LINEAR OPERATORS
BANACH CENTER PUBLICATIONS, VOLUME 38
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

GENERALIZED EIGENFUNCTION EXPANSIONS AND SPECTRAL DECOMPOSITIONS

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Abstract. The paper relates several generalized eigenfunction expansions to classical spectral decomposition properties. From this perspective one explains some recent results concerning the classes of decomposable and generalized scalar operators. In particular a universal dilation theory and two different functional models for related classes of operators are presented.

1. Introduction. The subject of eigenfunction expansions is as old as operator theory. The completeness of classical systems of eigenfunction expansions was originally related to mechanical problems and boundary value problems for differential operators. Later the study of eigenfunctions expansions has gained an independent and abstract status. From this perspective, the present paper has a very limited purpose. We will focus only on a few classes of abstract operators with rich spectral decompositions and some natural eigenfunction expansions related to them. The progress made in the last two decades in the theory of decomposable operators and some related classes of operators can at this moment be interpreted from the point of view of eigenfunction expansions. This approach has recently led to a new insight into several problems of abstract spectral theory and it produced a series of quite unexpected applications. It is the aim of the present paper to report a few results along this line.

First we review some classical spectral decomposition theories, and based on them, we discuss several natural classes of generalized eigenfunctions. To be more precise we distinguish four classes of proper vectors: the usual eigenvectors, generalized eigenvectors in the sense of measure theory, eigendistributions and generalized eigenvectors in the sense of analytic functionals. The general picture which will be made precise in Section 3 is the following: if a Banach space operator T has a complete system of generalized eigenvectors in one of the above classes, then its topological adjoint admits sufficiently

1991 *Mathematics Subject Classification*: 47A70, 47B40, 47A13.
Paper supported by the National Science Foundation grant DMS 9500954.
The paper is in final form and no version of it will be published elsewhere.

invariant subspaces to be spectrally localized. In such a way the classes of subscalar or subdecomposable operators appear naturally related to the completion of certain systems of eigenfunctions. Moreover, if both the operator and its dual have complete systems of generalized eigenvectors, then the respective operator can be even spectrally decomposed. Section 4 will contain various conditions on the resolvent of a Banach space operator which insure the completeness of a prescribed system of generalized eigenvectors. Among these conditions the so-called Bishop's property (β) has a distinguished place. Section 5 sketches a general dilation theory for arbitrary operators which puts into a geometrical form the relation between a given operator and the classes of scalar or decomposable operators. In Section 6 the sheaf model for subdecomposable operators is briefly recalled on the basis of a few examples. Finally, in Section 7 we present without entering into details the multivariable analogues of the results mentioned in the previous sections. These aspects of multivariable spectral theory will be developed in a forthcoming monograph, see Eschmeier and Putinar (1996).

The main results to be presented below have been obtained in the last decade by methods of homological algebra and sheaf theory. Although these tools are essentially needed only in the multivariable case of commuting n-tuple of operators, they enlighten even the single operator situation. We will try to avoid in the sequel any reference to resolutions, tensor products or derived functors; however, we would like to stress that the homological framework has suggested most of the constructions which are apparently new in the theory of abstract spectral decompositions. Each section below will contain precise references in this direction.

This article represents an expanded version of a conference given at Indiana University -Bloomington in honor of Professor Ciprian Foiaş sixtieth birthday. Most of the material discussed below is directly related to his early contributions to spectral theory. I dedicate the following pages to Ciprian Foiaş, with respect and admiration for his work and mathematical intuition.

2. Classical spectral decompositions. It is the aim of the present section to recall and review some well known examples of single operators which admit remarkable spectral decompositions. The general reference for this subject is the monograph Dunford and Schwartz (1958), (1963), (1971). For more special spectral decompositions the reader is referred to Colojoară and Foiaş (1968) and Berezanskiĭ (1968).

The best understood example of a spectral decomposition is the *Jordan form* of a linear operator acting on a finite-dimensional space. Let V be a finite-dimensional complex vector space and let T be a linear transformation of V. Then it is well known that there is a basis of V with respect to which T can be represented as a matrix with diagonal blocks of the form:

$$\lambda I_k + N_k,$$

where I_k is the identity matrix of order k and N_k is the nilpotent matrix of order k with *1* over the principal diagonal and zero elsewhere. The complex numbers $\lambda, \dots$ which appear in this representation form the *spectrum* of T, denoted in what follows by $\sigma(T)$.

The usual definition of the spectrum is

$$\sigma(T) = \{\lambda \in \mathbb{C}; T - \lambda \text{ is not invertible}\}.$$

A basis of the space V cannot in general be obtained from the solutions of the equation $(T-\lambda)\xi = 0$, known as the *eigenvectors* of T, but instead the *generalized eigenvectors* of T span V. These are the vectors ξ of V which satisfy the equation

$$(T-\lambda)^p \xi = 0 \tag{1}$$

for some natural number p.

The Jordan block structure of the matrix associated to T is unique up to order arrangements. More precisely the spectrum of T and the collection of the orders of the Jordan blocks coresponding to a fixed point of the spectrum form a complete system of invariants for the transformation T (up to similarity).

On infinite-dimensional topological spaces, the structure and classification of linear continuous operators is much more subtle. To fix ideas we will consider only bounded linear operators acting on a complex, infinite-dimensional Hilbert space H.

It is known for instance that if $T : H \to H$ is a compact operator, then its spectrum is discrete with *0* the single accumulation point and there is a finite-dimensional subspace of H which carries the whole spectral information in neighbourhoods of a non-zero point of the spectrum. However, whether these spaces span H, even up to completion in the uniform norm is a delicate question. (See Dunford and Schwartz (1958) and Gohberg and Krein (1969).) Moreover, the structure of the compact operator T in the neighbourhood of 0 is hardly classifiable.

The richest spectral theory developed and used so far in applications concerns another class of Hilbert space operators, namely the self-adjoint operators. The classical theory of Hahn and Hellinger represents any bounded self-adjoint operator A as the multiplication with the real variable x on the Hilbert space:

$$H = \bigoplus_{a=1}^{N} L^2(\mathbb{R}, \mu_a)$$

where N is a natural integer or infinity and μ_a are positive measures with compact support in R. Moreover, there is a spectral multiplicity function which classifies the operator A and which counts roughly how many of the above measures have a given point in their support.

Let us consider a simple example. Let M denote the multiplication with the variable x on $L^2[0,1]$, the latter space being considered with the usual Lebesgue measure. Then it is obvious that for any fixed point $\lambda \in \mathbb{R}$ the equation (1) has no solutions in $L^2[0,1]$. Consequently the operator M has no generalized eigenvectors and there is no hope to realize it in a Jordan form. It was Dirac who remarked that there are however vector valued measures which satisfy the eigenvector equation, and they are enough to span the whole underlying Hilbert space. (See Dirac (1930).) In modern terms this assertion can be translated as follows. The measure

$$u(x,y) = f(y)\delta(x-y) \in M(\mathbb{R}) \mathbin{\widehat{\otimes}} L^2[0,1]$$

satisfies the equation

$$(M - x)u(x, y) = 0 \tag{2}$$

for any continuous function f on $[0, 1]$. Moreover, the space $L^2[0, 1]$ contains as a dense linear subspace the elements of the form

$$\int u(x, y)\, dy = f(x).$$

In fact the above argument is valid for an arbitrary self-adjoint operator $A \in L(H)$. A standard way of expressing this fact leads to the spectral measure of A. Indeed, let us consider a closed set $F \subset \mathbb{C}$ and the space of all H-valued proper measures for A supported by F:

$$M_F = \{u \in M(\mathbb{R}) \mathbin{\widehat{\otimes}} H; (A - x)u(x) = 0,\ \mathrm{supp}(u) \subset F\}.$$

Then the closed subspace H_F of H generated by $\int u$ for all $u \in M_F$ defines a unique orthogonal projection $E(F)$ of H. It turns out that $E(\cdot)$ is a σ-additive, multiplicative measure with values projections of H, and as one very well knows

$$A = \int_{\mathbb{R}} xE(dx). \tag{3}$$

The measure E is uniquely determined by the self-adjoint operator A and in fact it carries in a flexible form the spectral data of A, see for details Dunford and Schwartz (1958).

