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SUBNORMAL OPERATORS OF HARDY TYPE

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Introduction. Links between operators and function theory are often fruitful in both directions. For example, in operator theory one tends to build models for various classes of Hilbert space operators, expressing them (up to some equivalence) as certain simple actions defined on suitable function spaces. Working in the opposite direction, one can view function-theoretic phenomena as statements about the related linear operators. Sometimes the interpretation of operations on functions via different functional models allows one to view them "at a more convenient angle". The last section is intended to exemplify this approach and despite its simplicity, the underlying idea seems worth further studying.

We begin with outlining the Hardy space model for a quite large class of subnormal operators S. Among various functional models preferred are those satisfying as much as possible of the following three postulates. Namely, they should be: 1° determined up to unitary equivalence, 2° acting by a simple formula, 3° defined on a space consisting of concrete functions (rather than, say, of distributions). All these three requirements are met by the model, introduced (under quite restrictive assumptions on the geometry of the spectrum $\sigma(S)$ of S) by Abrahamse and Douglas [AD1], [AD2]. The first substantial relaxation of these geometric assumptions (still in the case of finitely connected $\sigma(S)$) was presented in [R1], but the serious difficulty in extending the model to (any) infinitely connected $\sigma(S)$ was overcome in [R2] after the employment of new tools: W. Mlak's absolute continuity result [M] and M. Hasumi's and C. Neville's extension of the Beurling–Lax theorem [H].

In the present paper we use the results of [S] to simplify the earlier construction [R2]. These results eliminate the need for certain additional assumptions and provide a better explanation of the role of other requirements.

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1. Preliminaries. We shall consider *subnormal* operators S on complex separable Hilbert spaces H, which means the existence of a normal operator N on some bigger space $K \supset H$ with $N(H) \subset H$ and $N|_H = S$, a fact referred to as S being a *part* of N and N being a *normal extension* of S. In what follows we assume N to be a *minimal* normal extension (m.n.e.) of S; then the well known Spectral Inclusion Theorem says that

$$\partial\sigma(S) \subset \sigma(N) \subset \sigma(S), \tag{SI}$$

where ∂ stands for the topological boundary. It follows that $\sigma(S)$ is obtained from $\sigma(N)$ by *filling in* some *holes*. An equivalent conclusion is that *locally* (i.e. for each hole of $\sigma(N)$) one of the (SI) inclusions must be an equality. The second inclusion turning into equality is typical for Bergman Shifts, but this fragment of spectral picture (i.e. the equality $\sigma(N) = \sigma(S)$ and the concrete knowledge of this set) does not determine the nature of S, since adding orthogonally any other subnormal operator whose spectrum is contained in a given hole yields the same pair $(\sigma(N), \sigma(S))$. The situation on the left-hand side of (SI) is, in this regard, more interesting and therefore, from now on, we assume that (globally) the first (SI) is an equality, which in view of (SI) is equivalent to requiring that

$$\sigma(N) \subset \partial\sigma(S). \tag{1.0}$$

Abrahamse and Douglas proved in [AD1], [AD2]—assuming additionally (a rather strong) geometric regularity of $\sigma(S)$—that S is unitarily equivalent to the orthogonal sum of a normal operator (the *normal part* of S) and of some bundle shift T_E defined below. This model can be viewed as a *Generalised Wold Decomposition* of S.

Assume further that S is *pure subnormal*, meaning it has no nontrivial normal part. Then the model theorem says that S is, up to unitary equivalence, a *bundle shift* of some flat unitary bundle E spread over the interior, say Ω, of $\sigma(S)$. The latter acts as multiplication by the independent variable z on the Hardy space $H^2[E]$. The term "shift" comes from the case when Ω is the unit disc: the orthonormal basis $\{e_k\}$ of H^2, where $e_k(z) = z^k$, indeed undergoes a shift when multiplied by z, i.e. $ze_k(z) = e_{k+1}(z)$. The Abrahamse–Douglas model theorem explains therefore the name proposed in the title for our class of subnormal operators. Actually, it is more convenient to modify slightly the assumption (1.0) defining it, due to some topological difficulties (absent in [AD1], [AD2], where only finitely many smoothly bordered holes of Ω were admittted).

