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ON THE DIFFERENCES OF THE CONSECUTIVE POWERS OF BANACH ALGEBRA ELEMENTS

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Abstract. Let $\mathcal{A}$ denote a complex unital Banach algebra. We characterize properties such as boundedness, relative compactness, and convergence of the sequence $\{x^n(x-1)\}_{n\in\mathbb{N}}$ for an arbitrary $x \in \mathcal{A}$, using $\sigma(x)$ and resolvent conditions. Under these circumstances, we investigate elements in the peripheral spectrum, and give further conclusions, also involving the behaviour of $\{x^n\}_{n\in\mathbb{N}}$ and $\left\{\frac{1}{n}\sum_{k=0}^{n-1} x^k\right\}_{n\in\mathbb{N}}$.

1. Introduction. Throughout this paper, $\mathcal{A}$ denotes a *complex Banach algebra with identity element* 1 satisfying $\|1\| = 1$. For $x \in \mathcal{A}$, we denote by $\sigma(x)$, $\rho(x)$, and $r(x)$, the *spectrum*, the *resolvent set* $\mathbb{C} \setminus \sigma(x)$, and the *spectral radius* of x, respectively. The vector valued analytic function $\rho(x) \ni \lambda \mapsto r(\lambda, x) := (\lambda - x)^{-1} \in \mathcal{A}$ is called the *resolvent* of x.

A subset $\sigma \subset \sigma(x)$ is called a *spectral set* of x, if it is open and closed relative to $\sigma(x)$. For a spectral set σ, its complement $\sigma' := \sigma(x) \setminus \sigma$ is a spectral set, too. Via the *Riesz–Dunford functional calculus* for holomorphic functions [4, VII.4, Proposition 4.7], to each spectral set $\sigma \subset \sigma(x)$ we can assign a so-called *associated spectral idempotent* $p_\sigma \in \mathcal{A}$. For two complementary spectral sets the associated spectral idempotents satisfy (cf. [4, VII.4, Proposition 4.11], [7, Satz 99.5])

(i) $p_\sigma^2 = p_\sigma$, and $p_\sigma = 0$ if and only if $\sigma = \emptyset$,
(ii) $p_\sigma + p_{\sigma'} = 1$, $p_\sigma p_{\sigma'} = p_{\sigma'} p_\sigma = 0$,
(iii) $\sigma(xp_\sigma) \subset \sigma \cup \{0\}$, $\sigma(xp_{\sigma'}) \subset \sigma' \cup \{0\}$.

A complex number $\lambda \in \sigma(x)$ is called a *pole of order* p of the resolvent if $\{\lambda\}$ is a spectral set and for the associated spectral idempotent $p_\lambda \in \mathcal{A}$, $(\lambda - x)^p p_\lambda = 0$ and $(\lambda - x)^{p-1} p_\lambda \neq 0$ hold. We say that λ is a pole of order at most p if at least the first equation is satisfied. We will denote that by $\operatorname{ord}(\lambda) = p$ or $\operatorname{ord}(\lambda) \leq p$, respectively, and define $\operatorname{ord}(\lambda) = 0$ for all $\lambda \in \rho(x)$.

We write $\mathbb{D}$ for the open unit disk and Γ for the unit circle.

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2. The orbit. In this introductory section we collect some facts about the spectrum $\sigma(x)$ and the resolvent $r(\cdot, x)$ of $x \in \mathcal{A}$, provided its *orbit* $\{x^n\}_{n\in\mathbb{N}}$ shows a certain behaviour. The results are known from operator theory and formulated here in terms of Banach algebras. They will play an essential role in the next section, where by a reduction principle they will be brought up and serve to prove a lot of new statements.

In their joint paper [1], G. R. Allan and T. J. Ransford introduced the notion of power domination. They call $x \in \mathcal{A}$ *power dominated* by a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of positive real numbers if $\|x^n\| \le \mu_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu_{n+1}/\mu_n = 1$, and prove that a necessary and sufficient condition that x be power dominated is $r(x) \le 1$.

We will show that all elements that occur within the scope of our investigations fall into this very class. In the particular case where $x \in \mathcal{A}$ may be power dominated by a bounded sequence, so $\sup_{n\in\mathbb{N}} \|x^n\| < \infty$, we say that x is *power bounded.*

Obviously every power bounded element x satisfies $r(x) \le 1$, but the converse is false.

EXAMPLE 2.1. Let $\mathcal{M}(m,\mathbb{C})$ be the algebra of complex valued $m\times m$-matrices, and $B(X)$ the algebra of all bounded linear operators in the complex Banach space X provided with the operator norm. Then both are unital Banach algebras with identity elements $E \in \mathcal{M}(m,\mathbb{C})$ and $I \in B(X)$, respectively.

First, consider $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathcal{M}(2,\mathbb{C})$. Then $\sigma(A) = \{1\}$, so $r(A) = 1$, but since $A^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, A is not power bounded.

Now, let the linear operator $S\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Then $\sigma(S) = \overline{\mathbb{D}}$ and $\|S\| = 1$ imply $\|S^n\| = 1$ for all $n \in \mathbb{N}$. So, S is power bounded in $B(\ell^2(\mathbb{N}))$.

This example reveals that the set theoretical knowledge $\sigma(x) \subset \overline{\mathbb{D}}$ of the spectrum itself—in general—is insufficient to distinguish between power dominated and power bounded elements. For this purpose, conditions on the resolvent $r(\cdot, x)$ are necessary. But before passing to those more complicated things, we will handle the special case $r(x) < 1$.

THEOREM 2.2. *For $x \in \mathcal{A}$ the following are equivalent*:

(i) $\{p(n)x^n\}_{n\in\mathbb{N}} \to 0$ *for all polynomials p as $n \to \infty$*,
(ii) $\{p(n)x^n\}_{n\in\mathbb{N}} \to 0$ *for some non-zero polynomial p as $n \to \infty$*,
(iii) $r(x) < 1$,
(iv) $(1 - x)^{-1} \in \mathcal{A}$ *and* $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$.

The proof is based on Neumann's theorem and [7, Aufgabe 108.1].

Remark. In particular, every element $x \in \mathcal{A}$ with $r(x) < 1$ is power bounded and satisfies $\{x^n\}_{n\in\mathbb{N}} \to 0$ as $n \to \infty$.

As a corollary we can state:

COROLLARY 2.3. *For $x \in \mathcal{A}$, $\{(\mu x)^n\}_{n\in\mathbb{N}} \to 0$ for all $\mu \in \mathbb{C}$ as $n \to \infty$ if and only if x is quasi-nilpotent, i.e. $r(x) = 0$.*

The corollary expresses that in case that $r(x) = 0$ the decrease of $\|x^n\|$ is not only faster than any polynomial increase (as in the case $r(x) < 1$) but even faster than any exponential growth of the coefficients.

A rather technical way of characterizing power bounded elements is in the following "re-norming lemma", which comes from [2, Theorem 4.1].

LEMMA 2.4. *Let $x \in \mathcal{A}$ satisfy $r(x) = 1$. Then x is power bounded in $(\mathcal{A}, \|\cdot\|)$ if and only if there exists an equivalent algebra norm $\|\cdot\|^*$ on $\mathcal{A}$ such that $\|x\|^* = 1$.*

The first appearance of resolvent conditions now directly leads to a sufficient condition for power boundedness of $x \in \mathcal{A}$, a proof of which can be derived from [23, Satz 5].

PROPOSITION 2.5. *Let $x \in \mathcal{A}$ satisfy $r(x) \le 1$, and suppose that there exists $p \in \mathbb{N}$ such that $\sigma(x) \cap \Gamma$ is a set of poles of the resolvent of order not exceeding p. Then $\big\{\frac{1}{n^{p-1}} x^n\big\}_{n\in\mathbb{N}}$ is relatively compact.*

This proposition has a natural extension to the case $p = 0$ which was treated in Theorem 2.2. The case $p = 1$ gives the desired condition for power boundedness of x.

Remark. Since $\sigma(x) \cap \Gamma$ is a compact set it is finite whenever it consists only of poles of the resolvent.