Similarly a commuting system of bounded self-adjoint operators has a joint spectral measure which diagonalizes them simultaneously. In particular a normal operator admits a spectral decomposition as (3). The general theory of spectral measures on Banach spaces was developed by Dunford, see Dunford and Schwartz (1971). The Banach space operators which admit a representation like (3) are called *scalar* operators. Moreover, Dunford extended this concept to unbounded operators and he found important examples of scalar operators among differential operators.

The Jordan form of a matrix has been successfully generalized to the class of *spectral* operators, which are Banach space operators T of the form

$$T = S + Q,$$

where S is Dunford scalar and Q is a quasi-nilpotent operator, that is, $\sigma(Q) = \{0\}$. See for details Dunford and Schwartz (1971).

It was Ciprian Foiaş who, at the end of the fifties, investigated at the same abstract level operators with sufficiently many eigendistributions rather than only vector valued measures. From his work (mentioned in the reference list of the monograph Colojoară and Foiaş (1968)) has emerged the following definition.

A linear bounded operator T acting on the Banach space X is called *generalized scalar* if there is a continuous homomorphism of unital algebras

$$U : E(\mathbb{C}) \to L(X)$$

such that $T = U(z)$.

The space of smooth functions on the complex plane is denoted above by $E(\mathbb{C})$, while z denotes the complex coordinate. In other terms the operator T is generalized scalar if it admits a continuous functional calculus with smooth functions. If we regard U as an element of $E'(\mathbb{C}) \widehat{\otimes} L(X)$, then it is obvious from the above definition that the following equation holds in the sense of distributions:

$$(z - T)U = 0.$$

The theory of generalized scalar operators resembles much that of scalar operators. The monograph Colojoară and Foiaş (1968) is entirely devoted to this class of operators.

A typical example of a generalized scalar operator is the multiplication with a smooth function on a space of continuously differentiable (up to a given order) functions defined on a subset of the complex plane.

3. Decomposable operators. Going one step further and studying the eigenvector equation at the level of analytic functionals we are close to the works of Bishop (1959) and Foiaş (1963) in the area of operator theory called today local spectral theory. Although this approach with analytic functionals represents an a posteriori explanation of the early results in local spectral theory, it has certain advantages which will be outlined in the sequel.

We assume for simplicity that X is a reflexive Banach space and $T \in L(X)$ is a bounded linear operator on X. The space of X-valued analytic functions on the open set $U \subset \mathbb{C}$ is denoted by $O(U, X)$. It is a Fréchet space in the natural topology of uniform convergence on the compact subsets of X. Its topological dual $O(U, X)'$ is the space of analytic functionals supported by the closure of U.

To understand the subsequent constructions let us suppose at the beginning that the operator T is scalar and admits a spectral measure E. The space of those vectors of X which are supported by a closed set F of $\mathbb{C}$, relative to the operator T, is, as explained before, $E(F)X$. One easily remarks that any element $x \in E(F)X$ can be divided outside F by $T - z$:

$$x = (T - z)f(z).$$

Moreover, one can choose f to be an analytic function on $\mathbb{C} \setminus F$.

Thus we are naturally led to the following definition, for an arbitrary operator T this time. Let F be a closed subset of the complex plane and define the map:

$$J_F : X \to O(\mathbb{C} \setminus F, X)/(T - z)O(\mathbb{C} \setminus F, X),$$

by $J_F(x) = [1 \otimes x]$ (i.e. the class of the function identically equal to x in the respective quotient). A candidate for the spectral space $E(F)X$ proposed by Bishop (1959) would then be:

$$M(T, F) = \operatorname{Ker}(J_F).$$

Actually Bishop considered several similar possible spectral subspaces, but we will focus here only on the latter one.

A satisfactory spectral decomposition behaviour of T would be to have sufficiently many subspaces $M(F, T)$, so that, for a fixed partition of the plane into closed sets they span X. It was Bishop who discovered that this property is related to a remarkable condition involving the linear operator-valued analytic function $T' - z$.

Theorem (Bishop). *Let T be a bounded linear operator acting on a reflexive Banach space X. Suppose that the dual operator T' has the property that for each open set U of $\mathbb{C}$ the map $T'-z : O(U,X') \to O(U,X')$ is one to one with closed range. Then the subspace $\sum_{i\in I} M(\overline{U}_i,T)$ is dense in X for each finite open covering $(U_i)_{i\in I}$ of the complex plane.*

A sketch of a proof, which even yields a stronger result than stated, runs as follows. Since the sheaf $O^{X'}$ of germs of X'-valued analytic functions has vanishing cohomology on every open set of the complex plane (exactly as the sheaf O of analytic functions), for each open covering $(U_i)_{i\in I}$ of $\mathbb{C}$ one has an exact augmented Čech complex of alternating chains:

$$0 \to O(\mathbb{C},X') \to \prod_{i\in I} O(U_i,X') \to \prod_{i,j\in I} O(U_i\cap U_j,X') \to \dots$$

Let us suppose that the operator T' satisfies the condition in the statement. Then the complex obtained from the above complex by taking quotients modulo the range of $T'-z$ remains exact. Its first two terms are:

$$O(\mathbb{C},X')/(T'-z)O(\mathbb{C},X') \hookrightarrow \prod_{i\in I} O(U_i,X')/(T'-z)O(U_i,X') \to \dots \tag{4}$$

Moreover, a simple computation with power series identifies X' with the quotient $O(\mathbb{C},X')/(T'-z)O(C,X')$ via the map $J_\emptyset$.

Because of the reflexivity of X the space $O(U)'\,\widehat{\otimes}\,X$ can be regarded as the strong dual of the Fréchet space $O(U)\,\widehat{\otimes}\,X'$ for any open set U. Therefore

$$K_i = \operatorname{Ker}(T-z : O(U_i)'\,\widehat{\otimes}\,X \to O(U_i)'\,\widehat{\otimes}\,X)$$

can be identified, at least algebraically, with the dual space of

$$C_i = \operatorname{Coker}(T'-z : O(U_i)\,\widehat{\otimes}\,X' \to O(U_i)\,\widehat{\otimes}\,X').$$

Let us denote by $\sigma : \bigoplus_{i\in I} K_i \to X$ the adjoint of the operator $j : X' \to \prod_{i\in I} C_i$. By general duality theory the operator σ is onto if and only if j is injective with closed range. But each element $u_i \in K_i \subset O(U_i)'\,\widehat{\otimes}\,X$ is an X-valued analytic functional supported by the closure of U_i, which in addition is proper with respect to T:

$$(T-z)u_i = 0.$$

Moreover,

$$u_i(1) = (T-w)u_i(1/(z-w))$$

for any $w \in \mathbb{C}\setminus\overline{U}_i$, so that $u_i(1) \in M(\overline{U}_i,T)$. Hence the observation that

$$\sigma((u_i)_{i\in I}) = \sum_{i\in I} u_i(1)$$

implies that $X = \sum_i M(\overline{U}_i,T)$ under the stated conditions.

In fact Bishop also proved a reciprocal of the above theorem and he distinguished four degrees of "spectral dualities", each similar to the conclusion of the theorem. For later use we isolate from the preceding discussion the following definition.

Definition 1. An operator $T \in L(X)$ is said to have *property* (β) if for any open set U of $\mathbb{C}$ the map $T - z$ is one-to-one with closed range on the space $O(U, X)$ of X-valued analytic functions defined on U.

