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LINEAR PRESERVERS ON $\mathcal{B}(X)$

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1. Introduction. By a linear preserver we mean a linear map of an algebra $\mathcal{A}$ into itself which, roughly speaking, preserves certain properties of some elements in $\mathcal{A}$. Linear preserver problems concern the characterization of such maps. Automorphisms and anti-automorphisms certainly preserve various properties of the elements. Therefore, it is not surprising that these two types of maps often appear in the conclusions of the results.

In this paper, we shall concentrate on the case when $\mathcal{A} = \mathcal{B}(X)$, the algebra of all bounded linear operators on a complex *infinite dimensional* Banach space X. We should point out that a great deal of work has been devoted to the case when X is finite dimensional, that is, the case when $\mathcal{A}$ is a matrix algebra (see survey articles [1, 19, 20]), and that the first papers concerning this case date back to the previous century [12]. It seems that in the last few years the interest for the infinite dimensional situation grows. The emphasis in this paper will be on the description of the techniques that were developed recently.

Although the literature on linear preservers in the infinite dimensional case is far from being as vast as in the finite dimensional one, we make no attempt at a detailed survey in this short paper.

In Section 2 we show that some questions regarding linear preservers on $\mathcal{B}(X)$ can be reduced to the problem of determining all linear maps that carry the set of rank one operators onto themselves. This approach has been developed in the finite dimensional situation, and has later proved to be useful also in infinite dimensions. We outline how this

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method can be used to characterize bijective linear maps on $\mathcal{B}(X)$ preserving the spectrum of each operator (Theorem 2.5), preserving the nilpotency of operators (Theorem 2.6), and preserving the commutativity of operators (Theorem 2.7). The result on preservers of nilpotent operators leads to a characterization of bijective linear maps preserving the spectral radius of each operator (Theorem 2.8).

In Section 3 we show how a number of linear preserver problems can be solved using algebraic tools. A ring theory approach to commutativity preserving map problem is described (Theorem 3.1), and its consequences to linear preserver problems in $\mathcal{B}(X)$ are discussed (Corollaries 3.2, 3.4, 3.5). The applicability of algebraic results on Jordan homomorphisms is presented (Propositions 3.6, 3.7 and Theorems 3.8, 3.9).

Let us remark that for the sake of simplicity we consider only maps from $\mathcal{B}(X)$ into itself. Most of the results could be formulated for the maps from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$.

In the conclusions of most of the results we shall speak about automorphisms or antiautomorphisms. These results could be formulated more precisely. Namely, it is known that every automorphism of $\mathcal{B}(X)$ is inner, that is, it is of the form $A \longmapsto TAT^{-1}$ for some invertible operator $T \in \mathcal{B}(X)$, and every antiautomorphism is of the form $A \longmapsto TA'T^{-1}$ for some invertible bounded operator $T : X' \longrightarrow X$ (here, X' denotes the dual of X, and A' the adjoint of $A \in \mathcal{B}(X)$). In particular, this tells us that in a number of theorems the maps under consideration are automatically continuous.

Considering linear preservers on $\mathcal{B}(X)$ one usually assumes that the maps are at least surjective if not bijective. It seems that without this assumption the problems would become extremely difficult. Namely, even the question how to describe all (not necessarily bijective) endomorphisms of $\mathcal{B}(X)$ does not seem to have a simple answer.

Let us fix the notation. Throughout, X will be a complex Banach space with dual X', and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. Let $A \in \mathcal{B}(X)$. By A', $\sigma(A)$ and $r(A)$ we denote the adjoint of A, the spectrum of A and the spectral radius of A, respectively.

2. Reducing to the problem of determining rank one operator preservers. A number of questions regarding linear preservers on $\mathcal{B}(X)$ can be reduced to the problem of determining linear maps that carry the operators of rank one into themselves. We need the following lemmas to illustrate this approach.