A natural question in this context is now what can be said about the order of poles in $\sigma(x) \cap \Gamma$ if x is power bounded. The answer is given by

PROPOSITION 2.6. *Let $x \in \mathcal{A}$ be power bounded. If $\lambda \in \sigma(x) \cap \Gamma$ is a pole of order p of the resolvent then $p = 1$.*

The proof is straightforward and can be found in [18, Satz 3.13]. The statement can also be derived from [23, Satz 2] or [25, Lemma 2.2].

This proposition explains why in Example 2.1 the matrix A failed to be power bounded: The eigenvalue 1 is of double multiplicity, so 1 is a pole of the resolvent of A of order 2.

The special case $p = 1$ in Proposition 2.5 together with Proposition 2.6 leads to the characterization of the relative compactness of $\{x^n\}_{n\in\mathbb{N}}$, which was proven by M. A. Kaashoek and T. T. West in [10, Theorem 3].

THEOREM 2.7. *For $x \in \mathcal{A}$, $\{x^n\}_{n\in\mathbb{N}}$ is relatively compact if and only if $r(x) \le 1$ and every $\lambda \in \sigma(x) \cap \Gamma$ is a simple pole of the resolvent.*

Based on a different approach, A. Święch gave a proof of the same result in [20, Theorem 2]. From (the sufficiency part of) both proofs we can derive an explicit representation of the set $\mathrm{Acc}(x)$ of all accumulation points of $\{x^n\}_{n\in\mathbb{N}}$ in the case where the orbit is relatively compact, namely:

$$\mathrm{Acc}(x) = \Big\{ s = \sum_{k=1}^{m} \lambda_k^{(0)} p_k \in \mathcal{A} \Big\}, \tag{1}$$

where $\sigma(x) \cap \Gamma = \{\lambda_1, \ldots, \lambda_m\}$, $p_1, \ldots, p_m \in \mathcal{A}$ are the associated spectral idempotents of $\lambda_1, \ldots, \lambda_m$, and $\lambda_k^{(0)} = \lim_{j\to\infty} \lambda_k^{a_j}$, where $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ is chosen such that these limits exist for all $k = 1, \ldots, m$. (Such a sequence always exists, since $\sigma(x) \cap \Gamma$ is finite and each λ_k is of modulus 1.)

Some properties of the set $\mathrm{Acc}(x)$ are obvious:

(i) $\mathrm{Acc}(x)$ is finite if and only if $\lambda_1, \ldots, \lambda_m$ are roots of unity,

(ii) $\mathrm{Acc}(x)$ is closed under multiplication,

(iii) $0 \in \mathrm{Acc}(x) \Leftrightarrow \{x^n\}_{n \in \mathbb{N}} \to 0$ as $n \to \infty$,

(iv) $1 \in \mathrm{Acc}(x)$ if and only if $\sigma(x) \subset \Gamma$. Moreover, each of these two conditions is equivalent to the relative compactness of $\{x^n\}_{n \in \mathbb{Z}}$, provided $\{x^n\}_{n \in \mathbb{N}}$ is relatively compact ([18, Korollare 4.6 and 4.7]).

These results allow us to characterize convergence of $\{x^n\}_{n \in \mathbb{N}}$ very easily. In this case, $\mathrm{Acc}(x)$ must be a singleton which (due to (ii)) consists of an idempotent. And from (1) we derive that $\lambda_k^{(0)}$ must be independent of the choice of a suitable sequence $\{a_j\}_{j \in \mathbb{N}}$. This means $\lambda_k = 1$ for all $k = 1, \ldots, m$, so $\sigma(x) \cap \Gamma = \{1\}$, and 1 is a simple pole of the resolvent. We summarize these results in the next theorem.

THEOREM 2.8. *For $x \in \mathcal{A}$ the following are equivalent*:

(i) $\{x^n\}_{n \in \mathbb{N}} \to p \in \mathcal{A}$ *as* $n \to \infty$,

(ii) $\{x^n\}_{n \in \mathbb{N}}$ *is relatively compact and* $\sigma(x) \cap \Gamma \subset \{1\}$,

(iii) $\sigma(x) \subset \sigma_0 \cup \{1\}$, *where* $\overline{\sigma}_0 \subset \mathbb{D}$, *and if* $1 \in \sigma(x)$ *then* 1 *is a simple pole of the resolvent*,

(iv) $x = x_0 + p$, *where* $x_0, p \in \mathcal{A}$ *satisfy* $p^2 = p$, $x_0 p = p x_0 = 0$ *and* $r(x_0) < 1$.

The equivalence between (i) and (iii) is due to J. J. Koliha ([12, Theorem 0]).

Equipped with these results, we can state the following corollary.

COROLLARY 2.9. *For $x \in \mathcal{A}$ with $\{x^n\}_{n \in \mathbb{N}} \to p$ as $n \to \infty$ and $y_\lambda = \lambda x$ the following are equivalent*:

(i) $p = 0$, *i.e.* $r(x) < 1$,

(ii) $\{y_\lambda^n\}_{n \in \mathbb{N}} \to 0$ *for all* $\lambda \in \Gamma$ *as* $n \to \infty$,

(iii) $\{y_\lambda^n\}_{n \in \mathbb{N}} \to 0$ *for some* $\lambda \in \Gamma$ *as* $n \to \infty$,

(iv) $\{y_\lambda^n\}_{n \in \mathbb{N}}$ *converges for all* $\lambda \in \Gamma$ *as* $n \to \infty$,

(v) $\{y_\lambda^n\}_{n \in \mathbb{N}}$ *converges for some* $\lambda \in \Gamma \setminus \{1\}$ *as* $n \to \infty$.

Proof. It is enough to prove that (v) implies (i). Since $\{y_\lambda^n\}_{n \in \mathbb{N}}$ converges, $\sigma(y_\lambda) \cap \Gamma \subset \{1\}$ by Theorem 2.8. But $\sigma(y_\lambda) \cap \Gamma = \sigma(\lambda x) \cap \Gamma \subset \{\lambda\}$. Hence, from $\lambda \neq 1$ it follows $\sigma(y_\lambda) \cap \Gamma = \emptyset$, so $r(y_\lambda) < 1$. This implies $r(x) < 1$. ■

3. The sequence of consecutive differences. This section is devoted to the study of the sequence of *consecutive differences* $\{x^n(x-1)\}_{n \in \mathbb{N}}$. Similar to the preceding section, we analyse the spectrum $\sigma(x)$ and poles of the resolvent in cases where $\{x^n(x-1)\}_{n \in \mathbb{N}}$ shows particular properties (e.g. boundedness, relative compactness, convergence). Furthermore, we give conditions in order to ensure a certain behaviour of this sequence, and derive a characterization of those $x \in \mathcal{A}$ for which $\{x^n(x-1)\}_{n \in \mathbb{N}}^-$ is compact. Here, as in the whole text, the closure is w.r.t. the norm topology of $\mathcal{A}$.

Certainly, we restrict ourselves to the non-trivial case where $1 \in \sigma(x)$. We will show that for $1 \in \sigma(x)$ being a simple pole of the resolvent these properties can be transferred from $\{x^n(x-1)\}_{n\in\mathbb{N}}$ to $\{x^n\}_{n\in\mathbb{N}}$ (Proposition 3.4, Corollary 3.19, Corollary 3.12).

The quantitative description of the convergence of the sequence $\{x^n(x-1)\}_{n\in\mathbb{N}}$ to zero is treated in detail in the book of O. Nevanlinna [16] ([1]). Moreover, applications of these results in order to solve linear equations by iterations are presented there.

PROPOSITION 3.1. *Let $x \in \mathcal{A}$ be such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is bounded. Then x satisfies $r(x) \le 1$.*

Proof. Choose $\lambda \in \sigma(x)$. Define $M = \sup_{n\in\mathbb{N}} \|x^n(x-1)\| < \infty$; then by the spectral mapping theorem [4, VII.4, Proposition 4.10], $M \ge |\lambda^n(\lambda-1)| = |\lambda^n||\lambda-1|$ for all $n \in \mathbb{N}$. If $\lambda \ne 1$, this implies boundedness of $\{|\lambda^n|\}_{n\in\mathbb{N}}$, so $|\lambda| \le 1$. From the definition of the spectral radius, it follows that $r(x) \le 1$. ■

So, elements with bounded sequence of consecutive differences are power dominated. For them we now investigate poles of the resolvent.