Remark that, by the Maximum Principle for analytic functions, condition (β) is trivially satisfied whenever $U \supset \sigma(T)$.

We will see later that (β) is a central property in spectral theory, at least at this abstract level.

The most flexible class of operators which behave as in the statement of Bishop's theorem was introduced by Foiaş in 1963 and since then these operators have been constantly investigated from different perspectives. Thus we recall from Foiaş (1963) the following definition. (The original definition was in fact more sophisticated and less intrinsic; for the evolution towards the present definition see Vasilescu (1982)).

Definition 2. An operator $T \in L(X)$ is called *decomposable* if for every finite open covering $(U_i)_{i\in I}$ of the complex plane there are closed invariant subspaces $(X_i)_{i\in I}$ of T with the properties

$$X = \sum_{i\in I} X_i, \qquad \sigma(T|X_i) \subset U_i,\ i \in I. \tag{5}$$

One easily remarks that all self-adjoint, scalar, generalized scalar or quasi-nilpotent operators are decomposable. However, this class of operators is much larger and it may contain very patological examples, see Albrecht (1978). There are at present several simplifications of the above definition of a decomposable operator; for instance one can require that the set I has exactly two elements. A comprehensive reference, with ample hystorical remarks concerning the class of decomposable operators is Vasilescu (1982). Another very useful survey about decomposable operators is Radjabalipour (1978).

It is known that a decomposable operator satisfies condition (β). On the other hand, it is clear that property (β) is inherited from an operator to its restriction to a closed invariant subspace. In fact, the reciprocal is true, and moreover a characterization of decomposability in terms of (β) is possible. The next theorem makes precise these statements. Its proof, although a variation of Bishop's ideas, was found only recently. (See Lange (1981), Albrecht and Eschmeier (1987) and Eschmeier and Putinar (1984).)

Theorem 1. *Let T be a linear bounded operator acting on a not necessarily reflexive Banach space. Then*:

(a) *The operator T is decomposable if and only if T and T' have property (β)*;

(b) *The operator T is subdecomposable if and only if T has property (β).*

Corollary. *A Banach space operator is decomposable if and only if its topological adjoint is decomposable.*

This statement, for non-reflexive Banach spaces, has circulated a long period of time as an open question, see Radjabalipour (1978) and Vasilescu (1982).

More about the preceding theorem and similar phenomena will appear in the next section. Now we conclude with a few more definitions.

An operator is called *subdecomposable*, for short SD, if it is similar to the restriction of a decomposable operator to a closed invariant subspace. Similarly one defines a subgeneralized scalar operator, for short SGS. Dually, a quotient of a decomposable operator, for short QD, is an operator which is similar to the quotient of a decomposable operator modulo a closed invariant subspace. Similarly there is a notion of quotient of a generalized scalar operator, for short QGS.

To conclude this section, we remark in view of Bishop's Theorem that the completeness of the system of proper analytic functionals with arbitrary small support of an operator T is related by duality to property (β) for the adjoint operator. This relationship explains in fact the majority of the known results about decomposable or related classes of operators.

4. Conditions involving the resolvent. Traditionally the resolvent of an operator is the first object to look at when the spectral properties of the operator are needed. Again the case of a self-adjoint operator is best understood. Not far from this example Colojoară and Foiaş studied a series of growth conditions on the resolvent of an operator which imply the latter to be generalized scalar, see Colojoară and Foiaş (1963). Analogously, Lyubich and Matsaev (1962) exhibited a growth condition of the resolvent of an operator which implies its decomposability. To be more precise we recall a typical case of resolvent analysis.

Let T be a bounded linear operator acting on the Banach space X. Assume that the spectrum of T is contained in the unit torus $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ and assume that the following condition holds:

$$\|(T - z)^{-1}\| \leq C||z| - 1|^{-\alpha}, \qquad |z| \neq 1, \tag{6}$$

where C is a positive constant and $\alpha \geq 1$. Under that condition Colojoară and Foiaş (1968) prove that one can replace T in the Fourier series development of a function $f \in C^m(\mathbb{T})$, for $m > \alpha + 1$. Thus one proves that T is a generalized scalar operator. A good exercise for the reader is to use Theorem 1 above for proving that T is decomposable by checking property (β) for T and T'.

There is at present an explanation of results like the latter one, without involving the Fourier transform or any group structure on the support of the spectrum. Namely Dynkin (1972) discovered a functional calculus based on the Cauchy–Pompeiu formula with Whitney jets of analytic functions defined only on the spectrum, whenever the resolvent grows towards the spectrum not faster than a negative power of the distance to the spectrum.

Bishop's Theorem recalled in the preceding section suggests that some non-pointwise conditions on the resolvent of an operator T, or rather on $T - z$, would be more appropriate in the study of various spectral decomposition properties of T. This is indeed true, as will follow from the next theorem. First we need some more notation and terminology.

DEFINITION 3. An operator $T \in L(X)$ is said to have *property* (β_E) if $T - z$ is one-to-one with closed range on the space $E(\mathbb{C}, X)$ of X-valued smooth functions on the complex plane.

We remark that, by duality, if T satisfies condition (β_E), then, at the level of distributions the following map is onto:

$$T' - z : E(\mathbb{C}) \widehat{\otimes} X' \to E(\mathbb{C}) \widehat{\otimes} X'.$$

In fact, if the space X is reflexive this condition also implies β_E. In that case we will simply say that $T' - z$ divides all X'-valued distributions.

Similarly, property (β) for T is equivalent on reflexive spaces to the divisibility of all X'-valued analytic functionals by $T' - z$, this time with arbitrarily small support.

We can summarize the main results in this area in the following theorem.

THEOREM 2. *Let T be a linear bounded operator acting on a reflexive Banach space X. Then each column of the following table characterizes the respective class of operators.*

T	property	divisibility	complete system of eigen.
SD	(β)	$(T' - z) \mid$ an. funct.	an. funct. for T'
QD	$(\beta)'$	$(T - z) \mid$ an. funct.	an. funct. for T
D	(β), $(\beta)'$	$(T - z), (T' - z) \mid$ an. funct.	an. funct. for T, T'
SGS	$(\beta)_E$	$(T' - z) \mid$ distrib.	distrib. for T'
QGS	$(\beta)'_E$	$(T - z) \mid$ distrib.	distrib. for T

Looking at the above table, it is expected that the juxtaposition of the last two lines would characterize the generalized scalar operators. Unfortunately, this fact is not true, see Eschmeier and Putinar (1989) for an example.

Theorem 2 remains valid on arbitrary Banach space, with the only modification that the surjectivity of $T - z$ or $T' - z$ on the respective spaces of distributions must be changed into surjectivity plus the lifting property of bounded sets. For a complete proof see Eschmeier and Putinar (1996). Theorem 2 has a series of applications to the division of distributions by analytic functions. Some of these applications are also contained in Eschmeier and Putinar (1989).

Some of the above conditions may seem artificial and difficult to be verified. However, some quantitative versions are available, with quite unexpected applications. Next we mention only one such example.

PROPOSITION 1. *Let T be a bounded linear operator acting on the Banach space X and let $D \subset \overline{D} \subset G$ be two open sets which contain the spectrum of T. Then each of the following conditions implies property $(\beta)_E$ for T.*

(a) *There is a natural number k and a positive constant c such that*

$$c\|f\|_{2,D} \le \|(T - z)f\|_{2,G} + \|(T - z)\overline{\partial}^k f\|_{2,G}$$

for every smooth function $f \in E(\overline{G})$.

(b) *There is a linear bounded operator on X such that*

$$\|(S - \overline{z})x\| \le \|(T - z)x\|$$

for any complex number z in a neighbourhood of the spectrum of T and $x \in X$.

