UNITARY EXTENSIONS OF ISOMETRIES, GENERALIZED INTERPOLATION AND BAND EXTENSIONS

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Abstract. The aim of this paper is to give a very brief account of some applications of the method of unitary extensions of isometries to interpolation and extension problems.

I. Unitary extensions of isometries. A general method for solving several moment and interpolation problems can be summarized as follows: the data of the problem define an isometry, with range and domain in the same Hilbert space, in such a way that each unitary extension of that isometry gives a solution of the problem.

In this review paper, the method and some of its applications are briefly described. We now fix the notation and then specify the content of the following sections.

Unless otherwise specified, all spaces are assumed to be separable complex Hilbert spaces and all subspaces are closed; \( L(X,Y) \) denotes the set of all bounded linear operators from a space \( X \) to a space \( Y \); \( \mathcal{L}(X) \) is the same as \( L(X,X) \), and “\( \mathbb{V} \)” means “closed linear span”; \( P_E^X \equiv P_E \) denotes the orthogonal projection onto the subspace \( E \) of \( X \) and \( i_E^X \equiv i_E \) is the inclusion of \( E \) in \( X \). \( L^p(X) \) denotes the space of \( X \)-valued measurable functions on the unit disk \( \mathbb{T} \) with finite \( p \)-norm. \( L^p(X,Y) \) denotes the space of \( \mathcal{L}(X,Y) \)-valued measurable functions on \( \mathbb{T} \) with finite \( p \)-norm.

The isometry \( V \) acts in the Hilbert space \( H \) if its domain \( D \) and range \( R \) are (closed) subspaces of \( H \). We say that \((U,F)\) belongs to \( \mathcal{U} \), the set of equivalence classes of minimal unitary extensions of \( V \), if \( U \in \mathcal{L}(F) \) is a unitary extension of \( V \) to a space \( F \) that contains \( H \), such that \( F = \bigvee \{ U^n H : n \in \mathbb{Z} \} \); we consider two minimal unitary extensions to be equivalent, and write \((U,F) \approx (U',F')\) in \( \mathcal{U} \), if there exists a unitary operator \( X \in \mathcal{L}(F,F') \) such that \( XU = U'X \) and that its restriction to \( H \) equals the identity \( I_H \) in \( H \). An element \((U,F)\) of \( \mathcal{U} \) with special properties is given by the minimal unitary dilation \( U \in \mathcal{L}(F) \) of the contraction \( VP_D \in \mathcal{L}(H) \).

In Section II an isometry \( V \) is associated with a generalized interpolation problem in such a way that there is a bijection between \( \mathcal{U} \) and the set of all the solutions of

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the problem. A parametrization of that set by means of contractive analytic functions is described in Section III.

A general method for solving interpolation problems is given by the Nagy–Foiaş commutant lifting theorem ([Sz.-NF], [FF]). Parrott ([P]) has shown that a special lifting yields interpolation results for analytic functions with values in a von Neumann algebra. In Section IV each commutant is associated with an isometry \( V \) in such a way that there exists a bijection between the set of all the Nagy–Foiaş liftings and \( U \), and that a Parrott type lifting is given by \((U, F)\).

The band method is a general scheme for dealing with many extension problems. It has been developed in a series of papers including [DG.1], [DG.2], [GKW.1], [GKW.2] and [GKW.3]. In Section V the method of unitary extensions of isometries is applied to deal with one of the problems that in [GKW.1] is solved by the band method.

In Section VI, Schur analysis of the set of unitary extensions of an isometry is related with previously considered subjects.

A basic example of how the method can be applied is given by the problem of extending \( \partial \) \( \partial \) functions of positive type. Its bidimensional case is related to the problem of finding two commutative unitary extensions of two given isometries ([AF]).

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II. Generalized interpolation. The method of unitary extensions of isometries gives a proof of the following

**Theorem (1).** For \( j = 1, 2 \) let \( E_j \) be a Hilbert space, \( S_j \) the shift in \( L^2(E_j) \) and \( B_j \) a closed subspace of \( L^2(E_j) \) such that

\[
E_1 \subset B_1 \subset S_1^{-1}B_1 \quad \text{and} \quad S_2^{-1}E_2 \subset B_2 \subset S_2B_2.
\]

Let \( A \in \mathcal{L}(B_1, B_2) \) be such that \( AS_{1|B_1} = P_{B_2}S_2A \). Set

\[
\mathcal{F}_A = \{ w \in L^\infty(E_1, E_2) : P_{B_2}M_{w|B_1} = A, \ |w| = |A| \},
\]

with \( M_w \) the multiplication by \( w \). Then \( \mathcal{F}_A \) is nonempty.