PROPOSITION 3.2. *Let $x \in \mathcal{A}$ be such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is bounded. If $\lambda \in \sigma(x) \cap \Gamma$ is a pole of the resolvent then* $\mathrm{ord}(\lambda) \le 2$, *and* $\mathrm{ord}(\lambda) = 1$, *if* $\lambda \ne 1$.

Proof. If $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is bounded, then so is

$$\frac{1}{n}\sum_{k=0}^{n-1} x^k(x-1) = \frac{x^n-1}{n}.$$

In particular, $\|x^n\| = o(n^2)$ as $n \to \infty$, so $\mathrm{ord}(\lambda) \le 2$ for every pole $\lambda \in \sigma(x) \cap \Gamma$ by [23, Satz 2].

Now let $\lambda \in \sigma(x) \cap \Gamma \setminus \{1\}$ be a pole of order $p \le 2$. For the spectral idempotent $p \in \mathcal{A}$ corresponding to λ, $\{x^n(x-1)p\}_{n\in\mathbb{N}}$ is bounded, and for $x_0 := xp$ we have boundedness of $\{x_0^n(x_0-1)\}_{n\in\mathbb{N}}$. But $1 \notin \sigma(x_0) \subset \{0,\lambda\}$, so $\{x_0^n\}_{n\in\mathbb{N}}$ is bounded. This implies that λ is a simple pole of the resolvent $r(\cdot, x_0)$, so $\mathrm{ord}(\lambda) = 1$. ■

The next result is similar to Proposition 2.5.

PROPOSITION 3.3. *Let $x \in \mathcal{A}$ satisfy $r(x) \le 1$; if there exists $p \in \mathbb{N}$ such that $\sigma(x) \cap \Gamma$ consists only of poles of order* $\mathrm{ord}(\lambda) \le p$, *for* $\lambda \in \sigma(x) \cap \Gamma \setminus \{1\}$, *and* $\mathrm{ord}(1) \le p+1$, *then $\left\{\frac{1}{n^{p-1}} x^n(x-1)\right\}_{n\in\mathbb{N}}$ is relatively compact.*

Proof. Let $p_1 \in \mathcal{A}$ be the spectral idempotent associated with 1, and $p_0 = 1 - p_1$. Then $\left\{\frac{1}{n^{p-1}} x^n(x-1)p_0\right\}_{n\in\mathbb{N}}$ is bounded by virtue of Proposition 2.5. Now consider $\{x^n(x-1)p_1\}_{n\in\mathbb{N}}$ for $n \ge p$:

$$\begin{aligned} x^n(x-1)p_1 &= ((x-1)+1)^n(x-1)p_1 \\ &= \sum_{k=0}^{n}\binom{n}{k}(x-1)^{k+1}p_1 = \sum_{k=0}^{p-1}\binom{n}{k}(x-1)^{k+1}p_1. \end{aligned}$$

([1]) *Editorial note*: See also the paper by O. Nevanlinna in this volume.

From

$$\lim_{n\to\infty}\frac{1}{n^{p-1}}\binom{n}{k}=\begin{cases}\dfrac{1}{(p-1)!} & \text{if } k=p-1,\\ 0 & \text{if } k<p-1,\end{cases}$$

we get the convergence

$$\lim_{n\to\infty}\frac{1}{n^{p-1}}x^n(x-1)p_1=\frac{1}{(p-1)!}(x-1)^p p_1,$$

and so in particular relative compactness. This completes the proof. ∎

Proposition 3.4. *Let $x\in\mathcal{A}$ be such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is bounded and $1\in\sigma(x)$ is a simple pole of the resolvent. Then x is power bounded.*

Proof. Denote by $p_0,p_1\in\mathcal{A}$ the spectral idempotents associated with the spectral sets $\sigma(x)\setminus\{1\}$ and $\{1\}$, respectively. Then

$$x^n(x-1)=x^n(x-1)p_0+x^n\underbrace{(x-1)p_1}_{=0}=x^n(x-1)p_0=(xp_0)^n(xp_0-1).$$

Since $\{x^n(x-1)p_0\}_{n\in\mathbb{N}}$ is bounded by assumption and $1\notin\sigma(xp_0)$, $\{x^np_0\}_{n\in\mathbb{N}}$ is bounded. So, from $x^n=x^np_0+x^np_1=x^np_0+p_1$, boundedness of $\{x^n\}_{n\in\mathbb{N}}$ follows. ∎

The next step is to determine when $\{x^n(x-1)\}_{n\in\mathbb{N}}$ converges and what can be said about its limit and the spectrum $\sigma(x)$. As a highlight we present the famous theorem of Y. Katznelson and L. Tzafriri and one of its generalizations due to G. R. Allan and T. J. Ransford.

Proposition 3.5. *Let $x\in\mathcal{A}$ be such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ converges as $n\to\infty$. Then $\sigma(x)\subset\mathbb{D}\cup\{1\}$.*

Proof. Since $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is bounded, Proposition 3.1 yields $r(x)\le 1$, so $\sigma(x)\subset\overline{\mathbb{D}}$. It remains to show that $\lambda\in\sigma(x)\cap\Gamma$ implies $\lambda=1$, which is a consequence of the spectral mapping theorem. ∎

Proposition 3.6. *Let $x\in\mathcal{A}$ satisfy $\{x^n(x-1)\}_{n\in\mathbb{N}}\to q$ as $n\to\infty$. Then $\{n^{-1}x^n\}_{n\in\mathbb{N}}\to q$ as $n\to\infty$. In particular, $\{x^n(x-1)\}_{n\in\mathbb{N}}\to 0$ implies $\|x^n\|=o(n)$ as $n\to\infty$.*

Proof. We have

$$\frac{1}{n}x^n=\frac{1}{n}((x^n-1)+1)=\frac{1}{n}\Big(\sum_{k=0}^{n-1}x^k(x-1)+1\Big),$$

and since $\{x^k(x-1)\}_{k\in\mathbb{N}}\to q$ $(k\to\infty)$, the means $\big\{\frac{1}{n}\sum_{k=0}^{n-1}x^k(x-1)\big\}_{n\in\mathbb{N}}$ tend to q as $n\to\infty$ as well. ∎

This statement cannot be improved to the statement that $\{x^n(x-1)\}_{n\in\mathbb{N}}\to 0$ implies $\|x^n\|=O(1)$, i.e. power boundedness of x, as the following example due to B. Nagy shows [2].

Example 3.7. Consider $H=\ell^2(\mathbb{N})$, the Hilbert space of all square-summable sequences; then $B(X)$ with $X=H\oplus H$ is a Banach algebra with unity. In H we choose

[2] *Editorial note*: See also the paper by A. Święch in this volume.

orthonormal bases $\{e_k\}_{k\in\mathbb{N}}, \{f_k\}_{k\in\mathbb{N}}$, and define for $e = \sum_{k=1}^{\infty} \xi_k e_k$ and $f = \sum_{k=1}^{\infty} \eta_k f_k$ the operator $T : X \to X$ by

$$T(e \oplus f) = \sum_{k=1}^{\infty} (\alpha_k \xi_k + \beta_k \eta_k) e_k \oplus f \tag{2}$$

and linear continuation. The sequences $\{\alpha_k\}_{k\in\mathbb{N}}, \{\beta_k\}_{k\in\mathbb{N}} \in \ell^{\infty}(\mathbb{N})$ will be suitably chosen later.

First we prove $T \in B(X)$: From

$$\sum_{k=1}^{\infty} |\alpha_k\xi_k + \beta_k\eta_k|^2 \le 2\sum_{k=1}^{\infty} |\alpha_k\xi_k|^2 + |\beta_k\eta_k|^2$$

we get with $S := \max\{\|\{\alpha_k\}_{k\in\mathbb{N}}\|_{\infty}^2, \|\{\beta_k\}_{k\in\mathbb{N}}\|_{\infty}^2\}$:

$$\begin{aligned}\|T(e \oplus f)\|^2 &= \sum_{k=1}^{\infty} |\alpha_k\xi_k + \beta_k\eta_k|^2 + \|f\|^2 \\ &\le 2\Big(\sup_{k\in\mathbb{N}} \max\{|\alpha_k|, |\beta_k|\} \cdot \sum_{k=1}^{\infty} |\xi_k|^2 + |\eta_k|^2\Big) + \|f\|^2 \\ &\le (2S+1)(\|e\|^2 + \|f\|^2) = (2S+1)\|e \oplus f\|^2.\end{aligned}$$

Thus T is bounded.