LEMMA 2.1 [17, Theorem 1]. *Let $A \in \mathcal{B}(X)$, $A \neq 0$. Then A has rank one if and only if $\sigma(T + A) \cap \sigma(T + \lambda A) \subseteq \sigma(T)$ for every $T \in \mathcal{B}(X)$ and every $\lambda \in \mathbb{C}$, $\lambda \neq 1$.*

LEMMA 2.2 [31, Proposition 2.1]. *Let $\mathcal{N}(X)$ be the set of all nilpotent operators in $\mathcal{B}(X)$, and let $N \in \mathcal{N}(X)$, $N \neq 0$. Then N has rank one if and only if for every $A \in \mathcal{N}(X)$ satisfying $A + N \notin \mathcal{N}(X)$ we have $A + \lambda N \notin \mathcal{N}(X)$ for every nonzero $\lambda \in \mathbb{C}$.*

LEMMA 2.3 [22, Corollary 3.4]. *Suppose that a bijective linear map $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ preserves commutativity in both directions (that is, the operators A and B commute if and only if $\phi(A)$ and $\phi(B)$ commute). Then there exists $c \in \mathbb{C}$, $c \neq 0$, such that for every idempotent P of rank one there is an idempotent $\tilde{P}$ of rank one and $d \in \mathbb{C}$ such that $\phi(P) = c\tilde{P} + dI$.*

Note that Lemma 2.1 implies that a bijective linear map ϕ of $\mathcal{B}(X)$ that is spectrum–preserving (that is, $\sigma(\phi(A)) = \sigma(A)$ for every $A \in \mathcal{B}(X)$) is rank one operator preserving in both directions (that is, $A \in \mathcal{B}(X)$ has rank one if and only if $\phi(A)$ has rank one). Similarly, Lemma 2.2 shows that a bijective linear map preserving nilpotents in both directions also preserves nilpotents of rank one in both directions. It is implicit in Lemma 2.3 that maps preserving idempotents of rank one appear when studying maps preserving commutativity in both directions.

The above considerations show that the knowledge of the structure of bijective linear maps on $\mathcal{B}(X)$ preserving operators of rank one (idempotents of rank one, nilpotents of rank one) could be useful when studying some linear preserver problems. An interested reader can find results on the structure of such maps in [15, 25]. Here, we shall describe linear maps preserving rank one operators only in the finite–dimensional case as we believe that the proof in this special case already contains the essential ideas.

LEMMA 2.4. *Let X be finite–dimensional. If a bijective linear map $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ preserves rank one operators then there exist invertible $M, N \in \mathcal{B}(X)$ such that either $\phi(A) = MAN$ for every $A \in \mathcal{B}(X)$ or $\phi(A) = MA'N$ for every $A \in \mathcal{B}(X)$.*

Sketch of the proof. Given $x \in X$ and $f \in X'$ we define a rank one operator $x \otimes f$ by $(x \otimes f)y = f(y)x$. Consider the sets $L_x = \{x \otimes h | h \in X'\}$ and $R_f = \{u \otimes f | u \in X\}$. Each of L_x and R_f is a linear subspace of $\mathcal{B}(X)$ consisting of rank one operators and is maximal among such spaces. As ϕ preserves rank one operators it follows that for any nonzero $x \in X$, $\phi(L_x)$ is either L_y for some $y \in X$ or R_g for some $g \in X'$. In the next step one can prove that we cannot have $\phi(L_x) = L_y$ and $\phi(L_z) = R_f$ simultaneously for some nonzero x and z in X. Thus, we have two cases. Let us first consider the case when for every nonzero $x \in X$ there is $y \in X$ such that $\phi(L_x) = L_y$, so that $\phi(x \otimes f) = y \otimes g$ for every $f \in X'$. The map $f \longmapsto g$ is linear. Therefore, $g = C_x f$ for some linear operator $C_x : X' \longrightarrow X'$. It turns out that C_x and C_z are linearly dependent for every pair of nonzero vectors $x, z \in X$. Thus, by absorbing a constant in the first term of the tensor product, we may assume that C_x is independent of x. Whence, for every nonzero x there exists y such that $\phi(x \otimes f) = y \otimes Cf$, $f \in X'$. The map $x \longmapsto y$ is linear. Denoting it by M we thus have $\phi(x \otimes f) = Mx \otimes Cf$. The bijectivity of ϕ implies the bijectivity of M and C. Thus, there is a bijective linear operator $N : X \longrightarrow X$ such that $C = N'$, and we have $\phi(x \otimes f) = M(x \otimes f)N$, $x \in X$, $f \in X'$. That is, ϕ takes the first of the two forms described in the lemma.

