LINEAR OPERATORS
BANACH CENTER PUBLICATIONS, VOLUME 38
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

UNITARY EXTENSIONS OF ISOMETRIES, GENERALIZED INTERPOLATION AND BAND EXTENSIONS

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Abstract. The aim of this paper is to give a very brief account of some applications of the method of unitary extensions of isometries to interpolation and extension problems.

I. Unitary extensions of isometries. A general method for solving several moment and interpolation problems can be summarized as follows: the data of the problem define an isometry, with range and domain in the same Hilbert space, in such a way that each unitary extension of that isometry gives a solution of the problem.

In this review paper, the method and some of its applications are briefly described. We now fix the notation and then specify the content of the following sections.

Unless otherwise specified, all spaces are assumed to be separable complex Hilbert spaces and all subspaces are closed; $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from a space X to a space Y; $\mathcal{L}(X)$ is the same as $\mathcal{L}(X,X)$, and "$\bigvee$" means "closed linear span"; $P_E^X \equiv P_E$ denotes the orthogonal projection onto the subspace E of X and $i_E^X \equiv i_E$ is the inclusion of E in X. $L^p(X)$ denotes the space of X-valued measurable functions on the unit disk $\mathbb{T}$ with finite p-norm. $L^p(X,Y)$ denotes the space of $\mathcal{L}(X,Y)$-valued measurable functions on $\mathbb{T}$ with finite p-norm.

The isometry V acts in the Hilbert space H if its domain D and range R are (closed) subspaces of H. We say that (U,F) belongs to $\mathcal{U}$, the set of equivalence classes of minimal unitary extensions of V, if $U \in \mathcal{L}(F)$ is a unitary extension of V to a space F that contains H, such that $F = \bigvee\{U^n H : n \in \mathbb{Z}\}$; we consider two minimal unitary extensions to be equivalent, and write $(U,F) \approx (U',F')$ in $\mathcal{U}$, if there exists a unitary operator $X \in \mathcal{L}(F,F')$ such that $XU = U'X$ and that its restriction to H equals the identity I_H in H. An element $(\overline{U},\overline{F})$ of $\mathcal{U}$ with special properties is given by the minimal unitary dilation $\overline{U} \in \mathcal{L}(\overline{F})$ of the contraction $VP_D \in \mathcal{L}(H)$.

In Section II an isometry V is associated with a generalized interpolation problem in such a way that there is a bijection between $\mathcal{U}$ and the set of all the solutions of

1991 *Mathematics Subject Classification*: 47A57, 47A20.

the problem. A parametrization of that set by means of contractive analytic functions is described in Section III.

A general method for solving interpolation problems is given by the Nagy–Foiaş commutant lifting theorem ([Sz.-NF], [FF]). Parrott ([P]) has shown that a special lifting yields interpolation results for analytic functions with values in a von Neumann algebra. In Section IV each commutant is associated with an isometry V in such a way that there exists a bijection between the set of all the Nagy–Foiaş liftings and $\mathcal{U}$, and that a Parrott type lifting is given by $(\overline{U}, \overline{F})$.

The band method is a general scheme for dealing with many extension problems. It has been developed in a series of papers including [DG.1], [DG.2], [GKW.1], [GKW.2] and [GKW.3]. In Section V the method of unitary extensions of isometries is applied to deal with one of the problems that in [GKW.1] is solved by the band method.

In Section VI, Schur analysis of the set of unitary extensions of an isometry is related with previously considered subjects.

A basic example of how the method can be applied is given by the problem of extending functions of positive type. Its bidimensional case is related to the problem of finding two commutative unitary extensions of two given isometries ([AF]).

The author wants to thank the referee for his comments on a first version of this paper. An expanded version of it, as an introductory and essentially self-contained series of articles, is being published in “Publicaciones Matemáticas del Uruguay” ([A.5]).

II. Generalized interpolation. The method of unitary extensions of isometries gives a proof of the following

Theorem (1). *For $j = 1, 2$ let E_j be a Hilbert space, S_j the shift in $L^2(E_j)$ and B_j a closed subspace of $L^2(E_j)$ such that*

$$E_1 \subset B_1 \subset S_1^{-1} B_1 \quad \text{and} \quad S_2^{-1} E_2 \subset B_2 \subset S_2 B_2.$$

Let $A \in \mathcal{L}(B_1, B_2)$ be such that $AS_{1|B_1} = P_{B_2} S_2 A$. Set

$$\mathcal{F}_A = \{w \in L^\infty(E_1, E_2) : P_{B_2} M_{w|B_1} = A,\ \|w\|_\infty = \|A\|\},$$

with M_w the multiplication by w. Then $\mathcal{F}_A$ is nonempty.

