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A NOTE ON THE DIFFERENCES OF THE CONSECUTIVE POWERS OF OPERATORS

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Abstract. We present two examples. One of an operator T such that $\{T^n(T-I)\}_{n=1}^{\infty}$ is precompact in the operator norm and the spectrum of T on the unit circle consists of an infinite number of points accumulating at 1, and the other of an operator T such that $\{T^n(T-I)\}_{n=1}^{\infty}$ is convergent to zero but T is not power bounded.

Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$ be power bounded. Denote by Γ the unit circle in $\mathbb{C}$. The main aim of this note which may be regarded as an addendum to [2] is to answer a question stated there if the precompactness of $\{x^n(x-1)\}_{n=1}^{\infty}$ in $\mathcal{A}$ implies that $1 \notin \overline{\sigma(x) \cap (\Gamma \setminus \{1\})}$. The structure of $\sigma(x)$ in this case has been investigated in [1] and [2] (see the references quoted therein). It was proved in [1] that $\{x^n(x-1)\}_{n=1}^{\infty}$ is precompact if and only if $\sigma(x) \cap (\Gamma \setminus \{1\})$ consists of simple poles of x. The example that we present below shows that 1 can belong to the closure of $\sigma(x) \cap (\Gamma \setminus \{1\})$ and therefore the result quoted above is sharp.

Example 1. Let $\lambda_n = e^{2\pi i/n}$. Define $T : l^2 \to l^2$ by $Te_n = \lambda_n e_n$, $n = 1, 2, \ldots$, where $e_n = (0, \ldots, 0, 1, 0, \ldots)$ is the nth standard basis vector of l^2. Then $\|T\| = 1$ and $\sigma(T) = \{\lambda_1, \lambda_2, \ldots\}$. We claim that $\{T^n(T-I)\}_{n=1}^{\infty}$ is precompact in the algebra of bounded operators on l^2 equipped with the operator norm. This fact can be deduced from the above quoted result of [1], however we will give a simple and direct proof. We will show that for every increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers we can choose a subsequence $\{m_k\}_{k=1}^{\infty}$ such that $T^{m_k}(T-I)$ is convergent as $k \to \infty$. We construct $\{m_k\}_{k=1}^{\infty}$ as follows. Since $\lambda_2^{n_k}$ attains only a finite number of values we choose a subsequence $\{n_{2_k}\}_{k=1}^{\infty}$ such that $\lambda_2^{n_{2_k}} = \overline{\lambda}_2$ for some $\overline{\lambda}_2$. Then, since $\lambda_3^{n_{2_k}}$ takes on only a finite number of values, we choose from $\{n_{2_k}\}_{k=1}^{\infty}$ a subsequence $\{n_{3_k}\}_{k=1}^{\infty}$ such that $\lambda_3^{n_{3_k}} = \overline{\lambda}_3$ for some $\overline{\lambda}_3$. We continue this process and define $m_1 = n_1$ and $m_k = n_{k_k}$ for

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$k > 1$. For m_k so defined we have

$$\lambda_j^{m_k} = \overline{\lambda}_j \quad \text{for } k \geq j,\ k, j = 1, 2, \ldots \tag{1}$$

Notice that $m_k \geq k$ for $k = 1, 2, \ldots$ Let $\varepsilon > 0$. Choose n_0 such that $|\lambda_n - 1| < \varepsilon/2$ for $n \geq n_0$. Denote by P the orthogonal projection onto $\overline{\text{span}}\{e_{n_0+1}, e_{n_0+2}, \ldots\}$. Take any $k, l \geq n_0$. By the definition of T and (1) it follows that

$$\|(T^{m_k} - T^{m_l})(T - I)x\| = \|(T^{m_k} - T^{m_l})(T - I)Px\| \leq 2\|(T - I)P\|\|x\| \leq \varepsilon\|x\|,$$

where the last inequality follows from the choice of n_0. Therefore $\{T^{m_k}(T - I)\}_{k=1}^{\infty}$ is Cauchy. It converges to T_0 defined by $T_0 e_n = \overline{\lambda}_n e_n$, $n = 1, 2, \ldots$ The claim is proved.

An interesting feature of Example 1 is that the set $\{T^n\}_{n=1}^{\infty}$ is discrete. In fact, if $n - m = p > 0$, then

$$\|(T^n - T^m)e_{2p}\| = \|2e^{\pi m i/p} e_{2p}\| = 2.$$

However, if we set $\lambda_n = e^{\frac{2\pi i}{2^{n^2}}}$, $n = 1, 2, \ldots$, in the definition of T, then $\{T^n\}_{n=1}^{\infty}$ has an accumulation point. To show this observe that, if $x = \sum_{n=1}^{\infty} x_n e_n$, $\|x\| \leq 1$, $p_n = 2^{n^2}$, then

