)$ are arbitrary functions of t. They will be determined later. Since the right-hand side of (7) is known (if $a_{\pm 1}^{(0)}$ are given), we can find a solution $\Lambda_{\frac{1}{2}}$ of (7) of the form

$$\Lambda_{\frac{1}{2}} = a_2^{(\frac{1}{2})} e^{2\tau} + a_0^{(\frac{1}{2})} + a_{-2}^{(\frac{1}{2})} e^{-2\tau},$$

where

$$\begin{cases} a_2^{(\frac{1}{2})} &= \dfrac{{a_1^{(0)}}^2}{6\sqrt{s}}, \\ a_0^{(\frac{1}{2})} &= -\dfrac{a_1^{(0)} a_{-1}^{(0)}}{\sqrt{s}}, \\ a_{-2}^{(\frac{1}{2})} &= \dfrac{{a_{-1}^{(0)}}^2}{6\sqrt{s}}. \end{cases}$$

Hence the right-hand side of (7) is known. Now we impose the condition of non-secularity on the right-hand side: the coefficients of $e^{\pm\tau}$ must vanish. This yields a system of ordinary differential equations for $a_{\pm 1}^{(0)}$:

$$\begin{cases} \dfrac{\partial a_1^{(0)}}{\partial t} + \dfrac{5\sqrt{3}\, {a_1^{(0)}}^2 a_{-1}^{(0)}}{6 s^{\frac{3}{4}}} - \dfrac{a_1^{(0)}}{48 s} = 0, \\ \dfrac{\partial a_{-1}^{(0)}}{\partial t} - \dfrac{5\sqrt{3}\, a_1^{(0)} {a_{-1}^{(0)}}^2}{6 s^{\frac{3}{4}}} - \dfrac{a_{-1}^{(0)}}{48 s} = 0. \end{cases}$$

This system can be solved easily and we find

$$\begin{cases} a_1^{(0)} &= c_+ s^{5\sqrt{3}c_+c_- - \frac{1}{8}}, \\ a_{-1}^{(0)} &= c_- s^{-5\sqrt{3}c_+c_- - \frac{1}{8}}. \end{cases}$$

Here c_+ and c_- are arbitrary constants. Hence we have obtained Λ_0 and $\Lambda_{\frac{1}{2}}$.

Suppose we have $\Lambda_{k-\frac{1}{2}}$, then we can determine Λ_k except for the coefficients $a_{\pm 1}^{(k)}$ of $e^{\pm\tau}$. They are determined by the non-secularity condition of the equation for Λ_{k+1}:

$$\left(s\frac{d}{ds} - A \right) \begin{pmatrix} a_1^{(k)} \\ a_{-1}^{(k)} \end{pmatrix} = s^{-\frac{5}{4}k - \frac{1}{8}} \begin{pmatrix} s^{5\sqrt{3}c_+c_-} f_1^{(k)} \\ s^{-5\sqrt{3}c_+c_-} f_{-1}^{(k)} \end{pmatrix},$$

where we set

$$A = \begin{pmatrix} 5\sqrt{3}c_+c_- - \frac{1}{8} & 5\sqrt{3}c_+^2 \\ -5\sqrt{3}c_-^2 & -5\sqrt{3}c_+c_- - \frac{1}{8} \end{pmatrix}$$

and $f_{\pm 1}^{(k)}$ are written in terms of $a_l^{(j)}$ $(j \le k - \frac{1}{2})$. We can find $a_{\pm 1}^{(k)}$ in the form

$$\begin{pmatrix} a_1^{(k)} \\ a_{-1}^{(k)} \end{pmatrix} = s^{-\frac{5}{4}k - \frac{1}{8}} \begin{pmatrix} s^{5\sqrt{3}c_+c_-} b_1^{(k)} \\ s^{-5\sqrt{3}c_+c_-} b_{-1}^{(k)} \end{pmatrix}.$$

Hence we get Λ_k and $\Lambda_{k+\frac{1}{2}}$. Thus we have the following

THEOREM 2.2. *There is a two-parameter family of formal solutions to P_I of the form*

$$\lambda_I = \sqrt{-\frac{t}{6}} + \eta^{-\frac{1}{2}} \sum_{k=0}^{\infty} \eta^{-\frac{k}{2}} \Lambda_{\frac{k}{2}}.$$

Each $\Lambda_{\frac{k}{2}}$ *has the form*

$$\Lambda_{\frac{k}{2}} = \frac{1}{s^{\frac{5k+1}{8}}} \sum_{l=0}^{k+1} b_{k+1-2l}^{(\frac{k}{2})} e^{(k+1-2l)\Phi}.$$

Here $e^{\Phi} = s^{5\sqrt{3}c_+c_-} e^{\tau}$*;* c_+ *and* c_- *are arbitrary constants,* $\tau = \eta \int^t \sqrt{12\sqrt{s}}dt$ $\left(s = -\frac{t}{6}\right)$ *and* $b_j^{(\frac{k}{2})}$*'s are constants depending on* $c_\pm$*.*

Remark. If we set $c_+ = c_- = 0$ in the above solution, we recover the formal solution without free parameter constructed in Section 2.1.