Definition 1.1. Let Ω be a domain in the complex plane $\mathbb{C}$. We say that a bounded operator S is of *Hardy type with respect to* Ω if S is pure subnormal and the spectra of S and of its *minimal normal extension* $N = \text{m.n.e.}(S)$ satisfy the following condition:

$$\sigma(S) \subset \overline{\Omega} \quad \text{and} \quad \sigma(N) \subset \partial\Omega. \tag{1.2}$$

Using (SI) and the connectedness of Ω one easily deduces that such a set must necessarily be bounded, with $\overline{\Omega} = \sigma(S)$. Hence (1.2) implies (1.0). Moreover, the pure part ([AD1, Prop. 1.1]) of a subnormal operator satisfying (1.2) is of Hardy type (cf. [R2, (2.1b)]), which allows one to restrict considerations to the case when S itself is pure. Note that the set Ω is not determined by S satisfying (1.2) due, e.g., to possible slit indentations in Ω. Such indentations are, however, excluded by our next assumptions made in Proposition 2.8 below. In particular, if $\sigma(S)$ equals the closure of its interior, then (1.2) with $\Omega = \mathrm{int}(\sigma(S))$ is equivalent to our "provisional assumption" (1.0).

Essential to our analysis is the use of certain spaces of functions on Ω. Let us begin with recalling some notions. We say that E (more precisely, a pair (E, π), where $\pi : E \to \Omega$) is a *flat unitary bundle over* Ω if E is a topological space, its *fibers* $E_\lambda = \pi^{-1}\{\lambda\}$ over the points $\lambda \in \Omega$ are Hilbert spaces and Ω has a covering $\mathcal{W} = \{U; U \in \mathcal{W}\}$ by open sets U such that $\pi^{-1}U = E_U$ are homeomorphic via some mappings τ_U to trivial bundles $U \times K_U$. Here K_U are certain Hilbert spaces and $\tau_U : E_U \to U \times K_U$ are supposed to be compatible in such a way that the transition functions

$$\tau_U \circ \tau_V^{-1} : (U \cap V) \times K_V \to (U \cap V) \times K_U \tag{1.3}$$

map (λ, x) into $(\lambda, \tau_\lambda^{UV}(x))$ with $\tau_\lambda^{UV} : K_V \to K_U$ some unitary operators *depending holomorphically on* $\lambda \in U \cap V$. Consequently, τ_λ^{UV} are *constant* with respect to λ, which explains the term *flat* [AD1]. A mapping $f : \Omega \to E$ is called a *holomorphic cross-section* of E if $\pi(f(\lambda)) = \lambda$ $(\forall \lambda \in \Omega)$ and the mappings $\tau_U \circ f|_U : U \to U \times K_U$ are holomorphic $(\forall U \in \mathcal{W})$. Note that for $\lambda \in \Omega$ the norms $\|f(\lambda)\|_U$ in the coordinate spaces K_U are the same for all $U \in \mathcal{W}$ whenever $\lambda \in U$, so that the notation $\|f(\cdot)\|$ for the function $\Omega \ni \lambda \mapsto \|f(\lambda)\| \in \mathbb{R}_+$ is unambiguous.

DEFINITION 1.4. For $1 \le p < \infty$ the *Hardy space* $H^p[E]$ of a flat unitary bundle E is the set of all holomorphic cross-sections f of E such that the function $\|f(\cdot)\|^p$ possesses a harmonic majorant, i.e. a harmonic function $h : \Omega \to [0, +\infty)$ satisfying

$$\|f(\lambda)\|^p \le h(\lambda) \quad \forall \lambda \in \Omega.$$

The norm $\|f\|_p$ of $f \in H^p[E]$ is defined with respect to some fixed *norming point* $\lambda_0 \in \Omega$ as the quantity

$$\|f\|_p = (h(\lambda_0))^{1/p},$$

in which h is the *least harmonic majorant* of $|f|^p$. We define analytic *Toeplitz operators* T_φ for $\varphi \in H^\infty(\Omega)$, by

$$T_\varphi : H^p[E] \ni f \mapsto \varphi f \in H^p[E], \quad (T_\varphi f)(\lambda) = \varphi(\lambda) f(\lambda). \tag{1.5}$$

The *bundle shift* T_E is the multiplication by the independent variable: $T_E = T_\varrho$ for the function $\varrho(\lambda) = \lambda$, i.e.

$$(T_E f)(\lambda) = \lambda f(\lambda), \quad \lambda \in \Omega.$$

In particular, if E is a trivial bundle $\Omega \times K$ (so that $\pi(\lambda, k) = \lambda$), we identify its sections with functions $f : \Omega \to K$, denoting the corresponding Hardy class by $H_K^p(\Omega)$. In the scalar-valued case ($K = \mathbb{C}$), we write $H^p(\Omega)$. For functions defined on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the notation H_K^p (resp. H^p) replaces $H_K^p(\Omega)$ (resp. $H^p(\Omega)$).