It remains to consider the case when for every nonzero $x \in X$ there is $f \in X'$ such that $\phi(L_x) = R_f$. Using an analogous approach one can then prove that ϕ takes the second form. ∎

Using Lemmas 2.1, 2.2, 2.3 and certain extensions of Lemma 2.4, Jafarian and Sourour [17], Šemrl [31] and Omladič [22] have proved the following theorems.

THEOREM 2.5 [17, Theorem 2]. *A surjective spectrum–preserving linear map of $\mathcal{B}(X)$ onto $\mathcal{B}(X)$ is either an automorphism or an antiautomorphism.*

THEOREM 2.6 [31, Main theorem]. *Let $\mathcal{B}_0(X)$ be the linear span of all nilpotent operators in $\mathcal{B}(X)$. Suppose that a surjective linear map $\phi : \mathcal{B}_0(X) \longrightarrow \mathcal{B}_0(X)$ preserves* *nilpotent operators in both directions. Then* ϕ *is of the form* $\phi(A) = c\psi(A)$, $A \in \mathcal{B}_0(X)$, *where* $c \in \mathbb{C}$, $c \neq 0$, *and* ψ *is either an automorphism or an antiautomorphism of* $\mathcal{B}(X)$.

THEOREM 2.7 [22, Theorem 1.1]. *Suppose that the dimension of* X *is greater than* 2 *and* $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ *is a bijective linear map preserving commutativity in both directions. Then* ϕ *is of the form* $\phi(A) = c\psi(A) + f(A)I$, *where* $c \in \mathbb{C}$, $c \neq 0$, ψ *is either an automorphism or an antiautomorphism of* $\mathcal{B}(X)$, *and* f *is a linear functional on* $\mathcal{B}(X)$.

Let us mention the papers [23, 30] which also use the approach just described.

We remark that Theorems 2.5, 2.6 and 2.7 generalize a number of results existing in the literature (see [17, 31, 22] for details concerning background).

It should be mentioned that the assumption in Theorem 2.7 concerning the dimension of X is not superfluous (see an example in [34]). In the next section we shall see how Theorem 2.7 can be generalized using completely different approach.

Theorem 2.5 has been extended in several ways [3, 24, 32]. In these papers a similar approach based on some spectral characterizations of rank one operators has been used. Let us just mention the result of Sourour [32] stating that Theorem 2.5 remains true when assuming that ϕ is unital and preserves invertible operators (that is, $\sigma(\phi(A)) \subseteq \sigma(A)$ for every $A \in \mathcal{B}(X)$). For other results on invertibility preserving maps of various algebras we recommend [18, Section 9] and [33].

Let us state another generalization of Theorem 2.5, due to the present authors, which was obtained by quite different means.

THEOREM 2.8 [9, Theorem 1]. *Suppose that a surjective linear map* ϕ *of* $\mathcal{B}(X)$ *onto* $\mathcal{B}(X)$ *preserves the spectral radius* (*that is,* $r(\phi(A)) = r(A)$ *for every* $A \in \mathcal{B}(X)$). *Then* $\phi = c\psi$ *where* $c \in \mathbb{C}$, $|c| = 1$, *and* ψ *is either an automorphism or an antiautomorphism of* $\mathcal{B}(X)$.

Sketch of the proof. We shall outline only the main steps in the proof. It is easy to see that ϕ is injective, and that without loss of generality one can assume $\phi(I) = I$.

In the next step one proves that $B^k = 0$, $k \geq 2$, implies $\phi(B)^{2k-1} = 0$. In order to show this one considers the expression $r(A+\lambda Q)^k$ where $\lambda \in \mathbb{C}$, $Q = \phi(B)$, and $A \in \mathcal{B}(X)$ satisfies $AQ^iA = 0$ for $i = 0, 1, \ldots, k-1$. On the one hand we have

$$r(A + \lambda Q)^k = r((A + \lambda Q)^k) = |\lambda|^{k-1} r(B_1 + \lambda B_2)$$

where $B_1 = AQ^{k-1} + QAQ^{k-2} + \ldots + Q^{k-1}A$ and $B_2 = Q^k$. On the other hand, using $B^k = 0$ we get

$$r(A + \lambda Q)^k = r(\phi^{-1}(A + \lambda Q))^k = r((\phi^{-1}(A) + \lambda B)^k) = r(A_0 + \lambda A_1 + \ldots + \lambda^{k-1} A_{k-1})$$

