A NOTE ON THE DIFFERENCES OF THE CONSECUTIVE POWERS OF OPERATORS

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Abstract. We present two examples. One of an operator $T$ such that $\{T^n(T-I)\}_{n=1}^{\infty}$ is precompact in the operator norm and the spectrum of $T$ on the unit circle consists of an infinite number of points accumulating at 1, and the other of an operator $T$ such that $\{T^n(T-I)\}_{n=1}^{\infty}$ is convergent to zero but $T$ is not power bounded.

Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$ be power bounded. Denote by $\Gamma$ the unit circle in $\mathbb{C}$. The main aim of this note which may be regarded as an addendum to [2] is to answer a question stated there if the precompactness of $\{x^n(x-1)\}_{n=1}^{\infty}$ in $\mathcal{A}$ implies that $1 \notin \sigma(x) \cap (\Gamma \setminus \{1\})$. The structure of $\sigma(x)$ in this case has been investigated in [1] and [2] (see the references quoted therein). It was proved in [1] that $\{x^n(x-1)\}_{n=1}^{\infty}$ is precompact if and only if $\sigma(x) \cap (\Gamma \setminus \{1\})$ consists of simple poles of $x$. The example that we present below shows that 1 can belong to the closure of $\sigma(x) \cap (\Gamma \setminus \{1\})$ and therefore the result quoted above is sharp.

Example 1. Let $\lambda_n = e^{2\pi i / n}$. Define $T : l^2 \rightarrow l^2$ by $Te_n = \lambda_n e_n$, $n = 1, 2, \ldots$, where $e_n = (0, \ldots, 0, 1, 0, \ldots)$ is the $n$th standard basis vector of $l^2$. Then $\|T\| = 1$ and $\sigma(T) = \{\lambda_1, \lambda_2, \ldots\}$. We claim that $\{T^n(T-I)\}_{n=1}^{\infty}$ is precompact in the algebra of bounded operators on $l^2$ equipped with the operator norm. This fact can be deduced from the above quoted result of [1], however we will give a simple and direct proof. We will show that for every increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers we can choose a subsequence $\{m_k\}_{k=1}^{\infty}$ such that $T^{m_k}(T-I)$ is convergent as $k \rightarrow \infty$. We construct $\{m_k\}_{k=1}^{\infty}$ as follows. Since $\lambda_2^{m_k}$ attains only a finite number of values we choose a subsequence $\{n_{2k}\}_{k=1}^{\infty}$ such that $\lambda_2^{n_{2k}} = \lambda_2$ for some $\lambda_2$. Then, since $\lambda_3^{n_{2k}}$ takes on only a finite number of values, we choose from $\{n_{2k}\}_{k=1}^{\infty}$ a subsequence $\{n_{3k}\}_{k=1}^{\infty}$ such that $\lambda_3^{n_{3k}} = \lambda_3$ for some $\lambda_3$. We continue this process and define $m_1 = n_1$ and $m_k = n_{3k}$ for...
Notice that \( m_k \geq k \) for \( k = 1, 2, \ldots \). Let \( \epsilon > 0 \). Choose \( n_0 \) such that \( |\lambda_n - 1| < \epsilon/2 \) for \( n \geq n_0 \). Denote by \( P \) the orthogonal projection onto \( \overline{\text{span}} \{ e_{n_0+1}, e_{n_0+2}, \ldots \} \). Take any \( k, l \geq n_0 \). By the definition of \( T \) and (1) it follows that

\[
\| (T^{m_k} - T^{m_l})(T - I)x\| = \| (T^{m_k} - T^{m_l})(T - I)P x\| \leq 2\| (T - I)P \| \| x\| \leq \epsilon \| x\|
\]

where the last inequality follows from the choice of \( n_0 \). Therefore \( \{T^{m_k}(T - I)\}_{k=1}^{\infty} \) is Cauchy. It converges to \( T_0 \) defined by \( T_0 e_n = \lambda_n e_n, n = 1, 2, \ldots \). The claim is proved.