To show that T is not power bounded, it is sufficient to prove unboundedness of $\{\|T^n(e_k \oplus f_k)\|\}_{n\in\mathbb{N}}$ for some choice of $k \in \mathbb{N}$. Routine algebra yields

$$T^n(e_k \oplus f_k) = \left(\alpha_k^n + \frac{1-\alpha_k^n}{1-\alpha_k}\beta_k\right) e_k \oplus f_k.$$

Now we choose $\{\alpha_k\}_{k\in\mathbb{N}}$ with $0 < \alpha_k < 1$ for all $k \in \mathbb{N}$, $\alpha_k \nearrow 1$ as $k \to \infty$, and define

$$\beta_k = \sqrt{(1-\alpha_k)}; \tag{3}$$

so $0 < \beta_k < 1$ for all $k \in \mathbb{N}$ and $\{\beta_k\}_{k\in\mathbb{N}} \to 0$ as $k \to \infty$ (e.g. define $\alpha_k = 1 - k^{-2}$ and $\beta_k = k^{-1}$).

From this particular choice we conclude: for all $N \in \mathbb{N}$ there exists $k_0 \in \mathbb{N}$ such that $\frac{\beta_{k_0}}{1-\alpha_{k_0}} > 2N$. Since $0 < \alpha_k < 1$ and

$$\|T^n(e_{k_0} \oplus f_{k_0})\| = \left\|\left(\alpha_{k_0}^n + \frac{1-\alpha_{k_0}^n}{1-\alpha_{k_0}}\beta_{k_0}\right) e_k \oplus f_k\right\|,$$

there is $n \in \mathbb{N}, n > N$, such that

$$\|T^n(e_{k_0} \oplus f_{k_0})\| > \frac{\beta_{k_0}}{2(1-\alpha_{k_0})} > N.$$

Therefore $\{\|T^n(e_{k_0} \oplus f_{k_0})\|\}_{n\in\mathbb{N}} \to \infty$ as $n \to \infty$, which means that T is not power bounded.

To calculate $T^n(T-I)(e \oplus f)$ we note that

$$(T-I)(e \oplus f) = \sum_{k=1}^{\infty} [(\alpha_k - 1)\xi_k - \beta_k\eta_k] e_k \oplus 0;$$

consequently,

$$T^n(T-I)(e\oplus f)=\sum_{k=1}^{\infty}\alpha_k^n[(\alpha_k-1)\xi_k-\beta_k\eta_k]e_k.$$

Taking norms on both sides, we can estimate

$$\|T^n(T-I)(e\oplus f)\|^2\le 2\sum_{k=1}^{\infty}\alpha_k^{2n}[(1-\alpha_k)^2|\xi_k|^2+\beta_k^2|\eta_k|^2].$$

From (3) it follows that for all $\varepsilon>0$ there exists $N(\varepsilon)\in\mathbb{N}$ such that $\max\{1-\alpha_k,\beta_k\}<\varepsilon$ for all $k>N(\varepsilon)$. This implies

$$\sum_{k>N(\varepsilon)}\alpha_k^{2n}[(1-\alpha_k)^2|\xi_k|^2+\beta_k^2|\eta_k|^2]\le\varepsilon^2\sum_{k>N(\varepsilon)}|\xi_k|^2+|\eta_k|^2. \tag{4}$$

Since $0<\alpha_k<1$ for all $k\in\mathbb{N}$, there is $n_0(\varepsilon)\in\mathbb{N}$ such that $\alpha_{N(\varepsilon)}^n<\varepsilon$ for all $n>n_0(\varepsilon)$. From monotonicity of $\{\alpha_k\}_{k\in\mathbb{N}}$ we get $\alpha_k^{2n}<\varepsilon^2$ for all $k\in\{1,\dots,N(\varepsilon)\}$ and $n>n_0(\varepsilon)$. So for $n>n_0(\varepsilon)$,

$$\sum_{k=1}^{N(\varepsilon)}\alpha_k^{2n}[(1-\alpha_k)^2|\xi_k|^2+\beta_k^2|\eta_k|^2]\le\varepsilon^2\sum_{k=1}^{N(\varepsilon)}|\xi_k|^2+|\eta_k|^2. \tag{5}$$

Summing up (4) and (5), we obtain

$$\|T^n(T-I)(e\oplus f)\|^2\le 2\varepsilon^2\sum_{k=1}^{\infty}|\xi_k|^2+|\eta_k|^2=2\varepsilon^2\|e\oplus f\|^2$$

for $n>n_0(\varepsilon)$. This implies $\lim_{n\to\infty}\|T^n(T-I)\|=0$.

It would be interesting to know whether T has unbounded powers in the Calkin algebra, too.

Theorem 2.8 lists properties of the limit of $\{x^n\}_{n\in\mathbb{N}}$ in case it exists; it is an x-invariant idempotent that commutes with x. What can be said about $q:=\lim_{n\to\infty}\{x^n(x-1)\}_{n\in\mathbb{N}}$?

PROPOSITION 3.8. *Let $x\in\mathcal{A}$ be such that $q:=\lim_{n\to\infty}\{x^n(x-1)\}_{n\in\mathbb{N}}$ exists. Then $xq=qx=q$ and $q^2=0$, so q is nilpotent.*

Proof. The equality $xq=qx=q$ is obvious from the definition of q. From this we get $q^2=(\lim_{n\to\infty}x^n(x-1))^2=\lim_{n\to\infty}x^{2n}(x-1)(x-1)=q(x-1)=0$. ■

It is easy to see that in general $q\ne 0$. But in the special case of power bounded elements, the only possible limit of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is zero. This comes from the next corollary, which is a counterpart to Proposition 3.6.

COROLLARY 3.9. *Let $x\in\mathcal{A}$ satisfy $\{x^n(x-1)\}_{n\in\mathbb{N}}\to q$ and $\|x^n\|=o(n)$ as $n\to\infty$. Then $q=0$.*

A proof of this corollary for arbitrary scalar valued sequences, i.e. $\{x^n\}_{n\in\mathbb{N}}\subset\mathcal{A}$ and $\{x^n(x-1)\}_{n\in\mathbb{N}}\subset\mathcal{A}$ replaced by $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{F}$ and $\{x_{n+1}-x_n\}_{n\in\mathbb{N}}\subset\mathbb{F}$, respectively, where $\mathbb{F}$ is the field of real or complex numbers, can be found in [24, 1.7, Example 11]. For power bounded elements the statement is contained in Theorem 3.13.

Remark. To see that $\|x^n\| = O(n)$ is not sufficient in this corollary, consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}(2, \mathbb{C})$ with $A^n(A-E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for all $n \in \mathbb{N}$.