Proof. (a) We have to prove that the operator $T - z$ is one-to-one with closed range on the space $E(\mathbb{C})$. Let f_n be a sequence in $E(\mathbb{C})$ with the property that $(T - z)f_n$ converges to zero. Since $T - z$ is invertible on $\mathbb{C} \setminus D$, we can assume, after a partition of unity, that $\mathrm{Supp}(f_n) \subset G$ for any n. Then condition (a) in the statement implies that $\|\overline{\partial}^N f_n\|$ tends to zero for every positive integer N. Hence $\|f_n\|$ converges to zero in the norm of $E(D)$ which certainly is enough to complete the proof of property (β_E).

(b) This part is an application of (a), via the following observation. Let f be a smooth function with compact support in D. Let g denote the function

$$\overline{\partial}(f - (\overline{z} - S)\overline{\partial} f) = -(\overline{z} - S)\overline{\partial}^2 f.$$

If χ is a smooth function with compact support in $\mathbb{C}$, equal to 1 on $D - D$, then $h = \pi^{-1}\frac{\chi}{z} \in L^1(\mathbb{C})$ satisfies by the Cauchy–Pompeiu formula the following identity:

$$f(z) - (\overline{z} - S)\overline{\partial} f(z) = g * h(z), \qquad z \in D.$$

Hence the assertion follows in virtue of the hypothesis and (a).

Corollary. *Every M-hyponormal operator is subgeneralized scalar.*

By definition an *M-hyponormal operator* is a Hilbert space operator T which satisfies the inequality

$$\|(T - z)^* \xi\| \le M \|(T - z)\xi\|$$

for a fixed positive constant M, every complex number z and every vector ξ. Thus the corollary follows from Theorem 2 and the preceding Proposition. This was proved in Putinar (1984); in fact, the motivation for studying condition $(\beta)_E$ and its connection with subscalar operators comes from this result.

We continue this section with a remark concerning the quotients of generalized scalar operators. Namely, let $T \in L(X)$ be such an operator. Then we know from Theorem 2 that the map $T - z$ is onto on the space of vector valued distributions with values in X. In particular, for every $x \in X$ there is a distribution $u \in D'(\mathbb{C}) \mathbin{\widehat{\otimes}} X$ with the property

$$(T - z)u(z) = x, \quad z \in \mathbb{C}, \quad \text{and} \quad u(\mathbf{1}) = x.$$

Hence $\mathrm{Supp}(u)$ is contained in the spectrum of T and therefore we can form the convolution in the sense of distributions $U = u \star \frac{-1}{\pi z}$, so that $\overline{\partial} U = u$. Moreover, the usual development in series of $\frac{1}{w-z}$ yields for large values of $|z|$:

$$U(z) = \pi^{-1} \sum_{n=0}^{\infty} u(w^n) z^{-n-1}.$$

Since the distribution $(T - z)U(z)$ is annihilated by $\overline{\partial}$ it is an analytic function bounded at infinity. More exactly, the preceding series shows that $\lim_{z \to \infty} zU(z) = \pi^{-1}x$, hence by the Liouville Theorem

$$(T - z)U(z) \equiv -\pi^{-1}x.$$

Thus we have proved the following result.

Proposition 2. *Let $T \in L(X)$ be a quotient of a generalized scalar operator. Then for every $x \in X$ the resolvent $(T - z)^{-1}x$ extends across the spectrum $\sigma(T)$ in the sense of X-valued distributions.*

For some classes of self-adjoint operators this conclusion is well known and it was exploited since the classical period of operator theory, see Berezanskiĭ (1968). Also, hyponormal operators have some distinguished extensions of their resolvent evaluated at special vectors. Whenever such extensions exist, a functional model based on a distribution kernel obtained by putting together the extended resolvents is at hand. For a special case, see Martin and Putinar (1989).

A typical application of Theorem 2, more exactly of the characterization of quotients of generalized scalar operators is given by a new proof of the following classical result.

THEOREM (L. Schwartz). *Let U be an open connected subset of $\mathbb{C}^n$ and let f be an analytic function in U which is not identically zero. Then the multiplication by f on the space of distributions $D'(U)$ is surjective.*

The principal steps of a proof are the following. The problem is local, so we will work in neighbourhoods of a point, say $0 \in \mathbb{C}^n$. By, Weierstrass Preparation Lemma we can change the complex coordinates so that in a neighbourhood of 0 the function f has, up to an invertible factor, the form of a distinguished polynomial:

$$f(t,s) = t^k + a_1(s)t^{k-1} + \ldots + a_n(s),$$

where $t \in T, s \in S$, and $T \subset \mathbb{C}$, $S \subset \mathbb{C}^{n-1}$ are balls centered at zero.

Let $u \in E(T \times S)$ be a distribution supported by the product of the two small balls. Then we can write u as a sum of distributions of the form $(\partial/\partial\overline{s}_{i_1})\ldots(\partial/\partial\overline{s}_{i_p})v$, $(1 \le i_1 \le i_2 \le \ldots \le i_p \le n-1)$ and $v \in E'(T) \widehat{\otimes} L^2(S)$.

But the Weierstrass polynomial f can be factored into linear terms:

$$f(t,s) = (t - \alpha_1(s))\ldots(t - \alpha_k(s)),$$

with measurable, essentially bounded coefficients, $\alpha_i \in L^\infty(S), 1 \le i \le k$. Since the multiplication by each α_i is a generalized scalar operator on $L^2(S)$, $t - \alpha_i$ divides by Theorem 2 the distribution v $(1 \le i \le k)$. Thus f divides v, and being analytic it commutes with the $\overline{\partial}$-operators. In conclusion f divides the distribution u.

The original proof appeared in Schwarz (1955). The preceding proof and similar division results are detailed in Eschmeier and Putinar (1996).

In general the principal problems concerning the classes of operators mentioned in this section have been solved, with one intriguing exception.

OPEN PROBLEM. *Is the sum or product of two commuting decomposable operators still decomposable?*

For some partial results and different equivalent forms of this question see Eschmeier (1985).

5. Dilations and functional calculi. The area of operator theory which can be incorporated under the title of this section is ample and diverse. The celebrated Sz.-Nagy–Foiaş functional model for Hilbert space contractions is the most notable example in this direction. We will not discuss below this refined theory of contractions or any of its numerous ramifications. Instead, we will discuss briefly an idea which has recently appeared in the study of subdecomposable operators and which is able to produce a variety of apparently new dilation theorems for much general classes of operators.

The dictionary between operators and modules over algebras with a single generator is at this moment assimilated with clear benefits by all operator theorists. A natural continuation of this relationship, namely the correspondence between dilations of operators and resolutions of modules, was isolated for the first time in a preprint by Douglas and Foiaş (1976). Except the school of Hilbert modules led by R. G. Douglas (see the references in Douglas and Paulsen (1989)), very few mathematicians are nowadays aware of the potential hidden in this homological approach to dilation theory. It is the aim of the present section to sketch a framework for studying dilations of arbitrary operators via resolutions of modules, and then to apply the general construction in a couple of typical situations.

Let T be an operator acting on the Banach space X. Then one can regard X as a module over the algebra of entire functions $O(\mathbb{C})$, via the functional calculus map:

$$(f, x) \mapsto f(T)x, \quad x \in X, \ f \in O(\mathbb{C}).$$

We know from the previous sections that, in general, one cannot expect this functional calculus to extend to algebras with partitions of unity. However, changing the initial module into a *dilation* of it, that is, into a submodule of a quotient of a larger module, one can always assume that this dilation admits a rich functional calculus, for instance with smooth functions. The only problem is to control and shrink as much as possible the spectrum of the dilated module. First we explain the general construction of a dilation in a purely algebraic setting.