$$\begin{aligned}\|(T^{p_n} - I)x\| &= \Big\| \sum_{j=n+1}^{\infty} (\lambda_j^{p_j} - 1)x_j e_j \Big\| = \Big(\sum_{j=n+1}^{\infty} |\lambda_j^{p_j} - 1|^2 |x_j|^2 \Big)^{1/2} \\ &\leq \Big(\sum_{j=n+1}^{\infty} (\arg(\lambda_j^{p_j}))^2 \Big)^{1/2} \leq \Big(\sum_{j=n+1}^{\infty} \frac{16\pi^2}{2^{4j}} \Big)^{1/2} \\ &= \frac{4\pi}{\sqrt{15}} \frac{1}{2^{2n}} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore the existence of accumulation points depends also on the geometry of the spectrum and not only on its cardinality. Moreover, it trivially follows that if $\{x^n\}_{n=1}^{\infty}$ has an accumulation point then $\sigma(x) = \sigma_1 \cup \sigma_2$, where σ_1, σ_2 are two closed sets such that $\sigma_1 \subset \{z : |z| < 1\}$, $\sigma_2 \subset \Gamma$. To see this, suppose that x^{n_k} converges as $k \to \infty$ for some sequence $\{n_k\}_{k=1}^{\infty}$, i.e. for every $\varepsilon > 0$ there is n_0 such that $\|x^{n_{k_1}} - x^{n_{k_2}}\| < \varepsilon$ if $n_{k_1}, n_{k_2} > n_0$. Let $n_{k_2} > n_{k_1}$. We have $\sigma(x^{n_{k_1}} - x^{n_{k_2}}) = (z^{n_{k_1}} - z^{n_{k_2}})(\sigma(x))$. However,

$$|z^{n_{k_1}} - z^{n_{k_2}}| \geq |z|^{n_{k_1}} - |z|^{n_{k_2}} \tag{2}$$

and if we fix n_{k_1}, the right hand side of (2) can be made arbitrarily close to 1 by choosing $|z|$ sufficiently close to 1 and then n_{k_2} sufficiently large. Hence $\sigma(x)$ cannot approach Γ.

We would like to finish with an elementary example of an operator T such that $T^n(T - I) \to 0$ as $n \to \infty$ but T is not power bounded. It is similar in the spirit to Example 3.7 in [2], ours is however very explicit.

Example 2. Let $T : l^2 \to l^2$ be defined by

$$Te_{2n} = \frac{n-1}{n} e_{2n} + \frac{1}{\ln(n+1)} e_{2n-1}, \qquad Te_{2n-1} = e_{2n-1}$$

for $n = 1, 2, \ldots$ Let $\varepsilon > 0$, $x = \sum_{n=1}^{\infty} x_n e_n$. Then

$$\begin{aligned}\|(T^{k+1} - T^k)x\|^2 &= \Big\| \sum_{n=1}^{\infty} \Big(\frac{1}{\ln(n+1)} \Big(\frac{n-1}{n}\Big)^k x_{2n} e_{2n-1} - \frac{1}{n}\Big(\frac{n-1}{n}\Big)^k x_{2n} e_{2n} \Big) \Big\|^2 \\ &\le \sum_{n=1}^{\infty} \frac{1}{\ln^2(n+1)} \Big(\frac{n-1}{n}\Big)^{2k} |x_{2n}|^2 + \sum_{n=1}^{\infty} \frac{1}{n^2} \Big(\frac{n-1}{n}\Big)^{2k} |x_{2n}|^2 \\ &\le \sum_{n=1}^{n_0} \frac{1}{\ln^2(n+1)} \Big(\frac{n-1}{n}\Big)^{2k} |x_{2n}|^2 + \frac{1}{\ln^2(n_0+1)} \sum_{n=n_0+1}^{\infty} |x_{2n}|^2 \\ &\quad + \sum_{n=1}^{n_0} \frac{1}{n^2} \Big(\frac{n-1}{n}\Big)^{2k} |x_{2n}|^2 + \frac{1}{(n_0+1)^2} \sum_{n=n_0+1}^{\infty} |x_{2n}|^2 \le \varepsilon^2 \|x\|^2\end{aligned}$$

by first choosing n_0 sufficiently large and then taking k large enough. To see that T is not power bounded, observe that

$$\|T^{n+1} e_{2n}\| \ge \frac{1}{\ln(n+1)} \sum_{j=1}^{n} \Big(\frac{n-1}{n}\Big)^j.$$

Since $(\frac{n-1}{n})^n > \frac{1}{3}$ for large n (actually it is close to $1/e$), we obtain

$$\|T^{n+1} e_{2n}\| \ge \frac{n}{3\ln(n+1)}.$$

Hence $\|T^n\| \ge O(n^{\alpha})$ for every $0 < \alpha < 1$. On the other hand, we have $T^n = o(n)$.

References

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[2] H. C. Rönnefarth, *On the differences of the consecutive powers of Banach algebra elements*, this volume, 297–314.