3. Formal solution of the J-th Painlevé equation.

3.1 *Formal solution without free parameter.* We list up the Painlevé equations with the large parameter η:

$$P_I: \ \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t),$$

$$P_{II}: \ \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \alpha),$$

$$P_{III}: \ \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + 8\eta^2\left[2\alpha_\infty\lambda^3 + \frac{\alpha'_\infty}{t}\lambda^2 - \frac{\alpha'_0}{t} - 2\frac{\alpha_0}{\lambda}\right],$$

$$P_{IV}: \ \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + 2\eta^2\left[\frac{3}{4}\lambda^3 + 2t\lambda^2 + (t^2 + 4\alpha_1)\lambda - 4\frac{\alpha_0}{\lambda}\right],$$

$$\begin{aligned} P_V: \ \frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right) \\ &\quad + \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(\alpha_0 + \alpha_\infty) - \frac{\alpha_0}{\lambda^2} - \frac{\alpha_2 t}{(\lambda-1)^2} - \frac{\alpha_1 t^2(\lambda+1)}{(\lambda-1)^3}\right], \end{aligned}$$

$$\begin{aligned} P_{VI}: \ \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &\quad + \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right. \\ &\quad \left. + \eta^2\left\{(\alpha_0 + \alpha_1 + \alpha_t + \alpha_\infty) - \frac{\alpha_0 t}{\lambda^2} + \frac{\alpha_1(t-1)}{(\lambda-1)^2} - \frac{\alpha_t t(t-1)}{(\lambda-t)^2}\right\}\right]. \end{aligned}$$

Each Painlevé equation has quite a complicated form, but regarding η dependence, all the Painlevé equations have a common structure. That is, the J-th Painlevé equation can be expressed in the form

$$P_J: \quad \frac{d^2\lambda}{dt^2} = G_J\left(\lambda, \frac{d\lambda}{dt}, t\right) + \eta^2 F_J(\lambda, t), \tag{11}$$

where F_J and G_J are rational functions. Let us note that F_J does not contain $\frac{d\lambda}{dt}$. We seek a formal solution of P_J that has the following form:

$$\lambda = \lambda(t,\eta) = \sum_{j=0}^{\infty} \eta^{-j}\lambda_j(t). \tag{12}$$

Put this expression into (11) and compare the coefficients of the powers of η of the both sides. Then we easily see the following:

THEOREM 3.1 ([KT]). *There exists a formal solution λ of the form* (12) *that satisfies the followings:*

(i) *The leading term $\lambda_0(t)$ is a solution of the algebraic equation*

$$F_J(\lambda_0(t),t) = 0.$$

(ii) *Each $\lambda_j(t)$ is uniquely determined recursively once $\lambda_0(t)$ is fixed.*

(iii) $\lambda_{2j-1}(t) = 0$ *for all* $j = 1, 2, 3, \ldots$.

This particular solution is denoted by

$$\lambda_J^{(0)} = \lambda_J^{(0)}(t,\eta) = \sum_{j=0}^{\infty} \eta^{-2j}\lambda_{2j}(t).$$

3.2 *General formal solution of P_J.* In a similar manner as in Section 2.2, we can construct a family of formal solutions of P_J that have two arbitrary parameters:

THEOREM 3.2. *There is a family of formal solutions of P_J that contain two free parameters of the form*

$$\lambda_J = \lambda_{J,0} + \eta^{-\frac{1}{2}} \sum_{k=0}^{\infty} \eta^{-\frac{k}{2}} \Lambda_{\frac{k}{2}},$$

where $\lambda_{J,0}$ is a solution of $F_J(\lambda_{J,0},t) = 0$ and each $\Lambda_{\frac{k}{2}}$ has the form

$$\Lambda_{\frac{k}{2}} = \sum_{l=0}^{k+1} a_{k+1-2l}^{(\frac{k}{2})}(t) e^{(k+1-2l)\Phi}.$$

Here $\Phi = c_+c_-\theta(t) + \tau$; c_+ and c_- being arbitrary constants, θ is a function determined by F_J and G_J, $\tau = \eta\int^t \sqrt{\frac{\partial F_J}{\partial\lambda}(\lambda_{J,0}(t),t)}dt$, and each $a_j^{(\frac{k}{2})}(t)$ is a function depending on t and on $c_\pm$ that does not contain exponential terms.

We denote these formal solutions by

$$\lambda_J = \lambda_J(t,\eta;c_+,c_-).$$

If we set $c_+ = c_- = 0$, then we recover the formal solution $\lambda_J^{(0)}$ without free parameter:

$$\lambda_J(t,\eta;0,0) = \lambda_J^{(0)}(t,\eta).$$

4. Remarks. Our discussion is quite formal and giving some analytic meaning to the formal solutions constructed in Sections 2.2 and 3.2 is our next problem. But this formal objects are interesting from various points of view. Starting from the formal solution without free parameter (constructed in Sections 2.1 and 3.1), we have a formal solution of the form

$$\lambda_J(t,\eta;c_+,0)$$

after crossing a “Stokes curve” (non-linear Stokes phenomena). In fact, we can formally reduce all the $\lambda_J(t,\eta;\tilde{c}_+,\tilde{c}_-)$'s $(J = II,\dots,VI)$ to $\lambda_I(t,\eta;c_+,c_-)$ (for some $c_\pm$) and analyzing the non-linear Stokes phenomena for P_J can be reduced to that for P_I. In the case of the reduction of P_{II} to P_I, we see that the following relation of constants must hold:

$$\sqrt{2}\sqrt[4]{3}\,c_+ = (2^3 3^{\frac{5}{2}}\alpha)^{2\tilde{c}_+\tilde{c}_-}\,\tilde{c}_+,$$
$$\sqrt{2}\sqrt[4]{3}\,c_- = (2^3 3^{\frac{5}{2}}\alpha)^{-2\tilde{c}_+\tilde{c}_-}\,\tilde{c}_-.$$

See [AKT3] for the details.

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