The language of bundles is used to cope with multi-valued functions, providing one with more elegant formulations, but our analysis will be carried out mostly on equivalent objects known as *automorphic functions*, arising in the uniformisation technique. Here the basic fact is that the unit disc is (via some continuous mapping $t : \mathbb{D} \to \Omega$) a *universal covering space* for Ω. This means that Ω is a union of some open sets $W \subset \Omega$ for which the restriction of t to any connected component of $t^{-1}W$ maps the latter homeomorphically onto W. More importantly, the covering map t can be chosen to be holomorphic. Let $G = \mathrm{Aut}(\mathbb{D}, t)$ be the set of all Möbius automorphisms (i.e. fractional-linear bijections $A : \mathbb{D} \to \mathbb{D}$) such that $t \circ A = t$. We call G the *deck transformations group* of the cover $(\mathbb{D}, t)$. Let α be a unitary representation of this group in some Hilbert space K.

DEFINITION 1.6. We say that a function $f : \mathbb{D} \to K$ is *automorphic with factor* α, or α-automorphic, if

$$f(A(\lambda)) = \alpha(A)f(\lambda), \qquad \forall A \in G, \lambda \in \mathbb{D}.$$

In the same manner we distinguish α-automorphic functions among K-valued functions defined *almost everywhere* with respect to the *normalized Lebesgue measure* μ on the unit circle $\partial\mathbb{D}$. Here we extend the Möbius maps holomorphically to neighbourhoods of $\overline{\mathbb{D}}$ and we note that they preserve the class of μ-null subsets of the unit circle. If $\mathcal{F}$ is some space of K-valued functions on $\mathbb{D}$ (or defined a.e. $[d\mu]$ on $\partial\mathbb{D}$), we shall use the notation

$$\mathcal{F}/\alpha \stackrel{\text{def}}{=} \{f \in \mathcal{F} : f \text{ is } \alpha\text{-automorphic}\}.$$

In the case of the trivial representation ($\alpha(A) = I_K$, the identity operator $\forall A \in G$), we simply speak of *automorphic functions*, using the notation $\mathcal{F}/G$ in place of $\mathcal{F}/\alpha$.

Clearly, these subsets are closed linear subspaces. Using analogous notation H^∞/G for automorphic elements of the Banach algebra H^∞ of bounded analytic functions on $\mathbb{D}$, we see that H^p_K/α are even H^∞/G-submodules: Any function $\varphi \in H^\infty/G$ defines in the same manner as (1.5) a bounded linear operator

$$T_\varphi : H^p_K/\alpha \ni f \mapsto \varphi f \in H^p_K/\alpha.$$

In the case when φ is the covering map t, we use the notation T_α instead of T_t and the special role of this operator is explained by the following two results.

PROPOSITION 1.7 ([AD1, Theorem B]). *There is a one-to-one correspondence between the classes modulo unitary equivalence of unitary representations $\alpha : G \to \mathcal{U}(K)$ and the equivalence classes of flat unitary bundles E over Ω satisfying* $\dim(E_\lambda) = \dim(K)$.

Here α being equivalent to α_1 means the existence of a unitary $W \in \mathcal{U}(K)$ such that $\alpha(A) = W^{-1}\alpha_1(A)W$ $\forall A \in G$, while two bundles E, E^1 are considered equivalent if there is a homeomorphism $\Lambda : E \to E^1$ such that the transition functions $\tau_U \circ (\tau^1_V \circ \Lambda)^{-1}$ analogous to (1.3) are of the form $(\lambda, k) \mapsto (\lambda, W^{UV}k)$ for some unitary operators W^{UV} independent of $\lambda \in U \cap V$.