An interesting feature of Example 1 is that the set \( \{T^n\}_{n=1}^{\infty} \) is discrete. In fact, if \( n - m = p > 0 \), then

\[
\| (T^n - T^m)e_{2p} \| = \| 2e^{\pi mi/p} e_{2p} \| = 2.
\]

However, if we set \( \lambda_n = e^{2\pi i/n}, n = 1, 2, \ldots \), in the definition of \( T \), then \( \{T^n\}_{n=1}^{\infty} \) has an accumulation point. To show this observe that, if \( x = \sum_{n=1}^{\infty} x_n e_n, \| x\| \leq 1 \), \( p_n = 2^n \), then

\[
\| (T^n - I)x\| = \left\| \sum_{j=n+1}^{\infty} (\lambda_j^p - 1)x_j e_j \right\| = \left( \sum_{j=n+1}^{\infty} |\lambda_j^p - 1|^2 |x_j|^2 \right)^{1/2} \leq \left( \sum_{j=n+1}^{\infty} \left( \frac{16\pi^2}{2^{4j}} \right)^{1/2} \right)^{1/2} = \frac{4\pi}{\sqrt{15}} \frac{1}{2^{2n}} \to 0 \quad \text{as } n \to \infty.
\]

Therefore the existence of accumulation points depends also on the geometry of the spectrum and not only on its cardinality. Moreover, it trivially follows that if \( \{x^n\}_{n=1}^{\infty} \) has an accumulation point then \( \sigma(x) = \sigma_1 \cup \sigma_2 \), where \( \sigma_1, \sigma_2 \) are two closed sets such that \( \sigma_1 \subset \{z : |z| < 1\} \), \( \sigma_2 \subset \Gamma \). To see this, suppose that \( x^n \) converges as \( k \to \infty \) for some sequence \( \{n_k\}_{k=1}^{\infty} \), i.e. for every \( \epsilon > 0 \) there is \( n_0 \) such that \( \|x^{n_k_1} - x^{n_k_2}\| < \epsilon \) if \( n_{k_1}, n_{k_2} > n_0 \). Let \( n_{k_2} > n_{k_1} \). We have \( \sigma(x^{n_{k_1}} - x^{n_{k_2}}) = (z^{n_{k_1}} - z^{n_{k_2}})(\sigma(x)) \). However,

\[
|z^{n_{k_1}} - z^{n_{k_2}}| \geq |z|^{n_{k_1}} - |z|^{n_{k_2}}
\]

and if we fix \( n_{k_1} \), the right hand side of (2) can be made arbitrarily close to 1 by choosing \( |z| \) sufficiently close to 1 and then \( n_{k_2} \) sufficiently large. Hence \( \sigma(x) \) cannot approach \( \Gamma \).

We would like to finish with an elementary example of an operator \( T \) such that \( T^n(T - I) \to 0 \) as \( n \to \infty \) but \( T \) is not power bounded. It is similar in the spirit to Example 3.7 in [2], ours is however very explicit.

**Example 2.** Let \( T : l^2 \to l^2 \) be defined by

\[
Te_{2n} = \frac{n-1}{n} e_{2n} + \frac{1}{\ln (n+1)} e_{2n-1}, \quad Te_{2n-1} = e_{2n-1}
\]
for $n = 1, 2, \ldots$ Let $\varepsilon > 0$, $x = \sum_{n=1}^{\infty} x_n e_n$. Then

$$
\| (T^{k+1} - T^k)x \|^2 = \left\| \sum_{n=1}^{\infty} \left( \frac{1}{\ln(n+1)} \left( \frac{n-1}{n} \right)^k x_{2n} e_{2n-1} - \frac{1}{n} \left( \frac{n-1}{n} \right)^k x_{2n} e_{2n} \right) \right\|^2
$$

$$
\leq \sum_{n=1}^{\infty} \frac{1}{\ln^2(n+1)} \left( \frac{n-1}{n} \right)^{2k} |x_{2n}|^2 + \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{n-1}{n} \right)^{2k} |x_{2n}|^2
$$

$$
\leq \sum_{n=1}^{n_0} \frac{1}{\ln^2(n+1)} \left( \frac{n-1}{n} \right)^{2k} |x_{2n}|^2 + \sum_{n=n_0+1}^{\infty} \frac{1}{\ln^2(n_0+1)} \sum_{n=n_0+1}^{\infty} |x_{2n}|^2
$$

by first choosing $n_0$ sufficiently large and then taking $k$ large enough. To see that $T$ is not power bounded, observe that

$$
\| T^{n+1} e_{2n} \| \geq \frac{1}{\ln(n+1)} \sum_{j=1}^{n} \left( \frac{n-1}{n} \right)^j.
$$

Since $(\frac{n-1}{n})^n > \frac{1}{4}$ for large $n$ (actually it is close to $1/e$), we obtain

$$
\| T^{n+1} e_{2n} \| \geq \frac{n}{3\ln(n+1)}.
$$

Hence $\| T^n \| \geq O(n^\alpha)$ for every $0 < \alpha < 1$. On the other hand, we have $T^n = o(n)$.

References