for some $A_i \in \mathcal{B}(X)$. Comparing both results one can then show that the function $\lambda \longmapsto r(B_1 + \lambda B_2)$ is bounded. As this function is subharmonic by Vesentini's theorem [2, Theorem 3.4.7], it follows from the Liouville theorem for subharmonic functions [2, Theorem A.1.11] that it is a constant. Noting that $r(B_1) = 0$ we thus have

$$r(B_1 + \lambda B_2) = r(AQ^{k-1} + QAQ^{k-2} + \ldots + Q^{k-1}A + \lambda Q^k) = 0$$

for every $\lambda \in \mathbb{C}$. Chosing the operator A in a suitable way one then shows that this is impossible unless $Q^{2k-1} = 0$.

This step shows, in particular, that ϕ preserves nilpotent operators in both directions (namely, ϕ^{-1} also preserves the spectral radius). Therefore, Theorem 2.6 can be applied. Thus, in case $\mathcal{B}_0(X) = \mathcal{B}(X)$ (for instance, this is true when X is a Hilbert space [27]) we are done. In general, however, we still have some work to do. The rest of the proof is based on the observation that ϕ preserves a certain "small part" of the spectrum – more precisley, $\pi(\phi(A)) = \pi(A)$ for each $A \in \mathcal{B}(X)$ where $\pi(A) = \{\lambda \in \sigma(A) | r(A) = |\lambda|\}$. ∎

3. Using algebraic tools. We begin by stating a ring theory theorem which can be, as we shall see, applied to some linear preserver problems.

THEOREM 3.1 [4, Theorem 2]. *Let $\mathcal{A}$ and $\mathcal{A}'$ be centrally closed prime algebras over a field F. Suppose that the characteristic of $\mathcal{A}$ and $\mathcal{A}'$ is different from 2 and 3 and that neither $\mathcal{A}$ nor $\mathcal{A}'$ satisfies S_4. If a bijective linear map $\phi : \mathcal{A} \longrightarrow \mathcal{A}'$ satisfies $\phi(x^2)\phi(x) = \phi(x)\phi(x^2)$ for all $x \in \mathcal{A}$, then ϕ is of the form $\phi(x) = c\psi(x) + f(x)1$, where $c \in F$, $c \neq 0$, ψ is either an isomorphism or an antiisomorphism of $\mathcal{A}$ onto $\mathcal{A}'$, and f is a linear functional on $\mathcal{A}$.*

Let us explain roughly some notions appearing in the formulation of this theorem. An algebra $\mathcal{A}$ is called prime if $IJ = 0$, where I and J are ideals of $\mathcal{A}$, implies $I = 0$ or $J = 0$. A prime algebra $\mathcal{A}$ over a field F is said to be centrally closed over F if its extended centroid is equal to $F1$ (1 being the unit element of $\mathcal{A}$). The extended centroid of a prime ring is a certain field containing the center of a ring (it is often, but not always, the field of fractions of the center) – for an exact definition we refer the reader to [13] or [21]. Let us just mention that $\mathcal{B}(X)$ is centrally closed over $\mathbb{C}$ (the same is true, for instance, for subalgebras of $\mathcal{B}(X)$ containing the identity and all finite rank operators, for prime C^*-algebras, and finite dimensional prime normed algebras). The condition that $\mathcal{A}$ does not satisfy S_4, the standard polynomial identity of degree 4, is equivalent to the condition that $\mathcal{A}$ is noncommutative and does not embed in the ring of 2 by 2 matrices over a field.

Theorem 3.1 follows from a result which characterizes biadditive maps B of a prime ring R (satisfying some additional assumptions) such that $B(x,x)x = xB(x,x)$ for every $x \in R$. It turns out that there is $\lambda \in C$, the extended centroid of R, and maps $\mu, \nu : R \longrightarrow C$ such that $B(x,x) = \lambda x^2 + \mu(x)x + \nu(x)$ for every $x \in R$ [4, Theorem 1]. The proof of this result is elementary in a sense that only basic properties of prime rings and their extended centroids are used. The computations in the proof, however, are rather involved. The proof of Theorem 3.1 is based on the following observation. The condition $\phi(x^2)\phi(x) = \phi(x)\phi(x^2)$, $x \in \mathcal{A}$, can be expressed in the form $\phi(\phi^{-1}(y)^2)y = y\phi(\phi^{-1}(y)^2)$, $y \in \mathcal{A}'$. That is, $B(y,y)y = yB(y,y)$ where B is a bilinear map of $\mathcal{A}'$ defined by $B(y,z) = \phi(\phi^{-1}(y)\phi^{-1}(z))$. Now, of course, a result just mentioned applies.