Now we investigate the structure of $\sigma(x) \cap \Gamma$ for elements $x \in \mathcal{A}$ with convergent differences $\{x^n(x-1)\}_{n\in\mathbb{N}}$:

PROPOSITION 3.10. *Let $x \in A$ satisfy $r(x) = 1$, $\sigma(x) \cap \Gamma = \{1\}$, and let 1 be a pole of the resolvent of order p. Then $\{x^n(x-1)\}_{n\in\mathbb{N}}$ converges as $n \to \infty$ if and only if $p \le 2$. Moreover, $\{x^n(x-1)\}_{n\in\mathbb{N}} \to 0$ is equivalent to $p = 1$. Under these hypotheses, $\{x^n(x-1)\}_{n\in\mathbb{N}} \to (x-1)p_1$, where p_1 denotes the spectral idempotent associated with 1.*

Proof. If $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is convergent—and therefore bounded—$p \le 2$ follows from Proposition 3.2. If on the other hand $p \le 2$ is satisfied, then from $x^n(x-1) = \sum_{k=0}^{n-1} x^k(x-1)^2 + (x-1)$ we get

$$x^n(x-1)p_1 = \sum_{k=0}^{n-1} x^k \underbrace{(x-1)^2 p_1}_{=0} + (x-1)p_1 = (x-1)p_1,$$

and so with $p_0 = 1 - p_1$,

$$\lim_{n\to\infty} x^n(x-1) = \lim_{n\to\infty} (x^n(x-1)p_1 + \underbrace{x^n(x-1)p_0}_{\to 0}) = (x-1)p_1.$$

From this equation we conclude that $p = 1$ is satisfied if and only if $\{x^n(x-1)\}_{n\in\mathbb{N}}$ tends to zero as $n \to \infty$. ∎

As in Theorem 2.8, convergence and relative compactness of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ are closely related whenever the peripheral spectrum $\sigma(x) \cap \Gamma$ is small enough. To be more precise:

COROLLARY 3.11. *Let $x \in \mathcal{A}$ be such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact and $1 \in \sigma(x)$ is a pole of the resolvent. Then $\{x^n(x-1)\}_{n\in\mathbb{N}}$ converges as $n \to \infty$ if and only if $\sigma(x) \cap \Gamma = \{1\}$.*

Another consequence is the following:

COROLLARY 3.12. *For $x \in \mathcal{A}$ with $\{x^n(x-1)\}_{n\in\mathbb{N}} \to 0$ the following are equivalent:*

(i) *$1 \in \rho(x)$ or $1 \in \sigma(x)$ is a (simple) pole,*
(ii) *$\{x^n\}_{n\in\mathbb{N}}$ converges as $n \to \infty$.*

The next result is concerned with the question under what conditions convergence of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ can be guaranteed if $1 \in \sigma(x)$ is *not* a pole of the resolvent. The answer—for power bounded elements—was given by Y. Katznelson and L. Tzafriri [11, Theorem 1]:

THEOREM 3.13. *Let $x \in \mathcal{A}$ be power bounded. Then $\{x^n(x-1)\}_{n\in\mathbb{N}} \to 0$ as $n \to \infty$ if and only if $\sigma(x) \cap \Gamma \subset \{1\}$.*

A proof different from the original one can be found in [22]. The result was proven already by J. Esterle [5, Theorem 9.1] in the case where $\sigma(x) = \{1\}$. As well, Theorem 3.13 follows from a similar result for power dominated elements [1, Theorem 2.2]:

THEOREM 3.14. *Let $x \in \mathcal{A}$ be power dominated by $\{\mu_n\}_{n\in\mathbb{N}}$. Then $\|x^n(x-1)\| = o(\mu_n)$, i.e. $\left\{\frac{x^n(x-1)}{\mu_n}\right\}_{n\in\mathbb{N}} \to 0$ as $n \to \infty$, if $\sigma(x) \cap \Gamma \subset \{1\}$.*

We turn to the investigation of spectral properties of x when $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact. We start with a preparatory lemma.

LEMMA 3.15. *Let $x \in \mathcal{A}$ be such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact. Then $(\sigma(x)\cap\mathbb{D})^- \cap \Gamma \subset \{1\}$.*

Proof. Suppose, contrary to our claim, that $\sigma(x)\cap\mathbb{D} \supset \{\lambda_j\}_{j\in\mathbb{N}} \to \lambda \in \Gamma\setminus\{1\}$ as $j\to\infty$. Choose any subsequence $\{x^{n_k}(x-1)\}_{k\in\mathbb{N}}$ of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ and fix $\varepsilon = |\lambda-1|/3 > 0$. Let $k_0\in\mathbb{N}$ be arbitrary. From the assumption we conclude the existence of certain constants $n_0\in\mathbb{N}$, $l_0\ge k_0$ such that

$$|\lambda-\lambda_n|<\varepsilon \quad \text{for all } n\ge n_0, \quad |\lambda_{n_0}^{n_{k_0}}| > \frac{3}{4}, \quad |\lambda_{n_0}^{n_l}| < \frac{1}{4} \quad \text{for all } l\ge l_0.$$

Then for $l\ge l_0$ we have

$$(\lambda_{n_0}^{n_{k_0}} - \lambda_{n_0}^{n_l})(\lambda_{n_0}-1) \in \sigma\Big((x^{n_{k_0}} - x^{n_l})(x-1)\Big),$$

so by the spectral mapping theorem

$$\|(x^{n_{k_0}} - x^{n_l})(x-1)\| \ge |\lambda_{n_0}^{n_{k_0}} - \lambda_{n_0}^{n_l}||\lambda_{n_0}-1| > \frac{1}{2}|\lambda_{n_0}-1| \ge \frac{1}{2}\Big(|\lambda-1| - |\lambda-\lambda_{n_0}|\Big) > \varepsilon.$$

This clearly contradicts the relative compactness of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ since k_0 was arbitrary. ∎

The first result to uncover the spectral structure of x is the following.

PROPOSITION 3.16. *Let $x\in\mathcal{A}$ be such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact and $1\notin(\sigma(x)\cap\Gamma\setminus\{1\})^-$. Then $\sigma(x)\cap\Gamma\setminus\{1\}$ is a finite set all elements of which are simple poles of the resolvent.*

Proof. By assumption and Lemma 3.15, $\sigma(x) = \sigma_0 \,\dot\cup\, \sigma_1$, where $\sigma_0\cap\Gamma\subset\{1\}$ and $\sigma_1\subset\Gamma\setminus\{1\}$ are spectral sets. Let $p_0,p_1\in\mathcal{A}$ denote the associated spectral idempotents of σ_0,σ_1, respectively. Then for $x_1 = xp_1$, $\{x_1^n(x_1-1)\}_{n\in\mathbb{N}}$ is relatively compact, and since $1\notin\sigma(x_1) = \sigma(xp_1)\subset\sigma_1\cup\{0\}$, $(1-x_1)^{-1}$ exists. This implies relative compactness of $\{x_1^n\}_{n\in\mathbb{N}}$, and application of Theorem 2.7 yields the conclusion. ∎

Remark. If the condition $1\notin(\sigma(x)\cap\Gamma\setminus\{1\})^-$ in Proposition 3.16 is not fulfilled, then at least all $\lambda\in\sigma(x)\cap\Gamma$ which are not in the same component of the spectrum as 1 are simple poles of the resolvent.

QUESTION. Does relative compactness of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ already imply that $1\notin(\sigma(x)\cap\Gamma\setminus\{1\})^-$? (3) If it does, Propositions 3.16 and 3.17 give a full characterization of relative compactness of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ for power bounded elements.

One should compare this with a result recently obtained by S. Huang ([8, Corollary 4.1]).

Now it is time to give sufficient conditions on x for the relative compactness of $\{x^n(x-1)\}_{n\in\mathbb{N}}$.

PROPOSITION 3.17. *Let $x\in\mathcal{A}$ be power bounded with $\sigma(x)\cap\Gamma\setminus\{1\}$ being a finite set of poles. Then $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact.*

(3) *Editorial note*: See also the paper by A. Święch in this volume.

Proof. Denote by $p_0, p_1, \ldots, p_m \in \mathcal{A}$ the spectral idempotents associated with the spectral sets $\sigma(x) \setminus \{\lambda_1, \ldots, \lambda_m\}, \{\lambda_1\}, \ldots, \{\lambda_m\}$, respectively, where $\sigma(x) \cap \Gamma \setminus \{1\} = \{\lambda_1, \ldots, \lambda_m\}$. Then $\{x^n(x-1)p_0\}_{n\in\mathbb{N}} \to 0$ as $n \to \infty$ by virtue of Theorem 3.13. Since $\lambda_1, \ldots, \lambda_m$ are simple poles, by Proposition 2.6,

$$x^n(x-1) = x^n(x-1)p_0 + \sum_{k=1}^{m} \lambda_k^n(\lambda_k - 1)p_k,$$

and the second summand is a sum of elements of relatively compact sequences. So, the whole sequence is relatively compact, and the proof is complete. ∎

If x is not power bounded we have to assume some more:

PROPOSITION 3.18. *Let $x \in \mathcal{A}$ be such that $r(x) \le 1$, $\sigma(x) \cap \Gamma \setminus \{1\}$ consists of simple poles, and $1 \in \rho(x)$ or $1 \in \sigma(x)$ is a pole of order at most 2 of the resolvent. Then $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact.*

Proof. With the notation as in the preceding proposition, convergence of $\{x^n(x-1)p_0\}_{n\in\mathbb{N}}$ as $n \to \infty$ follows from Proposition 3.10. The rest of the proof now runs as in the previous proof since the sum of a convergent and a relatively compact sequence is again relatively compact. ∎

The following corollary summarizes the results given in the preceding propositions.