Let us also recall how one can view Ω and E as quotient spaces of $\mathbb{D}$ and $\mathbb{D} \times K$ respectively. Namely, points of Ω are identifiable as orbits of points $z \in \mathbb{D}$ under the equivalence $\zeta_1 \simeq \zeta_2$ iff $t(\zeta_1) = t(\zeta_2)$ (which, in turn, takes place iff for some $A \in G$ one has $\zeta_2 = A(\zeta_1)$). Similarly, (ζ, k) and (ζ_1, k_1) are identified in $\mathbb{D} \times K$ iff $(\zeta_1, k_1) = (A(\zeta), \alpha(A)k)$ for some $A \in G$. This also suggests how the bundle projection $\pi : E \to \Omega$ should arise. Furthermore, if $f : \mathbb{D} \to K$ is α-automorphic, then $\zeta_1 \simeq \zeta_2$ implies that $(\zeta_1, f(\zeta_1))$ and $(\zeta_2, f(\zeta_2))$ define the same point of E, say $F(\lambda)$, where $\lambda \in \Omega$ is identified with the coset of ζ_1. Thus F is the cross-section of E corresponding to f. Finally, this correspondence between cross-sections and automorphic functions is isometric. Here H^p_K/α is considered with the least harmonic majorant norm, but the latter coincides with the $L^p_K(\mu)$-norm, as shown in [AD1] (explaining why for $p = 2$ we get Hilbert spaces).

PROPOSITION 1.8. *Any automorphic function $\psi \in H^\infty/G$ is of the form $\psi = \varphi \circ t$ for a unique function $\varphi \in H^\infty(\Omega)$. Moreover, the Toeplitz operator T_ψ on $H^p[E]$ is isometrically equivalent to T_φ on H^p_K/α for the unitary representation $\alpha : G \to \mathcal{U}(K)$ corresponding to E in the above described manner.*

In particular, for $p = 2, \varrho(\lambda) = \lambda$, the bundle shift T_E is unitarily equivalent to T_α. This result follows from a slight modification of the analogous Theorem 5 of [AD1]. One shows that the mentioned correspondence between cross-sections and α-automorphic functions is isometric and carries T_φ onto T_ψ.

2. Behaviour at the boundary. In this section we explain the role played by condition (1.2) and the necessity of assuming some regularity for $\partial\Omega$ in the Hardy space model theory. We begin with recalling some facts on *harmonic measure* $dm = dm_\Omega$ for a bounded plane domain Ω. We fix a universal covering map $t : \mathbb{D} \to \Omega$, always assuming that the norming point $\lambda_0 \in \Omega$ is chosen so that $\lambda_0 = t(0)$ and that the harmonic measure m is taken *from the point* λ_0. In other words, $\int_{\partial\Omega} f\, dm$ evaluates at λ_0 the *generalised solution $\hat{f}$ of the Dirichlet problem for Ω*, given any continuous boundary data $f : \partial\Omega \to \mathbb{C}$. The set $\mathcal{I}$ of irregular ("bad") boundary points λ_b, where $\hat{f}(\lambda)$ fails to converge to $f(\lambda_b)$ as $\lambda \to \lambda_b, \lambda \in \Omega$, is of logarithmic capacity zero.

The harmonic measure for $\mathbb{D}$ taken from the origin is clearly the normalized Lebesgue measure $d\mu$ on the unit circle. The *principle of invariance of harmonic measure under holomorphic mappings* extends to boundary values $h(e^{i\theta}), \theta \in \mathbb{R}$, of bounded holomorphic mappings $h \in H^\infty$. In particular, t carries $d\mu$ onto dm (cf. [R3, p. 445]), i.e.

$$\int_{\partial\Omega} f\, dm = \int_{\partial\mathbb{D}} f \circ t\, d\mu \tag{2.1}$$

and the "lifting to the covering disk" operation

$$L^p_K(m) \ni f \mapsto f \circ t \in L^p_K(\mu) \tag{2.2}$$

is an isometry between these vector-valued L^p-spaces, whose image is precisely the subspace L^p_K/G of automorphic elements of L^p_K [AD1], [R2].

Actually, more important to our considerations will be the absolute continuity with respect to the harmonic measure—which, of course does not depend on the base point λ_0. It is here that some regularity conditions enter the picture. We begin with introducing one more concept.