The algebra $\mathcal{B}(X)$ satisfies all conditions of Theorem 3.1 provided that the dimension of X is not 1 or 2. As x and x^2 certainly commute, we have the following immediate consequence of Theorem 3.1.

COROLLARY 3.2. *Suppose that the dimension of X is greater than 2 and $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ is bijective linear map preserving commutativity. Then ϕ is of the form $\phi(A) =$*

$c\psi(A)+f(A)I$, where $c\in\mathbb{C}$, $c\neq 0$, ψ is either an automorphism or an antiautomorphism of $\mathcal{B}(X)$, and f is a linear functional on $\mathcal{B}(X)$.

Thus, this result tells us that in Theorem 2.7 one does not need to assume that ϕ preserves commutativity in both directions. We also mention that a similar approach gives a description of commutativity preserving maps on von Neumann algebras [5].

In order to obtain two other applications of Theorem 3.1 we need an easy lemma.

LEMMA 3.3. *Let X be a Hilbert space. Suppose that a map $\phi:\mathcal{B}(X)\longrightarrow\mathcal{B}(X)$ satisfies $\phi(S^2)\phi(S)=\phi(S)\phi(S^2)$ for every self–adjoint $S\in\mathcal{B}(X)$. Then $\phi(A^2)\phi(A)=\phi(A)\phi(A^2)$ for every $A\in\mathcal{B}(X)$.*

Proof. Let $S,T\in\mathcal{B}(X)$ be self–adjoint and let $t\in\mathbb{R}$. Replacing S by $S+tT$ in $[\phi(S^2),\phi(S)]=0$ (here, [X,Y] denotes the commutator $XY-YX$) we get

$$t([\phi(ST+TS),\phi(S)]+[\phi(S^2),\phi(T)])+t^2([\phi(T^2),\phi(S)]+[\phi(ST+TS),\phi(T)])=0.$$

Whence

$$[\phi(T^2),\phi(S)]+[\phi(ST+TS),\phi(T)]=0$$

for all self–adjoint $S,T\in\mathcal{B}(X)$. Decomposing an arbitrary $A\in\mathcal{B}(X)$ as $A=S+iT$ with S,T self–adjoint, we thus get

$$\begin{aligned}[\phi(A^2),\phi(A)]=i([\phi(S^2),\phi(T)]+[\phi(ST+TS),\phi(S)])\\ -([\phi(ST+TS),\phi(T)]+[\phi(T^2),\phi(S)])=0,\end{aligned}$$

proving the lemma. ∎

Clearly, for the case when X is a Hilbert space, Theorem 3.1 and Lemma 3.3 give the following slight generalization of Corollary 3.2.

COROLLARY 3.4. *Let X be a Hilbert space of dimension greater than 2. Suppose that $\phi:\mathcal{B}(X)\longrightarrow\mathcal{B}(X)$ is a bijective linear map preserving commutativity of self–adjoint operators (that is, $\phi(S)\phi(T)=\phi(T)\phi(S)$ whenever $ST=TS$ and S,T are self–adjoint). Then the conclusion of Corollary 3.2 holds.*

The third application of Theorem 3.1 is a description of normal–preserving maps.

COROLLARY 3.5 [7, Theorem 2]. *Let X be a Hilbert space of dimension greater than 2. Suppose that $\phi:\mathcal{B}(X)\longrightarrow\mathcal{B}(X)$ is a bijective linear map preserving normal operators. Then ϕ is of the form $\phi(A)=c\psi(A)+f(A)I$, where $c\in\mathbb{C}$, $c\neq 0$, ψ is either a $*-$automorphism or a $*-$antiautomorphism of $\mathcal{B}(X)$, and f is a linear functional on $\mathcal{B}(X)$.*