Let A be a commutative ring with unit and let M be an A-module. Let $L.$ be a free resolution of M (of course to the left):

$$\ldots \to L_1 \to L_0 \to M \to 0. \tag{7}$$

Let B denote a commutative unital A-algebra and suppose that A admits a finite resolution to the right, in the category of A-modules, with B-modules:

$$0 \to A \to N^0 \to N^1 \to \ldots \tag{8}$$

Consider the double complex $L. \otimes_A N^\cdot$ and the total complex $C.$ associated to it, that is,

$$C_n = \bigoplus_{p-q=n} L_p \oplus N^q,$$

endowed with the natural total differential. Then $C.$ is exact in all degrees, except the degree zero, and in degree zero its homology is isomorphic to M. Notice that the components of $C.$ are B-modules, but its differentials are only A-linear. In the terminology explained above, C_0 is a dilation to a B-module of the A-module M.

The first example we consider is in fact a construction which applies to an arbitrary operator and it provides some geometric support for the properties (β) and $(\beta)'$. Let T be a linear bounded operator acting on the Banach space X and let U be a bounded open set which contains the spectrum of T. Then it is well known that the sequence

$$0 \to O(U, X) \xrightarrow{T-z} O(U, X) \to X \to 0$$

is exact. Moreover, one can replace the space of all analytic functions above with the space $A^2(U, X)$ of X-valued, square summable, analytic functions on U. This exact sequence plays the rôle of the free resolution in the preceding general scheme. For the right resolution we choose the $\overline{\partial}$-complex given by the closed operator $\overline{\partial}$ on $L^2(U, X)$. More precisely, denoting by D the domain of this operator ($D = \{f \in L^2(U, X); \overline{\partial} f \in L^2(U, X)\}$) we have an exact sequence

$$0 \to A^2(U, X) \to D \xrightarrow{\overline{\partial}} L^2(U, X) \to 0,$$

which corresponds to the right resolution with B-modules in the above abstract construction.

Thus the total complex $C.$ is

$$0 \to D \to D \oplus L^2(U, X) \to L^2(U, X) \to 0,$$

with the boundary operators

$$d_1 = ((T - z) \oplus \overline{\partial}), \quad d_0 = (\overline{\partial}, z - T).$$

The homology in the middle is isomorphic, as $A = O(\mathbb{C})$-modules, to X, and elsewhere the complex is exact. Note that the components of $C.$ are naturally $C^1(\overline{U})$-modules. Let us denote by D^+ the A-module $C_0/\mathrm{Im}(d_1)$ and by D^- the A-module $\mathrm{Ker}(d_0)$, so that X is canonically a sub-A-module of D^+ and a quotient A-module of D^-.

Summing up, the multiplication operator S with the complex variable on the space C_0 is generalized scalar of order 1, with spectrum equal to the closure of U; it defines by restriction, respectively by quotient, the operators $S_- = S|D_-$, respectively $S_+ = S/\mathrm{Im}\, d_{-1}$. Moreover, S_+ is an extension of T and S_- is a co-extension of T. With these notations we have the following remarkable observation.

Theorem 3. *Let T be a linear bounded operator acting on the Banach space X and let U be a bounded open neighbourhood of the spectrum of T. Then with the above notations there is a C^1-scalar dilation S of T, an extension S_+ and a co-extension S_- of T with the following properties*:

(a) *The spectra of S, $S_\pm$ are contained in $\overline{U}$*;
(b) *T is subdecomposable if and only if S_+ is decomposable*;
(c) *T is q-decomposable if and only if S_- is decomposable*;
(d) *T is decomposable if and only if both $S_\pm$ are decomposable.*

The proof is a simple application of Theorem 2 and the fact that, in a short exact sequence of $O(\mathbb{C})$-modules, if two modules have property (β) then the third has the same property. Theorem 3 appeared in different forms in Albrecht and Eschmeier (1987), Eschmeier and Putinar (1988), and Putinar (1990).

This apparently too general dilation theorem has recently found an important application to the invariant subspace problem.

Corollary. *A subdecomposable operator with thick spectrum has non-trivial invariant subspaces.*

Here by a *thick set* we mean a compact set σ of the complex plane with the property that there is an open set G such that the H^∞-norm on G is attained on $G \cap \sigma$, for every element of $H^\infty(G)$. In particular, a set with interior points is thick in this sense. The proof of the above Corollary is due to Eschmeier and Prunaru (1990) and it exploits a method of Scott W. Brown, which in the last decade has produced a series of deep invariant subspace theorems.

Next we focus on a similar dilation theorem, more precise and more restrictive, which has the advantage of letting the spectrum of the dilated operator approach the spectrum of the initial operator.

THEOREM 4. *Let T be a linear bounded operator acting on the Banach space X and let U be a bounded set of the complex plane with smooth boundary. Suppose that there are positive constants C and b, such that*:

$$\|(T-z)^{-1}\| \le C\,\mathrm{dist}(z,\partial U)^{-b}, \qquad z \in \mathbb{C}\setminus \overline{U}.$$

Then T admits a dilation to a $\mathrm{Lip}_a(\overline{U})$-generalized scalar operator, where a is any non-integer greater than $b+2$.

Proof. Above Lip_a denotes the Lipschitz space of order a of differentiable functions. The proof repeats the construction oultined before, this time with Lipschitz spaces instead of Lebesgue spaces.

We can assume that the constant b in the statement is non-integral. (Otherwise we replace it with a slightly larger constant). Let $A^a(\overline{U})$ denote the space of analytic functions in U which have an extension to $\mathrm{Lip}_a(\overline{U})$. Under the conditions in the statement, there is an extension of the usual analytic functional calculus of T to an $A^{a+1}(\overline{U})$-functional calculus. (See Dynkin (1972) for the construction of this generalized functional calculus.) Thus the analytic module X has a topologically free resolution to the left, given for instance by the topological analogue of the classical Bar complex:

$$B.^A(A,X) \to X \to 0,$$

where we put for simplicity $A = A^{a+1}(\overline{U})$. (The Bar complex is the complex which defines the Hochschild cohomology of modules, see MacLane (1963). Its topological version appears in the cohomology theories of topological modules over topological algebras. Here we follow Taylor (1972a) in all definitions concerning topological and homological concepts.)

On the other hand, the boundary values of the Cauchy integral are linear bounded operators on any non-integral Lipschitz norm on the boundary of U (see for instance Henkin and Leiterer (1984)), whence the complex

$$0 \to A \to D \xrightarrow{\overline{\partial}} \mathrm{Lip}_{a+1}(\overline{U}) \to 0$$

is exact. As before D stands for the domain of $\overline{\partial}$ in the latter Lipschitz space. Moreover, the Cauchy integral gives a continuous $\mathbb{C}$-splitting of this complex. Thus each component of the Bar complex can be replaced by a two term exact complex and finally we obtain a double complex

$$B.^A(D,X) \to B.^A(\mathrm{Lip}_{a+1}(\overline{U}),X)$$

whose total complex is the abstract $C.$ above. To finish the sketch of the proof it remains to remark that the components of $C.$ are $\mathrm{Lip}_{a+2}(\overline{U})$-modules.

The universal extension and co-extension $S_\pm$ of T still exist inside C_0 and the conclusion of Theorem 3 remains valid in this case, too. It is worth remarking that in the statement of Theorem 4 the spectrum of T may intersect the boundary of the set U.

With the above proofs as a model, the reader can easily find other canonical dilations, starting with different refined functional calculi.

The same algebraic setting for a universal dilation of an operator explains with minor modifications the classical dilation theory of Sz.-Nagy and Foiaş, see Douglas and Paulsen (1989) for details. However, it is not expected that the principal analytic difficulties in this field, mainly related to the concept of spectral set and uniform estimates for functional calculi, will be solved by this homological approach. An instructive example of a positive solution to a difficult analytic question in dilation theory is Agler (1985).

An important open question related to the above mentiond work of Agler is to determine the Hilbert space operators which admit a normal dilation with spectrum contained in a prescribed closed set. So far only for the disk and an anulus there is a satisfactory solution.