DEFINITION 2.3. Let $R(\overline{\Omega})$ be the closure in the uniform convergence norm of the set of all rational functions having poles outside $\overline{\Omega}$. Let $\mathrm{Repr}(\lambda_0)$ be the set of all nonnegative Borel measures ν on $\partial\Omega$ representing the evaluation functional at λ_0, so that

$$f(\lambda_0) = \int_{\partial\Omega} f(\lambda)\, d\nu(\lambda) \qquad \forall f \in R(\overline{\Omega}).$$

We shall formulate below the conditions sufficient for the absolute continuity relation

$$\nu \ll m \qquad \forall \nu \in \mathrm{Repr}(\lambda_0). \tag{2.4}$$

The relevance of all these notions to Hardy type operators is explained by the following fundamental result (cf. [R2], with corrections in [R4]).

THEOREM 2.5. *Let S be a Hardy type operator with respect to a plane domain Ω whose boundary has zero area.*

(1) *The* scalar-valued spectral measure *of S is absolutely continuous with respect to some representing measure $\nu \in \mathrm{Repr}(\lambda_0)$.*

(2) *If, moreover, the absolute continuity* (2.4) *holds, then there exist a Hilbert space K and a subspace $\mathcal{M}$ of $L^2_K(m)$, pure invariant under multiplication by the functions from $R(\overline{\Omega})$, such that S is unitarily equivalent to the restriction $M_z|_{\mathcal{M}}$ to $\mathcal{M}$ of the multiplication by the independent variable on $L^2_K(m)$.*

Here we denote by M_φ the multiplication by φ defined exactly as in (1.5), and the pure invariance of $\mathcal{M}$ means that none of its nonzero subspaces can reduce all operators M_φ, $\varphi \in R(\overline{\Omega})$. The proof is analogous to that of [R2, Prop. 2.5] except that in order to apply the result of Mlak [M, Lemma 1] on absolute continuity of Szegő measures, one has to know that there is only one non-trivial Gleason part of the maximal ideal space of $R(\overline{\Omega})$. But since Ω is contained in one such part, all the remaining Gleason parts must be subsets of $\partial\Omega$, hence of zero area. However, nontrivial (i.e. having more than one point) parts must have positive area [G]. ∎

As we shall see, the absolute continuity condition (2.4) is one of the three function-theoretic fundamentals implying the model theorem. The failure of each of these three was shown in [R2, R3] to lead to the failure of the model theorem, i.e. to the existence of Hardy type operators with respect to Ω, not equivalent to any bundle shift over Ω. Let us recall the remaining two conditions.

DEFINITION 2.6. We say that $R(\overline{\Omega})$ is *pointwise boundedly dense* in $H^\infty(\Omega)$ if for any function $h \in H^\infty(\Omega)$ there exists a sequence of rational functions $r_n \in R(\overline{\Omega})$ with $|r_n(\lambda)| \le M(h)$ and $r_n(\lambda) \to h(\lambda)$ as $n \to \infty$ $(\forall \lambda \in \Omega)$. Here $M(h)$ is some constant depending only on h.

The last, perhaps most difficult condition is responsible for the validity of the Beurling–Lax theorem in $L^p_K(m)$, stated in Theorem 2.9(1) below.

DEFINITION 2.7. Ω is a *Parreau–Widom type domain* if any flat unitary bundle E admits nontrivial bounded holomorphic sections: $H^\infty[E] \neq \{0\}$ (here ∞ can be replaced by any $1 \le p < \infty$). For such domains we use as a further regularity condition the so called *Direct Cauchy Theorem* (cf. [H]), or briefly (DCT). If $\{a_k\}$ are the all critical points (counting according to their multiplicity) of the Green function G of Ω, put

$$\Lambda(z) = \exp\Big(-\sum G(z, a_k)\Big).$$

The assertion of (DCT) is that for any meromorphic function f on Ω such that $z \mapsto \Lambda(z)|f(z)|$ possesses a harmonic majorant on Ω and for any fixed base point $z_0 \in \Omega$ one has

$$f(z_0) = \int f(z)\, d\chi(z),$$

where we integrate over the Martin boundary the corresponding boundary values of f against the harmonic measure for z_0 on this boundary.

The importance of the (DCT) condition rests on the fact that it is equivalent to the validity of the Beurling–Lax theorem for $L^p_K(m)$. For $K = \mathbb{C}$ this result is due to Morisuke Hasumi [H].