COROLLARY 3.19. *Let $x \in \mathcal{A}$ satisfy $r(x) \le 1$ and let $1 \in \sigma(x)$ be a pole of order p. Then $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact if and only if $p \le 2$ and every $\lambda \in \sigma(x) \cap \Gamma \setminus \{1\}$ is a simple pole of the resolvent. Furthermore, $p = 1$ is equivalent to the relative compactness of $\{x^n\}_{n\in\mathbb{N}}$.*

We will finish this section showing the rotation non-invariance of the relative compactness of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ except for some "trivial" cases.

THEOREM 3.20 (non-rotation principle). *Let $x \in \mathcal{A}$ be such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact and suppose that $1 \notin (\sigma(x) \cap \Gamma \setminus \{1\})^-$. Then for $y_\lambda = \lambda x$ the following are equivalent:*

(i) *$\{x^n\}_{n\in\mathbb{N}}$ is relatively compact,*
(ii) *$\{y_\lambda^n\}_{n\in\mathbb{N}}$ is relatively compact for all $\lambda \in \Gamma$,*
(iii) *$\{y_\lambda^n\}_{n\in\mathbb{N}}$ is relatively compact for some $\lambda \in \Gamma$,*
(iv) *$\{y_\lambda^n(y_\lambda - 1)\}_{n\in\mathbb{N}}$ is relatively compact for all $\lambda \in \Gamma$,*
(v) *$\{y_\lambda^n(y_\lambda - 1)\}_{n\in\mathbb{N}}$ is relatively compact for some $\lambda \in \Gamma \setminus \{1\}$.*

Proof. It is sufficient to prove the implications (iii)⇒(i) and (v)⇒(i). We fix a suitable $\lambda \in \Gamma$. Since λ and its inverse $\overline{\lambda}$ are of modulus 1 and $\{y^n\}_{n\in\mathbb{N}}$ is relatively compact, the product $\{(\overline{\lambda}y)^n\}_{n\in\mathbb{N}}$ is relatively compact, too. But this is $\{x^n\}_{n\in\mathbb{N}}$.

For (v)⇒(i) we fix a suitable $\lambda \in \Gamma \setminus \{1\}$, and write y for y_λ. Then $\{x^n(x-1)\}_{n\in\mathbb{N}}$ and $\{y^n(y-1)\}_{n\in\mathbb{N}}$ are relatively compact. From Proposition 3.16 we derive the following unique decomposition of the spectrum $\sigma(x)$:

$$\sigma(x) = \sigma_0 \,\dot{\cup}\, \sigma_1, \quad \text{where } \sigma_0 \subset \mathbb{D} \cup \{1\},\ \sigma_1 \subset \Gamma \setminus \{1\}.$$

Since $1 \notin (\sigma(x) \cap \Gamma \setminus \{1\})^-$, σ_0, σ_1 are complementary (possibly empty) spectral sets, and any element of the finite set σ_1 is a simple pole of the resolvent $r(\cdot, x)$. From the spectral mapping theorem we get for $\sigma(y) = \sigma(\lambda x)$,

$$\sigma(y) = \widetilde{\sigma}_0 \mathbin{\dot\cup} \widetilde{\sigma}_1, \quad \text{where } \widetilde{\sigma}_0 \subset \mathbb{D} \cup \{\lambda\},\ \widetilde{\sigma}_1 \subset \Gamma \setminus \{\lambda\}.$$

Since $\{y^n(y-1)\}_{n\in\mathbb{N}}$ is relatively compact and $1 \notin (\sigma(y) \cap \Gamma \setminus \{1\})^-$ (since σ_1 is finite), we get another decomposition of $\sigma(y)$, namely

$$\sigma(y) = \sigma_0' \mathbin{\dot\cup} \sigma_1', \quad \text{where } \sigma_0' \subset \mathbb{D} \cup \{1\},\ \sigma_1' \subset \Gamma \setminus \{1\}.$$

Both decompositions must coincide, so $\sigma_1' \subset \Gamma \setminus \{1, \lambda\}$, i.e. $1 = \overline{\lambda}\lambda \notin \sigma(\overline{\lambda} y) = \sigma(x)$. Since $\{x^n(x-1)\}_{n\in\mathbb{N}}$ is relatively compact and $1 \in \rho(x)$, $\{x^n\}_{n\in\mathbb{N}}$ is relatively compact. ∎

Question. Do relative compactness of $\{x^n(x-1)\}_{n\in\mathbb{N}}$ and $\sigma(x) \cap \Gamma = \{1\}$ imply convergence of $\{x^n(x-1)\}_{n\in\mathbb{N}}$? If $1 \in \sigma(x)$ is a pole of the resolvent or x is power bounded then the answer is affirmative.

4. The Cesàro means.

For $x \in \mathcal{A}$, $p \in \mathbb{N}$, and $\lambda \in \Gamma$, we define the *Cesàro means* $c_{n,p}^{(\lambda)} = \frac{1}{n^p} \sum_{k=0}^{n-1} \left(\frac{x}{\lambda}\right)^k$; for sake of simplicity we let drop the indices (λ) or p where they equal one.

Proposition 4.1. *Let $x \in \mathcal{A}$ be such that $\{c_{n,p}^{(\lambda)}\}_{n\in\mathbb{N}}$ is bounded. Then $r(x) \le 1$.*

Proof. Assume $r(x) > 1$. Then there exists $\lambda_0 \in \sigma(x)$ with $|\lambda_0| > 1$. So, by the spectral mapping theorem

$$\|c_{n,p}^{(\lambda)}\| \ge \left| \frac{1}{n^p} \sum_{k=0}^{n-1} \left(\frac{\lambda_0}{\lambda}\right)^k \right| = \frac{1}{n^p} \left| \frac{1 - (\lambda_0/\lambda)^n}{1 - \lambda_0/\lambda} \right|,$$

which contradicts the boundedness of $\{c_{n,p}^{(\lambda)}\}_{n\in\mathbb{N}}$ since $|\lambda_0/\lambda| > 1$. ∎

The next step is the investigation of isolated points in $\sigma(x) \cap \Gamma$ when $\{c_{n,p}^{(\lambda)}\}_{n\in\mathbb{N}}$ converges.

Theorem 4.2. *Let $x \in \mathcal{A}$ satisfy $r(x) \le 1$ and suppose $p^{(\lambda)} = \lim_{n\to\infty} \{c_n^{(\lambda)}\}_{n\in\mathbb{N}}$ exists for some $\lambda \in \Gamma$. Then λ is at most a simple pole of the resolvent and $p^{(\lambda)}$ coincides with p_λ, the spectral idempotent associated with λ.*

A proof of this theorem can be found in [17, Theorem 2].

Remark. Theorem 4.2 does not extend to the case $p > 1$ (cf. [23]). But for $\mathcal{A} = B(X)$, L. Burlando [3] has proven extensions of this theorem assuming some properties of the range $R(\lambda - T)$ for $T \in B(X)$ ([4]).

The next proposition, which is due to H.-D. Wacker [23, Satz 6], answers which elements actually possess convergent Cesàro means $\{c_{n,p}^{(\lambda)}\}_{n\in\mathbb{N}}$.

([4]) *Editorial note*: See also L. Burlando, *A generalization of the uniform ergodic theorem to poles of arbitrary order*, Studia Math. 122 (1997), 75–98.