When $B_1 = H^2(E_1)$ and $B_2 = H^2_-(E_2) := L^2(E_2) \ominus H^2(E_2)$, the above is Page's extension of Nehari's theorem (see [N]). When $E_1 = E_2 = E$, $B_1 = H^2(E)$ and $B_2 = H^2_-(E) \oplus K$, with K a closed subspace of $H^2(E)$ such that $S[H^2(E) \ominus K] \subset H^2(E) \ominus K$, we have Sarason's general interpolation theorem [S]. For convenient choices of the data, $\mathcal{F}_A$ is the set of all the solutions of the Nevanlinna–Pick problem or of the Carathéodory–Fejér problem.

Lemma (2). *Let $A \in \mathcal{L}(B_1, B_2)$ be a contraction between Hilbert spaces. There exist a Hilbert space F and isometries $r_j \in \mathcal{L}(B_j, F)$, $j = 1, 2$, which are essentially unique, such that $F = (r_1 B_1) \vee (r_2 B_2)$ and $A = r_2^* r_1$. Moreover, if $U_j \in \mathcal{L}(B_j)$ is a unitary operator, $j = 1, 2$, and $U_2 A = A U_1$, there exists a unique unitary operator $W \in \mathcal{L}(F)$ such that*

$$W r_j = r_j U_j, \qquad j = 1, 2.$$

S k e t c h o f p r o o f o f (2). Let F be the Hilbert space generated by the linear space $B_1 \times B_2$ and the sesquilinear positive semidefinite form

$$\langle (b_1, b_2), (b_1', b_2') \rangle \equiv \langle b_1, b_1' \rangle_{B_1} + \langle Ab_1, b_2' \rangle_{B_2} + \langle b_2, Ab_1' \rangle_{B_2} + \langle b_2, b_2' \rangle_{B_2};$$

define r_1, r_2 by $b_1 \to (b_1, 0)$ and $b_2 \to (0, b_2)$, respectively; set $Wr_j b_j \equiv r_j U_j b_j$, etc.

S k e t c h o f p r o o f o f T h e o r e m (1). We may assume that $\|A\| = 1$. There exist H and two isometries $u_j \in \mathcal{L}(B_j, H)$, $j = 1, 2$, such that $A = u_2^* u_1$ and $H = (u_1 B_1) \vee (u_2 B_2)$; an isometry V acting in H with domain $D = (u_1 S_1 B_1) \vee (u_2 B_2)$ is defined, with obvious notation, by $V(u_1 S_1 b_1 + u_2 b_2) \equiv u_1 b_1 + u_2 S_2^{-1} b_2$.

If $(U, F) \in \mathcal{U}$, an isometric extension $r_j \in \mathcal{L}[L^2(E_j), F]$ of u_j such that $r_j S_j = U^* r_j$ is well defined; the following equalities hold: $r_1 S_1^{-n} b_1 = U^n u_1 b_1$, $n \geq 0$, $b_1 \in B_1$, and $r_2 S_2^k b_2 = U^{*k} u_2 b_2$, $k \geq 0$, $b_2 \in B_2$. Since $S_2 r_2^* r_1 = r_2^* r_1 S_1$, there exists $w \in L^\infty(E_1, E_2)$ such that $M_w = r_2^* r_1$; then $w \in \mathcal{F}_A$. Moreover:

Theorem (3). *In the same hypothesis of Theorem* (1) *assume* $\|A\| = 1$. *Set* $w_-(z) = \bar{z} P_{E_2} S_2 A (I - \bar{z} S_1)^{-1} i_{E_1}$. *There exist an isometry V acting in a Hilbert space H and two isometries $\pi_j \in \mathcal{L}(E_j, H)$, $j = 1, 2$, such that a bijection from $\mathcal{U}$ onto $\mathcal{F}_A$ is defined by associating with each $(U, F) \in \mathcal{U}$ the function $w \in \mathcal{F}_A$ given by* $w(z) = w_-(z) + \pi_2^* P_H U (I - zU)^{-1} i_H \pi_1$.