When \( B_1 = H^2(E_1) \) and \( B_2 = H^2(E_2) := L^2(E_2) \oplus H^2(E_2) \), the above is Page’s extension of Nehari’s theorem (see [N]). When \( E_1 = E_2 = E, B_1 = H^2(E) \) and \( B_2 = H^2(E) \oplus K \), with \( K \) a closed subspace of \( H^2(E) \) such that \( SH^2(E) \oplus K \subset H^2(E) \oplus K \), we have Sarason’s general interpolation theorem [S]. For convenient choices of the data, \( \mathcal{F}_A \) is the set of all the solutions of the Nevanlinna–Pick problem or of the Carathéodory–Fejér problem.

**Lemma (2).** Let \( A \in \mathcal{L}(B_1, B_2) \) be a contraction between Hilbert spaces. There exist a Hilbert space \( F \) and isometries \( r_j \in \mathcal{L}(B_j, F), \ j = 1, 2, \) which are essentially unique, such that \( F = (r_1B_1) \vee (r_2B_2) \) and \( A = r_2^*r_1 \). Moreover, if \( U_j \in \mathcal{L}(B_j) \) is a unitary operator, \( j = 1, 2, \) and \( U_2A = AU_1 \), there exists a unique unitary operator \( W \in \mathcal{L}(F) \) such that

\[
Wr_j = r_jU_j, \quad j = 1, 2.
\]
Sketch of proof of (2). Let $F$ be the Hilbert space generated by the linear space $B_1 \times B_2$ and the sesquilinear positive semidefinite form

$$\langle (b_1, b_2), (b'_1, b'_2) \rangle \equiv \langle b_1, b'_1 \rangle_{B_1} + \langle Ab_1, b'_2 \rangle_{B_2} + \langle b_2, Ab'_1 \rangle_{B_2} + \langle b_2, b'_2 \rangle_{B_2};$$

define $r_1, r_2$ by $b_1 \to (b_1, 0)$ and $b_2 \to (0, b_2)$, respectively; set $W r_j b_j \equiv r_j U_j b_j$, etc.

Sketch of proof of Theorem (1). We may assume that $\|A\| = 1$. There exist $H$ and two isometries $u_j \in \mathcal{L}(B_j, H)$, $j = 1, 2$, such that $A = u_2^* u_1$ and $H = (u_1 B_1) \lor (u_2 B_2)$; an isometry $V$ acting in $H$ with domain $D = (u_1 S_1 B_1) \lor (u_2 B_2)$ is defined, with obvious notation, by $V(u_1 S_1 b_1 + u_2 b_2) \equiv u_1 b_1 + u_2 S_2^{-1} b_2$.

If $(U, F) \in \mathcal{U}$, an isometric extension $r_j \in \mathcal{L}(L^2(E_j), F)$ of $u_j$ such that $r_j S_j = U^* r_j$ is well defined; the following equalities hold: $r_1 S_1^{-n} b_1 = U^* n b_1, n \geq 0$; $b_1 \in B_1$, and $r_2 S_2^{-k} b_2 = U^{*k} u_2 b_2, k \geq 0, b_2 \in B_2$. Since $S_2 r_2^* r_1 = r_2^* r_1 S_1$, there exists $w \in L^\infty(E_1, E_2)$ such that $M_w = r_2^* r_1$; then $w \in \mathcal{F}_A$. Moreover:

**Theorem (3).** In the same hypothesis of Theorem (1) assume $\|A\| = 1$. Set $w_\ldots(z) = \pi P E_2 S_2 A(I - \pi S_1)^{-1} i_{E_1}$. There exist an isometry $V$ acting in a Hilbert space $H$ and two isometries $\pi_j \in \mathcal{L}(E_j, H)$, $j = 1, 2$, such that a bijection from $\mathcal{U}$ onto $\mathcal{F}_A$ is defined by associating with each $(U, F) \in \mathcal{U}$ the function $w \in \mathcal{F}_A$ given by $w(z) = w_\ldots(z) + \pi_2^* P H U(I - z U)^{-1} i_{H^1}$. Concerning this section, details can be seen in [A.2].

**III. Parametrization formulas.** A set $\delta = \{E_1, E_2, X; A\}$, where $E_1$, $E_2$, $X$ are Hilbert spaces and $A = [A_{jk}]_{j,k=1,2}$ is a bounded operator from the space $X \oplus E_1$ to the space $E_2 \oplus X$, is called an operator colligation; it is unitary if $A$ is a unitary operator; a unitary colligation $\delta$ is called simple if the contraction $A_{21} = P X A_{|X}$ is completely nonunitary (c.n.u.), i.e., no nontrivial restriction of $A_{21}$ to an invariant subspace is unitary. The colligation $\delta' = \{E_1, E_2, X'; A'\}$ is equivalent to $\delta$ iff there exists a unitary operator $A \in \mathcal{L}(X, X')$ such that $A'(\lambda \oplus I_{E_1}) = (I_{E_2} \oplus \lambda) A$.