6. The sheaf model. We have encountered so far a few examples of spectral theories with data arranged in a discrete table (the Jordan form), a vector valued measure (self-adjoint and scalar operators) or a vector valued distribution (generalized scalar operators). A common feature of all these examples is the fact that the data are arranged into objects which carry certain multiplicities over the spectrum of the original operator. Thus it is not unexpected to put the same or similar data into a bundle or sheaf supported by the spectrum. As a matter of fact the decomposable and related families of operators are well fitted for such an approach. Although the corresponding sheaf model has originally appeared as a technical tool in the multivariable theory (of commuting systems of operators), it is a posteriori relevant even for a single operator.

To give a simple motivation for what follows we consider again a linear operator acting on a finite-dimensional space V. Regarded as an $O(\mathbb{C})$-module, V is isomorphic to $\mathrm{Coker}((T-z) : O(\mathbb{C}, V) \to O(\mathbb{C}, V))$. Obviously the latter module decomposes into its local factors $\mathrm{Coker}((T-z) : {O^V}_\alpha \to {O^V}_\alpha)$, where α is a point of the spectrum of T and ${O^V}_\alpha$ represents the module of germs at α of all V-valued analytic functions. Since each of these modules is of finite complex dimension, it is of the form $(O_\alpha/{m_\alpha}^{k_1}) \oplus \ldots \oplus (O_\alpha/{m_\alpha}^{k_p})$, where m_α is the maximal ideal of the local ring O_α. Thus another form of the Jordan decomposition of the operator T is to represent the analytic module V as follows:

$$V \cong \bigoplus_{\alpha\in\sigma(T)} [(O_\alpha/{m_\alpha}^{k_1}) \oplus \ldots \oplus (O_\alpha/{m_\alpha}^{k_{p(\alpha)}}].$$

By replacing the local rings of analytic functions with the sheaves of analytic functions on the complex plane one finds a so-called "sky-scraper sheaf" which is supported on $\sigma(T)$ and which localizes the space V with respect to the action of its linear endomorphism T. A similar construction holds for operators with property (β); in fact, it has already appeared in the proof of Bishop's Theorem discussed above in this paper.

DEFINITION 4. Let T be a linear bounded operator acting on the Banach space X. Suppose that T satisfies condition (β). The *sheaf model* of T is the sheaf associated to the presheaf

$$\mathcal{F}_T(U) = \mathrm{Coker}((T-z) : O(U,X) \to O(U,X)),$$

where U is an open subset of $\mathbb{C}$.

One easily finds that the preceding presheaf is actually a sheaf of Fréchet O-modules. Moreover, by its very definition we have a topologically free resolution of this sheaf on $\mathbb{C}$:

$$0 \to O^X \xrightarrow{T-z} O^X \to \mathcal{F}_T \to 0. \tag{9}$$

For that reason the sheaf model of the operator T has vanishing Čech cohomology on any open subset of the complex plane. Moreover, it is obvious that this sheaf is supported by the spectrum of T. According to (9), the space X can be realized as

$$X \cong \mathcal{F}_T(V),$$

whenever the open set V contains the spectrum of T. In this isomorphism T corresponds to the action of the complex coordinate on the sections of $\mathcal{F}_T$. This remark explains the name of sheaf model, as a parallel to various functional models of operators. The properties of the sheaf $\mathcal{F}_T$ reflect faithfully those of the original operator T. We mention below a few examples.

PROPOSITION 3. *Let T be a linear bounded operator with property (β) and let $\mathcal{F}_T$ be its sheaf model. Then*:

(a) $\mathrm{Supp}(\mathcal{F}_T) = \sigma(T)$;

(b) $M(T,A) = \{s \in \mathcal{F}_T(\mathbb{C}); \mathrm{supp}(s) \subset A\}$ *for every closed set A of* $\mathbb{C}$;

(c) *The operator T is Fredholm if and only if $\mathcal{F}_T$ is a coherent analytic sheaf in a neighbourhood of* 0. *In that case* $\mathrm{ind}(T) = -\mathrm{rk}_0\mathcal{F}_T$;

(d) *Two operators as in the statement are similar if and only if their sheaf models are isomorphic as Fréchet O-modules.*

For a proof of these facts see Putinar (1983), (1986).

A direct application of assertions (c) and (d) of the latter proposition concerns quasi-similar operators. We recall that the Banach space operators $T \in L(X)$ and $S \in L(Y)$ are said to be *quasi-similar* if there are bounded one-to-one operators $A : X \to Y$ and $B : Y \to X$ both with dense range such that $SA = AT$ and $BS = TB$. This is an equivalence relation which is more flexible than the similarity relation and it still preserves part of the properties of the operators T and S. In this direction we have the following result.

COROLLARY. *Let T and S be Banach space operators with property (β). If T and S are quasi-similar, then their spectra and essential spectra coincide.*

In the particular case of hyponormal operators the latter result has circulated as an open question for more than a decade. The complete proof of the corollary appears in Putinar (1992).

Thus, even if the partition of unity is missing from a functional calculus of the operator T, whenever property (β) is satisfied there is a satisfactory localization of the space, and the original operator, at the level of sheaves. Moreover, once we know that the corresponding module can be localized by a Fréchet analytic sheaf, there are simple criteria to verify that this is exactly the sheaf model, in the above sense. A typical example is contained in the following result. Next we adopt for convenience the equivalent language of modules instead of operators.

THEOREM 5. *A Banach $O(\mathbb{C})$-module X is decomposable if and only if there is an analytic Fréchet soft sheaf on $\mathbb{C}$ with the global sections space isomorphic to X as a Banach $O(\mathbb{C})$-module.*

In other words, if we know that an operator has a Fréchet soft sheaf model, then and only then it is decomposable. This implies in particular that any Fréchet soft sheaf model of an operator is unique, if it exists. The proof of Theorem 5 is also contained in Putinar (1983).

Thus the spectral subspaces $M(T, F)$ of a decomposable operator T can be arranged into a soft sheaf. This approach explains for instance the duality theory between these spaces (duality inherited from T and T'). It turns out that the sheaf models of T and T' reproduce formally the relation between the sheaves of test functions and distributions. This remark was the key in proving Theorem 2 and its main consequences.

Let us compute the sheaf models of a couple of operators which have appeared in the previous sections. Consider for instance the multiplication by z on the space $\mathrm{Lip}_\alpha(A)$, where A is a compact subset of $\mathbb{C}$. This analytic module is obviously decomposable. Its sheaf model is

$$\mathcal{F}(U) = \{f \in \mathrm{Lip}_\alpha(K \cap A); K \Subset U\},$$

that is, the local Lipschitz space of the same order, supported by A. Analogously the Bergman space $A^2(\Omega)$ of a bounded domain Ω of the complex plane is a subdecomposable analytic module, hence it can be localized. The corresponding sheaf model is

$$\mathcal{G}(U) = \{f \in O(U \cap \Omega); f \in A^2(V \cap \Omega),\ V \Subset U\}.$$

There are problems in function theory which require exactly such a localization approach. A simple example is discussed in the rest of this section.

Let Ω be a bounded domain of the complex plane. A system $f_1, \ldots, f_n$ of bounded analytic functions on Ω satisfies the assumption of the *Corona Problem* if

$$\inf_{z \in \Omega}(|f_1(z)| + \ldots + |f_n(z)|) > 0.$$

In that case the Corona Problem asks whether $(f_1, \ldots, f_n)$ generate $H^\infty(\Omega)$ as an algebra. A classical result of Carleson answers this question in the affirmative for the unit disk, and hence on any simply connected domain different from the entire plane. The problem is solvable on a series of other planar domains, but whether the Corona Problem has a positive solution on any domain of the complex plane is still an open question. For a comprehensive reference see Garnett (1981).