Sketch of the proof. Let $S\in\mathcal{B}(X)$ be self–adjoint. Then $S^2+\lambda S$ is normal for each $\lambda\in\mathbb{C}$. Whence $\phi(S^2)+\lambda\phi(S)$ is normal which further implies $[\phi(S^2),\phi(S)^*]=0$. As $\phi(S)$ is normal it follows from Fuglede's theorem [28, Corollary 1.18] that $[\phi(S^2),\phi(S)]=0$. Theorem 3.1 and Lemma 3.3 now tell us that ϕ is of the form $\phi(A)=c\psi(A)+f(A)I$ where $c\in\mathbb{C}$, f is a linear functional on $\mathcal{B}(X)$ and ψ is an automorphism or an antiautomorphism. It remains to prove that ψ is adjoint–preserving. But this is not too difficult. ∎

Both Corollary 3.4 and 3.5 were proved in [10] under the additional assumption that the map ϕ preserves adjoints. The proofs, however, are completely different from those outlined here.

Recall that a linear map ϕ from an algebra $\mathcal{A}$ into an algebra $\mathcal{A}'$ is called a *Jordan homomorphism* if $\phi(x^2) = \phi(x)^2$ for every $x \in \mathcal{A}$. As both homomorphisms and antihomomorphisms are obvious examples, it comes as no surprise that Jordan homomorphisms can appear when studying linear preserver problems. A well–known result of Herstein [14, Theorem 3.1] states that if ϕ is onto and $\mathcal{A}'$ is prime, then these two examples are in fact the only examples. This shows, in particular, that a bijective Jordan homomorphism on $\mathcal{B}(X)$ is either an automorphism or an antiautomorphism. In the next two results we illustrate how this fact can be used.

PROPOSITION 3.6. *Let X be a Hilbert space. Suppose that a bijective continuous linear map $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ preserves unitary operators. If $\phi(I) = I$ then ϕ is either a $*-$automorphism or a $*-$antiautomorphism.*

Proof. Pick a self–adjoint $S \in \mathcal{B}(X)$. Then e^{itS} is unitary for every $t \in \mathbb{R}$. Therefore,

$$\begin{aligned} I = \phi(e^{itS})\phi(e^{itS})^* &= \phi(I + itS - \frac{t^2}{2}S^2 + \ldots)\phi(I + itS - \frac{t^2}{2}S^2 + \ldots)^* \\ &= (I + it\phi(S) - \frac{t^2}{2}\phi(S^2) + \ldots)(I - it\phi(S)^* - \frac{t^2}{2}\phi(S^2)^* + \ldots) \\ &= I + t(i\phi(S) - i\phi(S)^*) + t^2(-\frac{1}{2}\phi(S^2) - \frac{1}{2}\phi(S^2)^* + \phi(S)\phi(S)^*) + \ldots \end{aligned}$$

Whence $\phi(S) = \phi(S)^*$ and $\frac{1}{2}\phi(S^2) + \frac{1}{2}\phi(S^2)^* = \phi(S)\phi(S)^*$ for every self–adjoint $S \in \mathcal{B}(X)$. The first relation implies that ϕ preserves adjoints. Therefore, the second relation can be written as $\phi(S^2) = \phi(S)^2$. Substituting $S + T$ for S we arrive at $\phi(ST + TS) = \phi(S)\phi(T) + \phi(T)\phi(S)$. Using the fact that every $A \in \mathcal{B}(X)$ can be written as $A = S + iT$ with S, T self–adjoint, we get $\phi(A^2) = \phi(A)^2$. Thus, Herstein's theorem now gives the desired conclusion. ∎

By a projection in $\mathcal{B}(X)$, where X is a Hilbert space, we mean a self–adjoint idempotent.

PROPOSITION 3.7. *Let X be a Hilbert space. Suppose that a bijective continuous linear map $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ preserves projections. Then ϕ is either a $*-$automorphism or a $*-$antiautomorphism.*

Proof. Let P and Q be orthogonal projections. Then $P + Q$ is a again a projection. Therefore, $\phi(P + Q)^2 = \phi(P + Q)$, which yields $\phi(P)\phi(Q) + \phi(Q)\phi(P) = 0$. Whence we see that $\phi(P)$ and $\phi(Q)$ are orthogonal projections. This implies that $\phi(H^2) = \phi(H)^2$ holds for every operator H which is a real–linear combination of mutually orthogonal projections. Observe that $\phi(H)$ is self–adjoint. As ϕ is continuous and the set of all such operators H is dense in the set of all self–adjoint operators in $\mathcal{B}(X)$, it follows that $\phi(S^2) = \phi(S)^2$ holds for every self–adjoint $S \in \mathcal{B}(X)$, and that ϕ preserves adjoints. Now continue in the same fashion as in the proof of the last proposition. ∎

It is quite clear that the proofs of both propositions work in a more general setting. It was our purpose, however, to present the methods rather than the results. For more general results see [29] and [11].