A colligation can be seen as a discrete linear system with response function $\Psi \equiv \Psi_3$ given by $\Psi(z) = A_{12} + z A_{11} (I - z A_{21})^{-1} A_{22}$, which is also called the characteristic function of the colligation. Two simple unitary colligations are equivalent iff they have the same characteristic function.

The space $H^\infty(E_1, E_2)$ is the set of analytic functions $\Psi : \mathbb{D} \to \mathcal{L}(E_1, E_2)$ on the unit disk such that $\|\Psi\|_\infty := \sup\{||\Psi(z)|| : z \in \mathbb{D}\} < \infty$. The characteristic function of a unitary colligation belongs to the set $\mathcal{B}(E_1, E_2) := \{\Psi \in H^\infty(E_1, E_2) : \|\Psi\|_\infty \leq 1\}$ of contractive analytic functions. The converse holds: if $\Psi \in \mathcal{B}(E_1, E_2)$, by applying Lemma (2) to the contraction $M_\Psi$, it can be proved that it is the characteristic function of a simple unitary colligation.

Let $V$ be any isometry with domain $D$, range $R$, and defect subspaces $N$ and $M$; that is, $N$ and $M$ are the orthogonal complements in $H$ of $D$ and $R$, respectively. To describe the set $\mathcal{U}$ of equivalence classes of minimal unitary extensions of $V$ is equivalent to describing the set of all (nonequivalent) simple unitary colligations $\{N, M, X; A\}$ with given $N$ and $M$. Thus, there exists a bijection between $\mathcal{U}$ and the set $\mathcal{B}(N, M)$ of contractive analytic functions:
THEOREM (4). Let $V$ be an isometry acting in a Hilbert space $H$ with defect subspaces $N$ and $M$. A bijection between the set $\mathcal{U}$ of equivalence classes of minimal unitary extensions of $V$ and the set $\mathcal{B}(N, M)$ of contractive analytic functions is obtained by associating with each $(U, F) \in \mathcal{U}$ the characteristic function of the simple unitary colligation $\{N, M, X; U|_{X \oplus N}\}$, with $X = F \ominus H$:

$$\Psi(z) = P_M U|_N + z P_M U|_X (I - z P_X U|_X)^{-1} P_X U|_N.$$ 

If $V$ is as in (4) and $D$ is its domain, a unitary extension $B \in \mathcal{U}(H \oplus M, N \oplus H)$ of $V$ is given by $B(h, m) = (P_N h, m + V P_D h)$, $\forall h, m \in M$. If $L$ is a closed subspace of $H$ and $L^\perp = H \ominus L$, set $\delta(V, L) = \{L \oplus M, N \oplus L, L^\perp; B\}$ and let $S(V, L) = [S_{jk}]_{j,k=1,2} \in \mathcal{B}(L \oplus M, N \oplus L)$ be the characteristic function of the unitary colligation $\delta(V, L)$. If $U \in \mathcal{L}(F)$ is a unitary operator, then $S(V, L) = \{L, L, F \ominus L; U\}$ and its characteristic function is $S(V, L)(z) = P_L U(I - z P_{F \ominus L} U)|_L$. Then:

**Theorem (5).** If $(U, F) \in \mathcal{U}$ corresponds to $\Psi \in \mathcal{B}(N, M)$ in the bijection given by Theorem (4), then, for every $z \in \mathbb{D}$,

$$S(V, L)(z) = S_{21}(z) + S_{22}(z)\Psi(z)[I - S_{12}(z)\Psi(z)]^{-1} S_{11}(z).$$

This formula was stated by Arov and Grossman ([AG]). As a consequence we obtain a parametrization of all solutions of the interpolation problems that can be solved by means of Theorem (1).

**Theorem (6).** In the same hypothesis and with the notation of Theorem (3), set $L = (\pi_1 E_1) \vee (\pi_2 E_2)$ and $[S_{jk}]_{j,k=1,2} = S(V, L)$. A bijection from $\mathcal{B}(N, M)$ onto $\mathcal{F}_A$ is given by associating with each $\Psi \in \mathcal{B}(N, M)$ the function $w \in \mathcal{F}_A$ defined by

$$w(z) = w_-(z) + \pi_1^* S(z)[I_L - z S(z)]^{-1} \pi_1,$$

$$S(z) = S_{21}(z) + S_{22}(z)\Psi(z)[I - S_{12}(z)\Psi(z)]^{-1} S_{11}(z).$$

Proofs of the Arov–Grossman formula and of the other statements in this section are given in [A.3].