THEOREM (Gamelin). *Let Ω be a bounded, locally simply connected planar domain. Then the Corona Problem is solvable on Ω.*

A sketch of the proof, based on the sheaf model described in this section, runs as follows. Let $f_i \in H^\infty(\Omega), 1 \le i \le n$, be the data of the Corona Problem on Ω. We have to prove that the Koszul complex $K.(f, H^\infty)$ is exact, where $f = (f_1, \ldots, f_n)$. The Banach $O(\mathbb{C})$-module $H^\infty(\Omega)$ is subdecomposable as a submodule of $L^\infty(\Omega)$, whence it can be localized by an analytic Fréchet sheaf $\mathcal{F}$. Since this sheaf has vanishing cohomology on C, it suffices to prove that the complex of sheaves $K.(f, \mathcal{F})$ is exact. But on neighbourhoods of a boundary point of Ω, $\mathcal{F}$ is isomorphic to the sheaf model of the H^∞-space of the unit disk. Therefore the above complex of sheaves is (locally) exact by Carleson's Theorem.

The original proof of Theorem 6 is contained in Gamelin (1970).

7. Multivariable spectral theory. With inherent but standard modifications the principal results outlined in the previous sections have an analogue for commutative systems of Banach space operators. The seminal ideas of J. L. Taylor (1969), (1972a), (1972b) have made this generalization possible. Next we focus only on the multivariable analogues of the notions of subdecomposable and subscalar operators. The catalyst at this abstract level of spectral theory is the concept of quasi-coherent analytic sheaf. It replaces the operators with property (β) and on the other hand it brings into the area the geometric intuition and some techniques proper to complex analytic geometry. Besides the expected operator theory statements which parallel those presented above, there are a few external applications of the multivariable spectral theory which we simply enumerate. First some division and interpolation problems in spaces of analytic functions depending on several complex variables are direct consequences of joint spectra computations. Then the index theory for commutative systems of operators is for instance used as a main ingredient in a proof the Riemann–Roch Theorem on complex spaces with singularities, cf. Levy (1987). Finally, the spectral localization, in the sense of the previous sections, explains the recent efforts in classifying the analytic invariant subspaces of a Bergman space of a domain in $\mathbb{C}^n$. These topics, including the essential prerequisite material, constitute the body of a forthcoming monograph Eschmeier and Putinar (1996). (In fact, the present paper is partially a preview of this book).

The multivariable analogue of Bishop's property (β) was discovered by Frunză: The commutative n-tuple T of bounded linear operators acting on the Banach space X satisfies by definition condition (β) if the Koszul complex $K.(T - z, O(U, X))$ is exact in positive degree and has Hausdorff homology in degree zero for any open polydisk U of $\mathbb{C}^n$, see Frunză (1975). Above z denotes the n-tuple of complex coordinates on $\mathbb{C}^n$. In fact, the latter definition has a coordinateless expression as follows: the Fréchet $O(\mathbb{C}^n)$-modules X and $O(U)$ are topologically transversal for every open polydisc U. That means, following Taylor (1972b), that the topological tensor product $O(U)\widehat{\otimes}_{O(\mathbb{C}^n)}X$ is Hausdorff in the natural quotient topology and its derived functors vanish. The joint spectrum of the system T is defined as the set on whose complement the same transversality occurs and in addition the above tensor product vanishes, see Taylor (1972a).

Let X be a Banach $O(\mathbb{C}^n)$-module with property (β). Then its associated sheaf model is, similarly to the single variable case

$$\mathcal{F}(U) = O(U) \,\widehat{\otimes}_{O(\mathbb{C}^n)}\, X = O(U, X)/\Big(\sum_{i=1}^{n} (T_i - z_i) O(U, X)\Big),$$

for every open polydisk U. This sheaf has a remarkable topologically free resolution given by the Koszul complex

$$K.(T-z,O^X)\to\mathcal{F}\to 0.$$

Thus $\mathcal{F}$ is a so-called Fréchet quasi-coherent analytic module. (The existence of a topologically free resolution for a Fréchet analytic sheaf is a possible definition of the quasi-coherence property; the original, equivalent, definition is explained in Putinar (1986).) Under the assumption (β) the conclusions of Proposition 3 remain all valid and in this way the sheaf $\mathcal{F}$ reflects the properties of the n-tuple of multiplication with the coordinate functions on the Banach module X.

In the multivariable setting the multivariable generalization of a decomposable operator is suggested by Theorem 5. Once this definition is accepted (i.e. a decomposable n-tuple is characterized by the existence of a Fréchet soft sheaf model) all properties of the decomposable operators extend to the case of commutative tuples of operators. (See for full details Eschmeier and Putinar (1996).)

Theorem 2 has in its turn some interesting analogues in several variables. It is the aim of the rest of this section to state a few results in this direction. First the construction from Section 5 of a universal dilation of an operator has the following straightforward generalization.

Let X be a Banach $O(\mathbb{C}^n)$-module and let U be a Stein open set of $\mathbb{C}^n$ which contains the joint spectrum of X. Then the augmented Koszul complex

$$K.(T-z,O(U,X))\to X\to 0$$

is exact (see Taylor (1969)). Moreover, the augmented Dolbeault complex

$$0\to O(U,X)\to E^{(0,0)}(U,X)\to\ldots\to E^{(0,n)}(U,X)\to 0$$

is exact. So, following Section 5 the total complex $C.$ attached to the double complex

$$K.(T-z,(E^{(0,.)}(U,X),\overline{\partial}^{(0,.)}))$$

is exact except in zero degree, where its homology is isomorphic to X. In particular, as in Theorem 3, there is a resolution to the right of the module X:

$$0\to X\to S_+\to D_1\to\ldots\to D_n\to 0,$$

where D_i $(1\le i\le n)$ are E-modules, hence decomposable modules in the above terminology. Moreover, one can change the Fréchet spaces appearing in this resolution to Hilbert spaces (of corresponding square summable functions). By repeating the proof of Theorem 3 in that case one obtains the equivalence of property (β) for X and the decomposability of the module S_+. See for details Putinar (1990).

The latter remarks allow us to follow Section 5 in constructing universal dilations for n-tuples of commuting operators. As a matter of fact a more invariant characterization of property (β) is available as a byproduct of the existence of the universal dilation.

THEOREM 6. *A Banach module over the algebra of entire functions in n variables has property (β) if and only if it has a finite resolution to the right with Banach soft modules.*

The proof of Theorem 6 is contained in Putinar (1990). A series of applications of this result appear in Eschmeier and Putinar (1996). At the level of analytic sheaves the preceding result has the following consequence.

Corollary. *Let $\mathcal{F}$ be a Fréchet analytic sheaf on a Stein manifold M. Suppose that the cohomology of $\mathcal{F}$ vanishes on M. Then the following conditions are equivalent*:

(a) *$\mathcal{F}$ is quasi-coherent*;

(b) *$\mathcal{F}$ admits on M a finite resolution to the right with Fréchet soft analytic modules*;

(c) *There is a finite complex of Fréchet E-modules on M which is exact except in a single degree*, *where its homology is isomorphic to $\mathcal{F}$.*

Thus a subdecomposable operator has an appropriate multidimensional analogue in an analytic module which admits a resolution to the right with decomposable modules. Analogously, to the class of subgeneralized scalar operators there corresponds the set of analytic modules with a right resolution with E-modules. See Putinar (1990) for the precise statements.

The typical example of an analytic module with property (β) is the Bergman space of a bounded pseudoconvex domain Ω in $\mathbb{C}^n$. More exactly, in that case the Dolbeault complex with coefficients square integrable functions on Ω is exact by a well known result of Hörmander (1965). Thus the Bergman space is localizable by a Fréchet quasi-coherent analytic sheaf. The investigation of this analytic module explains several known results of spectral analysis for analytic Toeplitz operators on Bergman spaces. See Eschmeier and Putinar (1996) for full details; the next results are reproduced from that book and they illustrate the versatility of the sheaf model approach in spectral analysis problems.