There are several geometric assumptions on $\partial\Omega$ implying the absolute continuity relations (2.4). In [R2, R3] Sarason's free arcs condition was assumed to the effect that "nearly all" of $\partial\Omega$ consists of free arcs γ, with free meaning that Ω is situated only on one side of γ. The "nearly all" phrase was used to mean "except some peak set K of *Hausdorff dimension* $\dim_{\mathrm{Hausd}}(K)$ less than one". Due to the results of Samokhin [S], we can do even better. Let us recall first that the *inner boundary* $\partial_{\mathrm{inn}}\Omega$ is the complement in $\partial\Omega$ of the union of all boundaries of holes in Ω. (Holes are defined as the connected components of $\mathbb{C}\backslash\overline{\Omega}$.)

PROPOSITION 2.8. *Assume that Ω is a Parreau–Widom domain satisfying the* (*DCT*) *condition. If* $\dim_{\mathrm{Hausd}}(\partial_{\mathrm{inn}}\Omega) < 1$, *then*

(1) *the absolute continuity condition* (2.4) *holds*,

(2) $R(\overline{\Omega})$ *is pointwise boundedly dense in* $H^\infty(\Omega)$.

Proof. The second assertion is the theorem of Davie and Øksendal cited in [R2]. In particular, the algebra $A(\overline{\Omega})$ containing $R(\overline{\Omega})$ is pointwise boundedly dense in $H^\infty(\Omega)$ and the assumptions of Theorem 10 in [S] are satisfied. Consequently, each measure κ orthogonal to $A(\overline{\Omega})$ (i.e. such that $\int f\,d\kappa = 0\ \forall f \in A(\overline{\Omega})$) is absolutely continuous with respect to m. Now, for any $\nu \in \mathrm{Repr}(\lambda_0)$ one has $m-\nu \perp A(\overline{\Omega})$, since $m-\nu \perp R(\overline{\Omega})$ and we can use pointwise bounded approximation. The conclusion from [S] is that $m-\nu \ll m$ and hence $\nu \ll m$. ∎

Now, as in [R2], we deduce the main result, stated now under weaker assumptions.

THEOREM 2.9. *Let Ω be a domain satisfying the assumptions of Proposition* 2.8 *and let* $1 \le p < \infty$.

(1) *For any closed subspace $\mathcal{M}$ of $L^p_K(m)$, pure invariant for $R(\overline{\Omega})$, there exists a Hilbert space M, a unitary representation $\alpha : G \to \mathcal{U}(M)$ and a decomposable isometry $\Psi : L^p_M \to L^p_K$ of multiplicative character α such that the canonical lift $\mathcal{M}\circ t = \{f\circ t : f \in \mathcal{M}\}$ of $\mathcal{M}$ is the image under Ψ of the Hardy class H^p_M/α*:

$$\mathcal{M}\circ t = \Psi(H^p_M/\alpha).$$

(2) *Any operator of Hardy type with respect to Ω is unitarily equivalent to some bundle shift T_E for a flat unitary bundle E over Ω.*

3. Multiplication operators. In this section an application of the bundle shift model to a concrete class of operators is presented. Its aim is to find some useful links between properties of functions and the behaviour of the related multiplication operators. One can argue that

$$T_\varphi f(\lambda) = \varphi(\lambda)f(\lambda) \tag{3.1}$$

is already a form simple enough. If we change it, however, to a form in which $\varphi(\lambda) = \lambda$ (with vectors f running through some different function space), spectral properties become much easier. We begin with a simple observation, proved essentially in [R3] and partially (the claim asserting purity) in [R4]. Let G be an arbitrary open set in $\mathbb{C}$ and fix a strictly nonconstant bounded analytic function φ on G (i.e. nonconstant on any component of G). This already involves a mild assumption on G so that ∂G supports a harmonic measure, say m_G, evaluating at fixed points the "local" solutions of the generalised Dirichlet problem. Here the term *local* refers to solutions considered separately in each connected component of G. Our formula (3.1) with $f \in H^p_K(G)$ and $\lambda \in G$ defines a bounded linear operator T_φ on $H^p_K(G)$, called *analytic Toeplitz*, or a *multiplication operator*.