PROPOSITION 4.3. *Let $x \in \mathcal{A}$ satisfy $r(x) \leq 1$ and suppose that there exists $p \in \mathbb{N}$ such that $\sigma(x) \cap \Gamma$ is a set of poles of the resolvent of order not exceeding p. Then*

$$c_{n,p}^{(\lambda)} = \frac{1}{n^p} \sum_{k=0}^{n-1} \left(\frac{x}{\lambda}\right)^k \to \begin{cases} 0 & \text{if } 0 \leq \operatorname{ord}(\lambda) < p \\ \dfrac{1}{p!}\left(\dfrac{x}{\lambda} - 1\right)^{p-1} p_\lambda & \text{if } \operatorname{ord}(\lambda) = p \end{cases} \quad \text{as } n \to \infty.$$

Here, p_λ denotes the associated spectral idempotent of λ.

Since the assumptions of this proposition coincide with those of Proposition 2.5, one might ask what relation exists between the power boundedness of $x \in \mathcal{A}$ and the convergence of its Cesàro means $\{c_n^{(\lambda)}\}_{n \in \mathbb{N}}$. For $\mathcal{A} = \mathcal{M}(m, \mathbb{C})$ both conditions are equivalent ([19]); this is no longer true for arbitrary operators in an infinite-dimensional Banach space. For Riesz operators we will deal with this question in Section 6.

A further conclusion, which can be found in terms of operators in [14], [15, Theorem 1], and [23, Satz 4], is the following.

COROLLARY 4.4. *For $x \in \mathcal{A}$ the following are equivalent*:

(i) $\left\{\frac{1}{n}\sum_{k=0}^{n-1} x^k\right\}_{n \in \mathbb{N}} \to p$ *as* $n \to \infty$,
(ii) $\|x^n\| = o(n)$ *as* $n \to \infty$ *and* $1 \in \rho(x)$ *or* $1 \in \sigma(x)$ *is a pole of the resolvent.*

If these equivalent conditions are fulfilled then $1 \in \sigma(x)$ is a simple pole and p coincides with the associated spectral idempotent.

Now we are able to get a first result that combines different sequences and properties:

COROLLARY 4.5. *Let $x \in \mathcal{A}$ satisfy $r(x) \leq 1$. Then $\{x^n\}_{n \in \mathbb{N}}$ is relatively compact if and only if $\{c_n^{(\lambda)}\}_{n \in \mathbb{N}}$ converges for all $\lambda \in \sigma(x) \cap \Gamma$ as $n \to \infty$.*

Proof. Proposition 4.3, Theorem 2.7. ∎ (5)

5. Further results. With the results of the preceding sections we can easily conclude the following equivalences.

COROLLARY 5.1. *For $x \in \mathcal{A}$ the following are equivalent*:

(i) $\{x^n\}_{n \in \mathbb{N}}$ *is relatively compact and* $\{x^n(x-1)\}_{n \in \mathbb{N}} \to 0$ *as* $n \to \infty$,
(ii) $\left\{\frac{1}{n}\sum_{k=0}^{n-1} x^k\right\}_{n \in \mathbb{N}} \to p \in \mathcal{A}$ *and* $\{x^n(x-1)\}_{n \in \mathbb{N}} \to 0$ *as* $n \to \infty$,
(iii) $\{x^n\}_{n \in \mathbb{N}} \to p$ *as* $n \to \infty$.

Proof. Corollary 4.5; Proposition 3.5, Corollary 4.4, Theorem 2.8. ∎

COROLLARY 5.2. *For $x \in \mathcal{A}$ the following are equivalent*:

(i) $\{x^n(x-1)\}_{n \in \mathbb{N}}$ *is relatively compact and* $\lim_{n\to\infty} \left\{\frac{1}{n}\sum_{k=0}^{n-1} x^k\right\}_{n \in \mathbb{N}}$ *exists*,
(ii) $\{x^n\}_{n \in \mathbb{N}}$ *is relatively compact.*

Proof. Corollary 4.4, Proposition 3.16; Theorem 2.7. ∎

(5) *Editorial note*: See also Yu. Lyubich and J. Zemánek, *Precompactness in the uniform ergodic theory*, Studia Math. 112 (1994), 89–97.

To state the next proposition, one more definition is needed.

DEFINITION 5.3. An element $x \in \mathcal{A}$ is said to be *doubly power bounded* if it is invertible and $\sup_{n\in\mathbb{Z}} \|x^n\| < \infty$.

The following lemma is a simple consequence of the spectral mapping theorem.

LEMMA 5.4. *If $x \in \mathcal{A}$ is doubly power bounded then $\sigma(x) \subset \Gamma$.*

Now we can state:

PROPOSITION 5.5. *Let $x \in \mathcal{A}$ satisfy $\sigma(x) \subset \Gamma$. Suppose that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ and $\big\{\frac{1}{n}\sum_{k=0}^{n-1} x^k\big\}_{n\in\mathbb{N}}$ converge as $n \to \infty$. Then $x = 1$.*

Proof. From Proposition 3.5 and Corollary 4.4 it follows that $\sigma(x) = \{1\}$, and 1 is a simple pole of the resolvent. Moreover, $p_1 = 1$, so $0 = (x-1)p_1 = x - 1$. ∎

In particular, double power boundedness of x and convergence of $\{x^n\}_{n\in\mathbb{N}}$ imply $x = 1$. The same result but under weaker assumptions was obtained by I. Gelfand in [6]:

THEOREM 5.6. *Let $x \in \mathcal{A}$ be doubly power bounded with $\sigma(x) = \{1\}$. Then $x = 1$.*

Remark. There does not exist any doubly power bounded element $x \in \mathcal{A}$ such that $\{x^n(x-1)\}_{n\in\mathbb{N}}$ and $\big\{\frac{1}{n}\sum_{k=0}^{n-1} x^k\big\}_{n\in\mathbb{N}}$ converge to zero simultaneously.

6. Riesz operators. We will apply the results of the preceding sections to the class of Riesz operators. On a complex Banach space X, an operator $T \in B(X)$ is called a *Riesz operator* if for all $\lambda \in \mathbb{C} \setminus \{0\}$, $\dim N(\lambda - A) = \operatorname{codim} R(\lambda - A)$ and $\alpha(\lambda - A) = \delta(\lambda - A) < \infty$. Here, $N(\lambda - A)$, $R(\lambda - A)$, $\alpha(\lambda - A)$, and $\delta(\lambda - A)$ denote the *nullspace*, the *range*, the *ascent*, and the *descent* of $\lambda - T$, respectively.

For such an operator T, every $\lambda \in \mathbb{C} \setminus \{0\}$ is either a regular value, so $\lambda \in \rho(T)$, or $\lambda \in \sigma(T)$ is a pole of the resolvent and the associated spectral projection $P_\lambda \in B(X)$ is finite-dimensional ([7, Satz 105.2]). In this case, $\operatorname{ord}(\lambda)$, the order of the pole, equals $\alpha(\lambda - A)$ and $\delta(\lambda - A)$ ([21, V.10, Theorem 10.1]).

THEOREM 6.1. *For a Riesz operator $T \in B(X)$ with $\sigma(T) \cap \Gamma \subset \{1\}$ the following are equivalent*:

1) $r(T) \le 1$ *and* $\operatorname{ord}(1) \le 2$,
2) $\{T^n(T-I)\}_{n\in\mathbb{N}}$ *is bounded*,
3) $\{T^n(T-I)\}_{n\in\mathbb{N}}$ *is relatively compact*,
4) $\{T^n(T-I)\}_{n\in\mathbb{N}} \to (T-I)P$ *as* $n \to \infty$,
5) $\big\{\frac{T^n}{n}\big\}_{n\in\mathbb{N}} \to (T-I)P$ *as* $n \to \infty$,
6) $\big\{\frac{T^n}{n^2}\big\}_{n\in\mathbb{N}} \to 0$ *as* $n \to \infty$,
7) $\big\{\frac{1}{n^2}\sum_{k=0}^{n-1} T^k\big\}_{n\in\mathbb{N}} \to \frac{1}{2}(T-I)P$ *as* $n \to \infty$,
8) $\|T^n\| = O(n)$ *as* $n \to \infty$.