Concerning this section, details can be seen in [A.2].

III. Parametrization formulas. A set $\delta = \{E_1, E_2, X; A\}$, where E_1, E_2, X are Hilbert spaces and $A = [A_{jk}]_{j,k=1,2}$ is a bounded operator from the space $X \oplus E_1$ to the space $E_2 \oplus X$, is called an *operator colligation*; it is *unitary* if A is a unitary operator; a unitary colligation δ is called *simple* if the contraction $A_{21} = P_X A_{|X}$ is completely nonunitary (c.n.u.), i.e., no nontrivial restriction of A_{21} to an invariant subspace is unitary. The colligation $\delta' = \{E_1, E_2, X'; A'\}$ is equivalent to δ iff there exists a unitary operator $\lambda \in \mathcal{L}(X, X')$ such that $A'(\lambda \oplus I_{E_1}) = (I_{E_2} \oplus \lambda) A$.

A colligation can be seen as a discrete linear system with response function $\Psi \equiv \Psi_\delta$ given by $\Psi(z) = A_{12} + z A_{11} (I - z A_{21})^{-1} A_{22}$, which is also called the *characteristic function* of the colligation. Two simple unitary colligations are equivalent iff they have the same characteristic function.

The space $H^\infty(E_1, E_2)$ is the set of analytic functions $\Psi : \mathbb{D} \to \mathcal{L}(E_1, E_2)$ on the unit disk such that $\|\Psi\|_\infty := \sup\{\|\Psi(z)\| : z \in \mathbb{D}\} < \infty$. The characteristic function of a unitary colligation belongs to the set $\mathcal{B}(E_1, E_2) := \{\Psi \in H^\infty(E_1, E_2) : \|\Psi\|_\infty \leq 1\}$ of contractive analytic functions. The converse holds: if $\Psi \in \mathcal{B}(E_1, E_2)$, by applying Lemma (2) to the contraction M_Ψ, it can be proved that it is the characteristic function of a simple unitary colligation.

Let V be any isometry with domain D, range R, and defect subspaces N and M; that is, N and M are the orthogonal complements in H of D and R, respectively. To describe the set $\mathcal{U}$ of equivalence classes of minimal unitary extensions of V is equivalent to describing the set of all (nonequivalent) simple unitary colligations $\{N, M, X; A\}$ with given N and M. Thus, there exists a bijection between $\mathcal{U}$ and the set $\mathcal{B}(N, M)$ of contractive analytic functions:

THEOREM (4). *Let V be an isometry acting in a Hilbert space H with defect subspaces N and M. A bijection between the set $\mathcal{U}$ of equivalence classes of minimal unitary extensions of V and the set $\mathcal{B}(N,M)$ of contractive analytic functions is obtained by associating with each $(U,F)\in\mathcal{U}$ the characteristic function of the simple unitary colligation $\{N,M,X;U_{|X\oplus N}\}$, with $X=F\ominus H$*:

$$\Psi(z)=P_MU_{|N}+zP_MU_{|X}(I-zP_XU_{|X})^{-1}P_XU_{|N}.$$

If V is as in (4) and D is its domain, a unitary extension $B\in\mathcal{U}(H\oplus M,N\oplus H)$ of V is given by $B(h,m)=(P_Nh,m+VP_Dh)$, $\forall h\in H$, $m\in M$. If L is a closed subspace of H and $L^\perp=H\ominus L$, set $\delta^{(V,L)}=\{L\oplus M,N\oplus L,L^\perp;B\}$ and let $S^{(V,L)}=[S_{jk}]_{j,k=1,2}\in\mathcal{B}(L\oplus M,N\oplus L)$ be the characteristic function of the unitary colligation $\delta^{(V,L)}$. If $U\in\mathcal{L}(F)$ is a unitary operator, then $\delta^{(U,L)}=\{L,L,F\ominus L;U\}$ and its characteristic function is $S^{(U,L)}(z)=P_LU(I-zP_{F\ominus L}U)^{-1}_{|L}$. Then:

THEOREM (5). *If $(U,F)\in\mathcal{U}$ corresponds to $\Psi\in\mathcal{B}(N,M)$ in the bijection given by Theorem* (4), *then, for every $z\in\mathbb{D}$*,

$$S^{(U,L)}(z)=S_{21}(z)+S_{22}(z)\Psi(z)[I-S_{12}(z)\Psi(z)]^{-1}S_{11}(z).$$

This formula was stated by Arov and Grossman ([AG]). As a consequence we obtain a parametrization of all solutions of the interpolation problems that can be solved by means of Theorem (1).