**IV. A lifting theorem.** Parrott’s extension of the Nagy–Foiaş theorem follows from:

**Theorem (7).** Let $T_j \in \mathcal{L}(E_j)$ be a contraction with minimal unitary dilation $U_j \in \mathcal{L}(F_j)$, $j = 1, 2$, and $X \in \mathcal{L}(E_1, E_2)$ such that $XT_1 = T_2 X$. Set $(A_1, A_2) \in \mathcal{A}$ if $A_j \in \mathcal{L}(E_j)$ bicommutes with $T_j$, $j = 1, 2$, and $X A_1 = A_2 X$, $X A_1^* = A_2^* X$; let $A_j \in \mathcal{L}(F_j)$ be the extension of $A_j$ that commutes with $U_j$ and is such that $\|\hat{A}_j\| = \|A_j\|$, $j = 1, 2$. There exists $\tau \in \mathcal{L}(F_1, F_2)$ such that $\tau U_1 = U_2 \tau$, $P_{E_2} \tau|_{E_1} = X$, $\|\tau\| = \|X\|$ and $\tau \hat{A}_1 = \hat{A}_2 \tau$, $\forall (A_1, A_2) \in \mathcal{A}.

Assume $\|X\| = 1$. Set $M_1 = \bigvee\{U_j^* E_1 : n \geq 0\}$ and $M_2' = \bigvee\{U_j^* E_2 : n \leq 0\}$. Let $H$ be a Hilbert space such that $H = M_1 \vee M_2'$ and $P_{M_2'|M_1} = X' := XP^{M_2} E_1$. Every $(A_1, A_2) \in \mathcal{A}$ defines an operator $A \in \mathcal{L}(H)$ by $A(g_2^2 + g_1) = \hat{A}_2 g_2^2 + \hat{A}_1 g_1$, $\forall g_2^2 \in M_2'$ and $g_1 \in M_1$. Set $D = U_2^* M_2' \vee M_1$; define the isometry $V$ by $V(U_2^* g_2^2 + g_1) = g_2^2 + U_1 g_1$. Let $\overline{U} \in \mathcal{L}(\overline{F})$ be the minimal unitary dilation of the contraction $VP_D \in \mathcal{L}(H)$. We may assume that $\overline{F} = F_1 \vee F_2$ and that $\overline{U}|_{F_j} = U_j$. Then $A$ extends to $\hat{A} \in \mathcal{L}(\overline{F})$. 


such that $\tilde{A}U = U\tilde{A}$, so $\tilde{A}_{F_j} = \tilde{A}_j$. Setting $\tau = P_{F_1|F_1}$ the result follows. Proofs and two-dimensional generalizations can be seen in [A.4].

V. A band extension problem. We are given the integers $N$ and $p$ such that $0 \leq p < N-1$, the Hilbert spaces $G_j$, $1 \leq j \leq N$, and the operators $A_{ij} \in \mathcal{L}(G_i, G_j)$, $1 \leq i, j \leq N$, $|i - j| \leq p$. The band $A^{(p)} := \{A_{ij} : |i - j| \leq p\}$ is positive if the operators $[A_{k,j}]_{i|k,j|k,j+i+p} \in \mathcal{L}[\bigoplus(G_j : i \leq j \leq i + p)]$ are positive for $1 \leq i \leq N - p$; $A^{(p)}$ is positive definite (p.d.) if $[A_{k,j}]_{i|k,j|k,j+i+p}$ is positive definite for $1 \leq i \leq N - p$. Recall that an operator in a Hilbert space is positive definite if it is positive and boundedly invertible. Set $G = \bigoplus(G_j : i \leq j \leq N)$. A positive operator $B = [B_{k,j}]_{1 \leq k,j \leq N} \in \mathcal{L}(G)$ such that $B_{ij} = A_{ij}$ whenever $|i - j| \leq p$ is called a positive extension of the given band. The following statement is related to one of the problems that are solved in [GKW] 1.

**Theorem (8).** Every positive band $A^{(p)}$ has positive extensions. If $A^{(p)}$ is positive definite, it has positive definite extensions and there exists one of them, $\tilde{A}$, such that $[\tilde{A}^{-1}]_{rs} = 0$ if $|s - r| > p$.