Furthermore, the condition

4′) $\{T^n(T-I)\}_{n\in\mathbb{N}} \to 0$ *as* $n\to\infty$

is equivalent to each of the following:

9) $r(T)\le 1$ *and* $\mathrm{ord}(1)\le 1$,
10) $T = T_0 + P$, *where* $T_0, P \in B(X)$ *with* $r(T_0) < 1$, $P^2 = P$, $T_0P = PT_0 = 0$,
11) $\{T^n\}_{n\in\mathbb{N}} \to P$ *as* $n\to\infty$,
12) $\{T^n\}_{n\in\mathbb{N}}$ *is relatively compact*,
13) T *is power bounded*,
14) $\{\frac{1}{n}\sum_{k=0}^{n-1} T^k\}_{n\in\mathbb{N}} \to P$ *as* $n\to\infty$,
15) $\|T^n\| = o(n)$ *as* $n\to\infty$,
16) $r(T)\le 1$ *and* $\alpha(T-I)\le 1$,
17) $r(T)\le 1$ *and* $\delta(T-I)\le 1$,
18) $r(T)\le 1$ *and* $X = \mathrm{Ker}(T-I)\oplus R(T-I)$.

P r o o f. We first prove 1)⇒ ... ⇒7)⇒1), then 5)⇒8)⇒1) and finish with 4′)⇒9)⇒ ... ⇒ 18)⇒4′).

1)⇒2): Proposition 3.3;
2)⇒3): Proposition 3.18;
3)⇒4): Proposition 3.2, 3.10;
4)⇒5): Proposition 3.6;
5)⇒6): trivial;
6)⇒7): [3, Theorem 3];
7)⇒1): [3, Theorem 3]
5)⇒8): trivial;
8)⇒1): [1, Proposition 2.1], [23, Satz 2];
4′)⇒9): Proposition 3.10;
9)⇒10): [12, Theorem 1];
10)⇒11): Theorem 2.2;
11)⇒12): trivial;
12)⇒13): trivial;
13)⇒14): Proposition 4.3;
14)⇒15): trivial;
15)⇒16): [26, Theorem 7];
16)⇒17): trivial;
17)⇒18): [21, V.6, Theorem 6.2];
18)⇒4′): [7, Satz 72.4], Proposition 3.10. ∎

R e m a r k. If we do not assume $\sigma(T)\cap\Gamma\subset\{1\}$ the conditions 4′), 10), and 11) are equivalent and each of them implies the conditions 1)∼18) and $\sigma(T)\cap\Gamma\subset\{1\}$.

Some more equivalences, also dealing with properties w.r.t. the strong topology, can be found in [26, Theorem 7]. There, also a growth condition on the resolvent $(T-\lambda I)^{-1}$

for $|\lambda| > 1$ equivalent to all the statements 4′) and 9)∼18) is given. This generalizes a result of H.-O. Kreiss for the finite-dimensional case ([13, Satz 4.1]) ([6]).

Together with Proposition 5.5 it follows from the preceding theorem that for a Riesz operator T with $\sigma(T) = \{1\}$ the statement of Gelfand's Theorem 5.6 holds already if T is assumed to be only power bounded.

7. Matrices. For all positive integers p, $\mathcal{M}(p, \mathbb{C})$ is a (finite-dimensional) Banach algebra with unity. So, it is natural to investigate elements of this algebra from the point of view of properties introduced in the preceding sections and to characterize matrices of particular behaviour by the structure of their Jordan canonical form. Since transformation into this form is nothing but a similarity transformation it leaves invariant all the relevant properties of A; so the behaviour of $\{A^n\}_{n\in\mathbb{N}}$, $\{A^n(A-E)\}_{n\in\mathbb{N}}$, and $\{\frac{1}{n}\sum_{k=0}^{n-1} A^k\}_{n\in\mathbb{N}}$ for $A \in \mathcal{M}(p, \mathbb{C})$ is uniquely determined by the Jordan canonical form of A.

Table 1

Property (as $n \to \infty$)	Necessary and sufficient conditions
$\{A^n\}_{n\in\mathbb{N}}$:	
• $\|A^n\| = O(n)$	$\delta_j \cdot \delta_{j+1} = 0$ for $j = 1, \ldots, l-2$, $\varepsilon_j \cdot \varepsilon_{j+1} = 0$ for $j = 1, \ldots, m-2$ (*)
• $\{n^{-1}A^n\}$ convergent	$\delta_1 = \ldots = \delta_{l-1} = 0$, $\varepsilon_j \cdot \varepsilon_{j+1} = 0$ for $j = 1, \ldots, m-2$ (*)
• $\|A^n\| = o(n)$	$\delta_1 = \ldots = \delta_{l-1} = \varepsilon_1 = \ldots = \varepsilon_{m-1} = 0$ (**)
• bounded (***)	$\delta_1 = \ldots = \delta_{l-1} = \varepsilon_1 = \ldots = \varepsilon_{m-1} = 0$
• convergent	$l = 0, \quad \varepsilon_1 = \ldots = \varepsilon_{m-1} = 0$
• convergent to zero	$l = m = 0$
$\{A^n(A-E)\}_{n\in\mathbb{N}}$:	
• bounded (***)	$\delta_1 = \ldots = \delta_{l-1} = 0$, $\varepsilon_j \cdot \varepsilon_{j+1} = 0$ for $j = 1, \ldots, m-2$
• convergent	$l = 0, \quad \varepsilon_j \cdot \varepsilon_{j+1} = 0$ for $j = 1, \ldots, m-2$
• convergent to zero	$l = 0, \quad \varepsilon_1 = \ldots = \varepsilon_{m-1} = 0$
$\{\frac{1}{n}\sum_{k=0}^{n-1} A^k\}_{n\in\mathbb{N}}$:	
• bounded (***)	$\delta_j \cdot \delta_{j+1} = 0$ for $j = 1, \ldots, l-2$, $\varepsilon_1 = \ldots = \varepsilon_{m-1} = 0$
• convergent	$\delta_1 = \ldots = \delta_{l-1} = \varepsilon_1 = \ldots = \varepsilon_{m-1} = 0$
• convergent to zero	$m = 0, \quad \delta_1 = \ldots = \delta_{l-1} = 0$
$\{A^n\}_{n\in\mathbb{Z}}$:	
• bounded (***)	$k = 0, \quad \delta_1 = \ldots = \delta_{m-1} = \varepsilon_1 = \ldots = \varepsilon_{m-1} = 0$

(*) This follows from [26, Theorem 8].
(**) This follows from [26, Theorem 7].
(***) Since $\dim \mathcal{M}(p, \mathbb{C}) < \infty$ the bounded sets are exactly the relatively compact ones.

([6]) *Editorial note*: See also the paper by J. C. Strikwerda and B. A. Wade in this volume.

All matrices occurring in this paper are at least power dominated ([1, Proposition 2.1]), so all of them are similar to

$$\begin{pmatrix}
\kappa_1 & \gamma_1 & & & & & & & & & & & \mathbf{0} \\
& \ddots & \ddots & & & & & & & & & & \\
& & \ddots & \gamma_{k-1} & & & & & & & & & \\
& & & \kappa_k & 0 & & & & & & & & \\
& & & & \lambda_1 & \delta_1 & & & & & & & \\
& & & & & \ddots & \ddots & & & & & & \\
& & & & & & \ddots & \delta_{l-1} & & & & & \\
& & & & & & & \lambda_l & 0 & & & & \\
& & & & & & & & \mu_1 & \varepsilon_1 & & & \\
& & & & & & & & & \ddots & \ddots & & \\
& & & & & & & & & & \ddots & \varepsilon_{m-1} & \\
\mathbf{0} & & & & & & & & & & & \mu_m &
\end{pmatrix}$$

where

(i) $k, l, m \in \mathbb{N}_0$, $k + l + m = p$,

(ii) $\kappa_1, \ldots, \kappa_k \in \mathbb{D}$; $\lambda_1, \ldots, \lambda_l \in \Gamma \setminus \{1\}$; $\mu_1 = \ldots = \mu_m = 1$,

(iii) $\gamma_1, \ldots, \gamma_{k-1}, \delta_1, \ldots, \delta_{l-1}, \varepsilon_1, \ldots, \varepsilon_{m-1} \in \{0, 1\}$.

Table 1 lists the interesting cases and gives their characterizations using the above notation.

Part of the statements of this table can also be found in [9] where the proofs are based on matrix theory only.

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Note. In the references we only included material that was referred to in this paper. For a reference list covering the whole topic the reader may consult [26].

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