In the next theorem we do not assume the continuity of the map. This makes the proof quite more nontrivial.

THEOREM 3.8 [8, Theorem 1]. *Let X be a Hilbert space. Suppose that a bijective linear map $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ preserves idempotents. Then ϕ is either an automorphism or an antiautomorphism.*

Sketch of the proof. Every self–adjoint operator of finite rank is a real–linear combination of mutually orthogonal projections. Using this fact and arguing similarly as in the proof of Proposition 3.7, one shows that $\phi|\mathcal{F}(X)$, the restriction of ϕ to the algebra $\mathcal{F}(X)$ of all finite rank operators, is a Jordan homomorphism. As $\mathcal{F}(X)$ is a locally matrix algebra, a result of Jacobson and Rickart [16, Theorem 8] tells us that $\phi|\mathcal{F}(X) = \psi + \theta$ where ψ is a homomorphism and θ is an antihomomorphism. In the next step one shows that ϕ preserves rank one idempotents. This readily implies that at least one of ψ and θ has a nonzero kernel. Since the kernels of homomorphisms and antihomomorphisms are ideals and $\mathcal{F}(X)$ is a simple algebra, it follows that $\psi = 0$ or $\theta = 0$. It is easy to see that without loss of generality we may assume that $\phi|\mathcal{F}(X)$ is a homomorphism. Now fix $x \in X$ with norm 1 and consider $x \otimes x$ (by $z \otimes w$ we denote the rank one operator defined by $(z \otimes w)y =< y, w > z$). We have $\phi(x \otimes x) = u \otimes v$ for some $u, v \in X$ with $< u, v >= 1$. Define a linear operator $T : X \longrightarrow X$ by $Ty = \phi(y \otimes x)u$. It is straightforward to verify that $TF = \phi(F)T$ holds for every $F \in \mathcal{F}(X)$. Using this one can show that $TP = \phi(P)T$ for every idempotent $P \in \mathcal{B}(X)$. As every operator in B(X) is a linear combination of idempotents [27] it follows that $TA = \phi(A)T$ holds for every $A \in \mathcal{B}(X)$. It is now easy to see that T is bijective and so ϕ is an automorphism. ∎

If X is a Banach space, Theorem 3.8 remains true under the additional assumption that ϕ is continuous in the weak operator topology [6, Theorem 3.3].

We close this paper by an application of Theorem 3.8. By a potent operator we mean an operator A satisfying $A^r = A$ for some integer $r \geq 2$.

THEOREM 3.9 [26, Theorem 4]. *Let X be a Hilbert space. Suppose that a surjective linear map $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ preserves potent operators in both directions. Then ϕ is of the form $\phi(A) = c\psi(A)$ where $c \in \mathbb{C}$ is a root of unity and ψ is either an automorphism or an antiautomorphism.*

Sketch of the proof. The most difficult part of the proof is to show that $\phi(I) = cI$ for some root of unity c. Once we know this, there is clearly no loss of generality in assuming $\phi(I) = I$. According to Theorem 3.8 it suffices to prove that $\phi(P)$ is an idempotent whenever $P \in \mathcal{B}(X)$ is an idempotent. Since $\phi(P)$ is a potent, it is enough to show that every $\lambda \in \sigma(\phi(P))$ is either 0 or 1. Since $P, I - P$, and $I - 2P$ are potents, $\phi(P)$, $I - \phi(P)$, and $I - 2\phi(P)$ are potents, too. But then each of the numbers λ, $1 - \lambda$, and $1 - 2\lambda$ is either a root of unity or 0. But then $\lambda = 0$ or $\lambda = 1$. ∎