Assume $p \geq 1$. If $r \land s$ denotes the minimum of $r$ and $s$, set $\mathcal{C} = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq N, i \leq j \leq (i + p) \land N\}$, $G_{ij} = G_j$ for every $(i,j) \in \mathcal{C}$ and $\tilde{G} = \bigoplus\{G_{ij} : (i,j) \in \mathcal{C}\}$; thus, every $f \in \tilde{G}$ is given by $[f_{ij}]_{(i,j) \in \mathcal{C}}$, $f_{ij} \in G_j$; its support is the set $\text{supp } f := \{(i,j) \in \mathcal{C} : f_{ij} \neq 0\}$.

Let $H$ be the Hilbert space generated by the vector space $\tilde{G}$ and the sesquilinear hermitian positive semidefinite form in $\tilde{G}$ given by

$$[f,f'] = \sum\{(A_{ijk}f_{ik}, f'_{jk})_{G_j} : (i,j), (i,k) \in \mathcal{C}\}$$

For any $f \in \tilde{G}$ such that $f_{00} = 0$, $1 \leq i \leq N$, let $g = \tau f \in \tilde{G}$ be given by $g_{ij} = f_{i-1,j}$ if $(i,j), (i-1,j) \in \mathcal{C}$ and $g_{ij} = 0$ if $(i,j) \in \mathcal{C}$ but $(i-1,j) \not\in \mathcal{C}$. In a natural way, $\tau$ defines an isometry $V$ acting in $H$.

For any $v \in G_i$, $1 \leq t \leq N$, let $\lambda tv \in H$ be given by $v' \in \tilde{G}$ such that $v' = \{(t,t)\}$ and $v'_{tt} = v$. Then $\lambda^t V^{i-1} \lambda = A_{ij}$, $\forall (i,j) \in \mathcal{C}$.

For any $(U,F) \in \mathcal{U}$, a positive extension $A = A(U,F)$ of the band $A^{(p)}$ is given by $A_{ij} = (i_H^p \lambda)^* U^{i-1} (i_H^p \lambda)$, $1 \leq i,j \leq N$, and every positive extension of the band $A^{(p)}$ is obtained in this way.

Assume that $A^{(p)}$ is p.d.; then $\tilde{A} := A(U,F)$ is a positive extension of $A^{(p)}$ and $[\tilde{A}^{-1}]_{rs} = 0$ if $|s - r| > p$.

Proofs of the above assertions can be seen in [A.5].

VI. Schur analysis. Let $V$ be any isometry acting in $H$, with domain $D$ and defect subspaces $N$ and $M$. If $(U,F) \in \mathcal{U}$ set $H_1 = H \lor UH$ and $V_1 = U_{H_1}$, let $N_1$ and $M_1$ be the defect subspaces of the isometry $V_1$ that acts in $H_1$, and set $\nu_1 = P_{M_1}U_{|N_1}$. By iteration a sequence of contractions $\{\nu_k : k > 0\}$ is associated with each $(U,F) \in \mathcal{U}$, and a Schur type analysis of the unitary extensions of an isometry is established ([A.1]). In fact, when the method of unitary extensions of isometries is applied to the Carathéodory–Fejér problem, those $\{\nu_k : k > 0\}$ are the classical sequences of Schur parameters.
In general, a bijection between $\mathcal{U}$ and the set of $(N,M)$-choice sequences (see [FF]) is established. If $(\mathcal{U}, F) \in \mathcal{U}$ is given by the minimal unitary dilation of the contraction $VP_D \in \mathcal{L}(H)$, then the corresponding sequence of “Schur parameters” is such that $\nu_k = 0$ for every $k > 0$, and the corresponding solution of a generalized interpolation problem can be considered as the maximum entropy solution.

There exists a bijection between the set of all the Nagy–Foiaş intertwining liftings of a commutant and a set of choice sequences (see [FF] and references therein); that result can be proved by means of the above sketched Schur analysis of the unitary extensions of an isometry. In [FFG], a “central intertwining lifting” is studied; it may be conjectured that it corresponds to the Parrott type lifting we considered in Section IV.

Concerning the band extension problem considered in Section V, this kind of Schur analysis of unitary extensions of isometries gives another proof of the following facts ([GKW1]): each positive extension $A$ of $A^{(p)}$ is bijectively associated with an $(N - p)$-tuple of contractions $\{T^{(k)} : p \leq k \leq N - 1\}$; when $A^{(p)}$ is p.d., $A$ is p.d. iff $\|T^{(k)}\| < 1$, $p \leq k \leq N - 1$, and $\overline{A}$ corresponds to $T^{(k)} = 0$, $p \leq k \leq N - 1$.

References