THEOREM (6). *In the same hypothesis and with the notation of Theorem* (3), *set $L=(\pi_1E_1)\vee(\pi_2E_2)$ and $[S_{jk}]_{j,k=1,2}=S^{(V,L)}$. A bijection from $\mathcal{B}(N,M)$ onto $\mathcal{F}_A$ is given by associating with each $\Psi\in\mathcal{B}(N,M)$ the function $w\in\mathcal{F}_A$ defined by*

$$\begin{aligned}w(z)&=w_-(z)+\pi_2^*S(z)[I_L-zS(z)]^{-1}\pi_1,\\ S(z)&=S_{21}(z)+S_{22}(z)\Psi(z)[I-S_{12}(z)\Psi(z)]^{-1}S_{11}(z).\end{aligned}$$

Proofs of the Arov–Grossman formula and of the other statements in this section are given in [A.3].

IV. A lifting theorem. Parrott's extension of the Nagy–Foiaş theorem follows from:

THEOREM (7). *Let $T_j\in\mathcal{L}(E_j)$ be a contraction with minimal unitary dilation $U_j\in\mathcal{L}(Fj)$, $j=1,2$, and $X\in\mathcal{L}(E_1,E_2)$ such that $XT_1=T_2X$. Set $(A_1,A_2)\in\mathcal{A}$ if $A_j\in\mathcal{L}(E_j)$ bicommutes with T_j, $j=1,2$, and $XA_1=A_2X$, $XA_1^*=A_2^*X$; let $\widehat{A}_j\in\mathcal{L}(F_j)$ be the extension of A_j that commutes with U_j and is such that $\|\widehat{A}_j\|=\|A_j\|$, $j=1,2$. There exists $\tau\in\mathcal{L}(F_1,F_2)$ such that $\tau U_1=U_2\tau$, $P_{E_2}\tau_{|E_1}=X$, $\|\tau\|=\|X\|$ and $\tau\widehat{A}_1=\widehat{A}_2\tau$, $\forall(A_1,A_2)\in\mathcal{A}$.*

Assume $\|X\|=1$. Set $M_1=\bigvee\{U_1^nE_1:n\ge 0\}$ and $M_2'=\bigvee\{U_2^nE_2:n\le 0\}$. Let H be a Hilbert space such that $H=M_1\vee M_2'$ and $P^H_{M_2'|M_1}=X':=XP^{M_1}E_1$. Every $(A_1,A_2)\in\mathcal{A}$ defines an operator $A\in\mathcal{L}(H)$ by $A(g_2'+g_1)=\widehat{A}_2g_2'+\widehat{A}_1g_1$, $\forall g_2'\in M_2'$ and $g_1\in M_1$. Set $D=U_2^*M_2'\vee M_1$; define the isometry V by $V(U_2^*g_2'+g_1)=g_2'+U_1g_1$. Let $\overline{U}\in\mathcal{L}(\overline{F})$ be the minimal unitary dilation of the contraction $VP_D\in\mathcal{L}(H)$. We may assume that $\overline{F}=F_1\vee F_2$ and that $\overline{U}_{|F_j}=U_j$. Then A extends to $\widehat{A}\in\mathcal{L}(\overline{F})$

such that $\widehat{AU} = \overline{U}\widehat{A}$, so $\widehat{A}_{|F_j} = \widehat{A}_j$. Setting $\tau = P^{\overline{F}}_{F_2|F_1}$ the result follows. Proofs and two-dimensional generalizations can be seen in [A.4].