References

[1] *A survey of linear preserver problems*, Linear and Multilinear Algebra 33 (1992), 1–129.
[2] B. Aupetit, *A primer on spectral theory*, Universitext, Springer, New York, 1991.
[3] B. Aupetit and H. du T. Mouton, *Spectrum–preserving linear mappings in Banach algebras*, Studia Math. 109 (1994), 91–100.
[4] M. Brešar, *Commuting traces of biadditive mappings, commutativity–preserving mappings and Lie mappings*, Trans. Amer. Math. Soc. 335 (1993), 525–546.
[5] M. Brešar and C. R. Miers, *Commutativity preserving mappings of von Neumann algebras*, Canad. J. Math. 45 (1993), 695–708.
[6] M. Brešar and P. Šemrl, *Mappings which preserve idempotents, local automorphisms, and local derivations*, Canad. J. Math. 45 (1993), 483–496.
[7] —, —, *Normal–preserving linear mappings*, Canad. Math. Bull. 37 (1994), 306–309.
[8] —, —, *On local automorphisms and mappings that preserve idempotents*, Studia Math. 113 (1995), 101–108.
[9] —, —, *Linear maps preserving the spectral radius*, J. Funct. Anal. 142 (1996), 360–368.
[10] M. D. Choi, A. A. Jafarian and H. Radjavi, *Linear maps preserving commutativity*, Linear Algebra Appl. 87 (1987), 227–241.
[11] H. A. Dye, *On the geometry of projections in certain operator algebras*, Ann. of Math. (2) 61 (1955), 73–89.
[12] G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen*, Sitzungsber. Deutsch. Akad. Wiss. Berlin (1897), 994–1015.
[13] I. N. Herstein, *Rings with involution*, University of Chicago Press, Chicago, 1976.
[14] —, *Topics in ring theory*, University of Chicago Press, Chicago, 1969.
[15] J. C. Hou, *Rank preserving linear maps on $\mathcal{B}(X)$*, Sci. China Ser. A 32 (1989), 929–940.
[16] N. Jacobson and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. 69 (1950), 479–502.
[17] A. A. Jafarian and A. R. Sourour, *Spectrum–preserving linear maps*, J. Funct. Anal. 66 (1986), 255–261.
[18] I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, Regional Conference Series in Mathematics 1, Amer. Math. Soc., Providence, 1970.
[19] C.–K. Li and N.–K. Tsing, *Linear preserver problems: A brief introduction and some special techniques*, Linear Algebra Appl. 162–164 (1992), 217–235.
[20] M. Marcus, *Linear operations on matrices*, Amer. Math. Monthly 69 (1962), 837–847.
[21] W. S. Martindale, *Prime rings satisfying a generalized polynomial identity*, J. Algebra 12 (1969), 576–584.
[22] M. Omladič, *On operators preserving commutativity*, J. Funct. Anal. 66 (1986), 105–122.
[23] —, *On operators preserving the numerical range*, Linear Algebra Appl. 134 (1990), 31–51.
[24] M. Omladič and P. Šemrl, *Spectrum–preserving additive maps*, ibid. 153 (1991), 67–72.
[25] —, —, *Additive mappings preserving operators of rank one*, ibid. 182 (1993), 239–256.
[26] —, —, *Linear mappings that preserve potent operators*, Proc. Amer. Math. Soc. 123 (1995), 1069–1074.
[27] C. Pearcy and D. Topping, *Sums of small numbers of operators*, Michigan Math. J. 14 (1967), 453–465.
[28] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Ergeb. Math. Grenzgeb. 77, Springer, Berlin, 1973.

[29] B. Russo and H. A. Dye, *A note on unitary operators in C^*-algebras*, Duke Math. J. 33 (1966), 413–416.

[30] P. Šemrl, *Two characterizations of automorphisms on $\mathcal{B}(X)$*, Studia Math. 105 (1993), 143–149.

[31] —, *Linear maps that preserve the nilpotent operators*, Acta Sci. Math. (Szeged) 61 (1995), 523–534.

[32] A. R. Sourour, *Invertibility preserving linear maps on $\mathcal{L}(X)$*, preprint.

[33] —, *The Gleason–Kahane–Żelazko theorem and its generalizations*, Banach Center Publ. 30 (1994), 327–331.

[34] W. Watkins, *Linear maps that preserve commuting pairs of matrices*, Linear Algebra Appl. 14 (1976), 29–35.