LEMMA 3.2. *For $p = 2$ the multiplication operator T_φ by a strictly nonconstant function $\varphi \in H^\infty(G)$ is pure subnormal. This operator is of Hardy type if the following "preservation of the boundary" condition is satisfied by φ:*

$$\mathcal{C}\ell_{\mathrm{n}}(\varphi, \lambda) \subset \partial\Omega \quad \text{for } [dm_G]\text{-almost all points } \lambda \in \partial G. \tag{3.3}$$

If, moreover, G is connected and if $\psi = \varphi \circ \tau \in H^\infty$ is a lifting of φ to $\mathbb{D}$ via some fixed universal covering map $\tau : \mathbb{D} \to G$, then (3.3) *is equivalent to the* $[d\mu]$*-essential image* (cf. [AK]) *of ψ being a subset of $\partial\Omega$.*

Here $\mathcal{C}\ell_{\mathrm{n}}(\varphi, \lambda_0)$ stands for the "non-tangential" cluster set of φ at λ defined to consist of all limit points of $\{\varphi(\lambda_n)\}_{n=1}^\infty$, for all sequences $\{\lambda_n\} \subset G$ converging to λ_0 non-tangentially in the following sense: If G_0 is a connected component of G satisfying $\lambda \in \overline{G}_0$, then the λ_n should all stay within the image of some Stolz region in $\mathbb{D}$ under an (analytic) universal cover of G_0. The set of analogous limits without the "non-tangency" constraint, called the (full) cluster set, is denoted by $\mathcal{C}\ell(\varphi, \lambda_0)$.

Note that although proved in [R3, R4] in the case of connected G, the above result holds without such restrictions. What is even more important, the properties of Hardy-type multiplication operators T_φ discussed here (i.e. satisfying the assumption (3.3)) depend solely on the ranges of functions φ—and not on their domains of definition, G.

Since we have here a concrete normal extension N (namely, multiplication by ψ on some L^2-space of automorphic vector-valued functions on $\partial\mathbb{D}$), we may directly establish in this case the absolute continuity of N with respect to the harmonic measure dm_Ω, by using the disintegration methods of [AK] rather than Samokhin's result [S]. As follows from [R3], a conformally invariant version of the assumptions from Prop. 2.8 suffices:

THEOREM 3.4 [R3, Thm.1.3]. *If a strictly nonconstant function $\varphi \in H^\infty(G)$ satisfies* (3.3) *and if $\Omega = \varphi(G)$ is conformally equivalent to a domain satisfying the conditions of Proposition* 2.8, *then T_φ is unitarily equivalent to a bundle shift T_E over Ω.*

To show an example of application, consider the following classical boundary value principle for holomorphic maps.

PROPOSITION 3.5 [P, Thm.1.9]. *If Ω is a plane domain consisting of points interior to a Jordan curve J and $\varphi : G \to \Omega$ is a nonconstant holomorphic function defined on some domain G of the extended plane such that the cluster values of φ at points of ∂G*

belong to $\partial\Omega = J$, *then* $\varphi(G) = \Omega$. *If*, *moreover*, *some value* $\lambda \in \Omega$ *is attained by* φ *only once* (*counting multiplicity*), *then* φ *must be* (*globally*) *injective on* G.

Our methods allow for a considerable weakening of the above assumptions.

PROPOSITION 3.6. *The conclusions of* 3.5 *remain valid if we assume that the domain* Ω *and function* φ *satisfy the conditions of Theorem* 3.4.

Proof. Since T_φ is by 3.4 unitarily equivalent to some bundle shift T_E, it suffices to give a suitable interpretation for the fact that the value ζ is attained at some point of G. This is equivalent to $\bar{\zeta}$ being an eigenvalue of $T_\varphi^\star$. The eigenvalues of $T_E^\star$ are the points of bounded evaluations, hence $\Omega \subset \varphi(G)$. Likewise, the injectivity can be interpreted in terms of unitary invariants, namely as the multiplicity of eigenvalues being equal to one, and must be locally constant, by the well-known properties of the index. ■

Another application can be found in [R3], where the following dichotomy for cluster values of functions φ satisfying (3.3) above is established: Denote by $\mathcal{C}\ell(\varphi, \partial G)$ the union over all boundary points $\lambda \in \partial G$ of the cluster sets $\mathcal{C}\ell(\varphi, \lambda)$. Then assuming 3.4 we see that only two possibilities occur:

$$\text{either} \quad \mathcal{C}\ell(\varphi, \partial G) \subset \partial\Omega, \quad \text{or else} \quad \mathcal{C}\ell(\varphi, \partial G) = \overline{\Omega}.$$

Analogously, one can prove (cf. [R5]) the corresponding local statements (for individual cluster sets at given points $\lambda \in \partial G$). The author has learned recently about analogous results of [S1], where more restrictive assumptions about the domains of definition (implying our conditions) were taken. This explains again the general phenomenon that only the regularity of the ranges is essential.

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