V. A band extension problem. We are given the integers N and p such that $0 \le p < N-1$, the Hilbert spaces G_j, $1 \le j \le N$, and the operators $A_{ij} \in \mathcal{L}(G_j, G_i)$, $1 \le i, j \le N$, $|i-j| \le p$. The band $A^{(p)} := \{A_{ij} : |i-j| \le p\}$ is *positive* if the operators $[A_{kj}]_{i\le k,j\le i+p} \in \mathcal{L}[\bigoplus(G_j : i \le j \le i+p)]$ are positive for $1 \le i \le N-p$; $A^{(p)}$ is *positive definite* (p.d.) if $[A_{kj}]_{i\le k,j\le i+p}$ is positive definite for $1 \le i \le N-p$. Recall that an operator in a Hilbert space is *positive definite* if it is positive and boundedly invertible. Set $G = \bigoplus(G_j : i \le j \le N)$. A positive operator $B = [B_{kj}]_{1\le k,j\le N} \in \mathcal{L}(G)$ such that $B_{ij} = A_{ij}$ whenever $|i-j| \le p$ is called a *positive extension* of the given band. The following statement is related to one of the problems that are solved in [GKW1].

THEOREM (8). *Every positive band $A^{(p)}$ has positive extensions. If $A^{(p)}$ is positive definite, it has positive definite extensions and there exists one of them, $\overline{A}$, such that $[\overline{A}^{-1}]_{rs} = 0$ if $|s-r| > p$.*

Assume $p \ge 1$. If $r \wedge s$ denotes the minimum of r and s, set $\mathcal{C} = \{(i,j) \in \mathbb{Z}^2 : 1 \le i \le N,\ i \le j \le (i+p)\wedge N\}$, $G_{ij} = G_j$ for every $(i,j) \in \mathcal{C}$ and $\widetilde{G} = \bigoplus\{G_{ij} : (i,j) \in \mathcal{C}\}$; thus, every $f \in \widetilde{G}$ is given by $[f_{ij}]_{(i,j)\in\mathcal{C}}$, $f_{ij} \in G_j$; its support is the set $\operatorname{supp} f := \{(i,j) \in \mathcal{C} : f_{ij} \ne 0\}$.

Let H be the Hilbert space generated by the vector space $\widetilde{G}$ and the sesquilinear hermitian positive semidefinite form in $\widetilde{G}$ given by

$$[f, f'] \equiv \sum\{\langle A_{jk} f_{ik}, f'_{ij}\rangle_{G_j} : (i,j), (i,k) \in \mathcal{C}\}.$$

For any $f \in \widetilde{G}$ such that $f_{ii} = 0$, $1 \le i \le N$, let $g = \tau f \in \widetilde{G}$ be given by $g_{ij} = f_{i-1,j}$ if $(i,j), (i-1,j) \in \mathcal{C}$ and $g_{ij} = 0$ if $(i,j) \in \mathcal{C}$ but $(i-1,j) \notin \mathcal{C}$. In a natural way, τ defines an isometry V acting in H.

For any $v \in G_t$, $1 \le t \le N$, let $\lambda_t v \in H$ be given by $v' \in \widetilde{G}$ such that $\operatorname{supp} v' = \{(t,t)\}$ and $v'_{tt} = v$. Then $\lambda_i^* V^{i-j} \lambda_j = A_{ij}$, $\forall (i,j) \in \mathcal{C}$.

For any $(U,F) \in \mathcal{U}$, a positive extension $A = A(U,F)$ of the band $A^{(p)}$ is given by $A_{ij} = (i_H^F \lambda_j)^* U^{i-j} (i_H^F \lambda_i)$, $1 \le i,j \le N$, and every positive extension of the band $A^{(p)}$ is obtained in this way.

Assume that $A^{(p)}$ is p.d.; then $\overline{A} := A(\overline{U}, \overline{F})$ is a positive extension of $A^{(p)}$ and $[\overline{A}^{-1}]_{rs} = 0$ if $|s-r| > p$.

Proofs of the above assertions can be seen in [A.5].

VI. Schur analysis. Let V be any isometry acting in H, with domain D and defect subspaces N and M. If $(U,F) \in \mathcal{U}$ set $H_1 = H \vee UH$ and $V_1 = U_{|H_1}$, let N_1 and M_1 be the defect subspaces of the isometry V_1 that acts in H_1, and set $\nu_1 = P_{M_1} U_{|N1}$. By iteration a sequence of contractions $\{\nu_k : k > 0\}$ is associated with each $(U,F) \in \mathcal{U}$, and a Schur type analysis of the unitary extensions of an isometry is established ([A.1]). In fact, when the method of unitary extensions of isometries is applied to the Carathéodory–Fejér problem, those $\{\nu_k : k > 0\}$ are the classical sequences of Schur parameters.

In general, a bijection between $\mathcal{U}$ and the set of (N, M)-choice sequences (see [FF]) is established. If $(\overline{U}, \overline{F}) \in \mathcal{U}$ is given by the minimal unitary dilation of the contraction $VP_D \in \mathcal{L}(H)$, then the corresponding sequence of "Schur parameters" is such that $\nu_k = 0$ for every $k > 0$, and the corresponding solution of a generalized interpolation problem can be considered as the maximum entropy solution.

There exists a bijection between the set of all the Nagy–Foiaş intertwining liftings of a commutant and a set of choice sequences (see [FF] and references therein); that result can be proved by means of the above sketched Schur analysis of the unitary extensions of an isometry. In [FFG], a "central intertwining lifting" is studied; it may be conjectured that it corresponds to the Parrott type lifting we considered in Section IV.

Concerning the band extension problem considered in Section V, this kind of Schur analysis of unitary extensions of isometries gives another proof of the following facts ([GKW1]): each positive extension A of $A^{(p)}$ is bijectively associated with an $(N-p)$-tuple of contractions $\{T^{(k)} : p \le k \le N-1\}$; when $A^{(p)}$ is p.d., A is p.d. iff $\|T^{(k)}\| < 1,\ p \le k \le N-1$, and $\bar{A}$ corresponds to $T^{(k)} = 0,\ p \le k \le N-1$.

References

[AF] R. Arocena and F. Montans, *On a general bidimensional extrapolation problem*, Colloq. Math. 64 (1993), 3–12.

[A.1] R. Arocena, *Schur analysis of a class of translation invariant forms*, in: Analysis and Partial Differential Equations: A Collection of Papers dedicated to Mischa Cotlar, C. Sadosky (ed.), Marcel Dekker, New York and Basel, 1990.

[A.2] —, *Some remarks on lifting and interpolation problems*, Rev. Un. Mat. Argentina 37 (1991), 200–211.

[A.3] —, *Unitary colligations and parametrization formulas*, Ukrainian Math. J. 46 (1994), 147–154.

[A.4] —, *On the Nagy–Foiaş–Parrott commutant lifting theorem*, in: Contemp. Math. 189, Amer. Math. Soc., 1995, 55–64.

[A.5] —, *Unitary extensions of isometries and interpolation problems*: (1) *dilation and lifting theorems*, Publ. Mat. Uruguay 6 (1995), 137–158.

[AG] D. Z. Arov and L. Z. Grossman, *Scattering matrices in the theory of unitary extensions of isometric operators*, Soviet Math. Dokl. 27 (1983), 518–522.

[DG.1] H. Dym and I. Gohberg, *Extensions of kernels of Fredholm operators*, J. Analyse Math. 42 (1982/83), 51–97.

[DG.2] —, —, *A new class of contractive interpolants and maximun entropy principles*, in: Oper. Theory Adv. Appl. 29, Birkhäuser, 1988, 117–150.

[FF] C. Foiaş and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Birkhäuser, Basel, 1990.

[FFG] C. Foiaş, A. E. Frazho and I. Gohberg, *Central intertwining lifting*, *maximum entropy and their permanence*, Integral Equations Operator Theory 18 (1994), 166–201.

[GKW.1] I. Gohberg, M. A. Kaashoek and H. J. Woerdeman, *The band method for positive and contractive extension problems*, J. Operator Theory 22 (1989), 109–115.

[GKW.2] I. Gohberg, M. A. Kaashoek and H. J. Woerdeman, *The band method for positive and contractive extension problems. An alternative version and new applications*, Integral Equations Operator Theory 12 (1989), 343–382.

[GKW.3] —, —, —, *A maximum entropy principle in the general framework of the band method*, J. Funct. Anal. 95 (1991), 231–254.

[N] N. K. Nikol'skiĭ, *Treatise on the Shift Operator*, Springer, New York, 1986.

[P] S. Parrott, *On a quotient norm and the Sz.-Nagy–Foiaş lifting theorem*, J. Funct. Anal. 30 (1978), 311–328.

[S] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. 127 (1967), 179–203.

[Sz.-NF] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.