Let Ω be a bounded strictly pseudoconvex domain of $\mathbb{C}^n$ with smooth boundary. Let $A^2(\Omega)$ denote the corresponding *Bergman space*, that is the space of analytic functions in Ω which are square summable with respect to the volume Lebesgue measure in the same domain. Two general problems of operator theory on that space are the computation of (joint) spectra of Toeplitz operators and the classification of analytic invariant subspaces. Although both problems have recently been investigated by various methods, they remain far from being completely solved. Next we list a few results obtained with the sheaf model of the Bergman space as a main tool.

1. *Let $f = (f_1, \ldots, f_m)$ be an m-tuple of uniformly bounded analytic functions in Ω and let λ be a point disjoint of $\overline{f(\Omega)}$. Then the Koszul complex $K.(f(z) - \lambda, A^2(\Omega))$ is exact.*

This result is a weak analogue in several complex variables of the Corona Problem approach mentioned in Section 6. Whether Corona Problem has a positive solution on classical domains of $\mathbb{C}^n$ is not yet known.

2. *Let $S \subset A^2(\Omega)$ be a finite-codimensional subspace of the Bergman space which is invariant under multiplication with bounded analytic functions in Ω. Then there are polynomials $P_1, \ldots, P_k$ with finitely many common zeroes inside Ω such that $S = P_1 A^2(\Omega) + \ldots + P_k A^2(\Omega)$.*

3. *Let M be a complex submanifold of an open neighbourhood of $\overline{\Omega}$ which intersects transversally the boundary of Ω. Let $f_1, \ldots, f_k$ be the generators of the radical ideal of M. If an element $F \in A^2(\Omega)$ vanishes on $\Omega \cap M$, then it belongs to $f_1 A^2(\Omega) + \ldots + f_k A^2(\Omega)$.*

The previous statement is an analogue with L^2-estimates of the classical Nullstellensatz in complex geometry. In particular the space

$$V^2(M) = \{f \in A^2(\Omega); f|\Omega \cap M = 0\} = f_1 A^2(\Omega) + \ldots + f_k A^2(\Omega),$$

is an analytically invariant closed subspace of the Bergman space.

4. *Let I and J be two ideals of analytic functions defined in a neighbourhood of $\overline{\Omega}$. Assume that I and J are topologically transversal to the Bergman space $A^2(\Omega)$ over the algebra of anaytic functions on $\overline{\Omega}$ and that the zeroes of I and J are analytic sets of codimension greater than* 1 *inside Ω. Then the* (*automatically closed*) *invariant subspaces $I.A^2(\Omega)$ and $J.A^2(\Omega)$ are isomorphic as analytic Hilbert modules if and only if $I = J$.*

The latter result is a typical rigidity phenomenon in the classification of analytic invariant subspaces. Details about the origins of this phenomenon and a series of generic examples can be found in Douglas and Paulsen (1989).

References

J. Agler (1985), *Rational dilation in an annulus*, Ann. of Math. 121, 537–564.

E. Albrecht (1978), *On two questions of Colojoară and Foiaş*, Manuscripta Math. 25, 1–15.

E. Albrecht and J. Eschmeier (1987), *Functional models and local spectral theory*, preprint.

Yu. M. Berezanskiĭ (1968), *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc. Transl., Providence, R.I.

E. Bishop (1959), *A duality theorem for an arbitrary operator*, Pacific J. Math. 9, 379–394.

I. Colojoară and C. Foiaş (1968), *Theory of Generalized Scalar Operators*, Gordon and Breach, New York.

P. A. M. Dirac (1930), *The Principles of Quantum Mechanics*, Clarendon Press, Oxford.

R. G. Douglas and C. Foiaş (1976), *A homological view in dilation theory*, preprint.

R. G. Douglas and V. Paulsen (1989), *Hilbert modules over function algebras*, Pitman Res. Notes in Math. 219, Harlow.

N. Dunford and J. Schwartz, Linear Operators, p. I. (1958), p. II. (1963), p. III. (1971), Wiley-Interscience, New York.

E. M. Dynkin (1972), *Functional calculus based on Cauchy–Green's formula*, in: Research in linear operators and function theory. III, Leningrad Otdel. Mat. Inst. Steklov, 33–39 (in Russian).

J. Eschmeier (1985), *Spectral decompositions and decomposable multipliers*, Manuscripta Math. 51, 201–224.

J. Eschmeier and B. Prunaru (1990), *Invariant subspaces for operators with Bishop's property* (β) *and thick spectrum*, J. Funct. Anal. 94, 196–222.

J. Eschmeier and M. Putinar (1984), *Spectral theory and sheaf theory. III*, J. Reine Angew. Math. 354, 150–163.

J. Eschmeier and M. Putinar (1988), *Bishop's condition* (β) *and rich extensions of linear operators*, Indiana Univ. Math. J. 37, 325–348.

J. Eschmeier and M. Putinar (1989), *On quotients and restrictions of generalized scalar operators*, J. Funct. Anal. 84, 115–134.

J. Eschmeier and M. Putinar (1996), *Spectral Decompositions and Analytic Sheaves*, Oxford University Press, Oxford.

C. Foiaş (1963), *Spectral maximal spaces and decomposable operators in Banach spaces*, Arch. Math. (Basel) 14, 341–349.

S. Frunză (1975), *The Taylor spectrum and spectral decompositions*, J. Funct. Anal. 19, 390–421.

T. Gamelin (1970), *Localization of the corona problem*, Pacific J. Math. 34, 73–81.

J. B. Garnett (1981), *Bounded Analytic Functions*, Academic Press, New York.

I. Gohberg and M. G. Krein (1969), *Introduction to the Theory of Linear Non-selfadjoint Operators*, Transl. Math. Monographs 18, Amer. Math. Soc., Providence, R.I.

G. Henkin and J. Leiterer (1984), *Theory of Functions on Complex Manifolds*, Birkhäuser, Basel-Boston-Stuttgart.

L. Hörmander (1965), *L^2-estimates and existence theorems for the $\overline{\partial}$-operator*, Acta Math. 113, 89–152.

R. Lange (1981), *A purely analytic criterion for a decomposable operator*, Glasgow Math. J. 21, 69–70.

R. N. Levy (1987), *The Riemann–Roch theorem for complex spaces*, Acta Math. 158, 149–188.

Yu. I. Lyubich and V. I. Matsaev (1962), *Operators with separable spectrum*, Mat. Sb. 56, 433–468 (in Russian).

S. MacLane (1963), *Homology*, Springer, Berlin.

M. Martin and M. Putinar (1989), *Lectures on Hyponormal Operators*, Birkhäuser, Basel.

M. Putinar (1983), *Spectral theory and sheaf theory. I*, in: Dilation Theory, Toeplitz Operators and Other Topics, Birkhäuser, Basel.

M. Putinar (1985), *Hyponormal operators are subscalar*, J. Operator Theory 12, 385–395.

M. Putinar (1986), *Spectral theory and sheaf theory. II*, Math. Z. 192, 473–490.

M. Putinar (1990), *Spectral theory and sheaf theory. IV*, in: Proc. Sympos. Pure Math. 51, part 2, Amer. Math. Soc., 273–293.

M. Putinar (1992), *Quasi-similarity of tuples with Bishop's property (β)*, Integral Equations Operator Theory 15, 1040–1052.

M. Radjabalipour (1978), *Decomposable operators*, Bull. Iranian Math. Soc. 9, 1–49.

L. Schwartz (1955), *Division par une fonction holomorphe sur une variété analytique complexe*, Summa Brasil. Math. 3, 181–209.

B. Sz.-Nagy and C. Foiaş (1967), *Analyse harmonique des opérateurs de l'espace de Hilbert*, Akad. Kiadó, Budapest.

J. L. Taylor (1970), *A joint spectrum for several commuting operators*, J. Funct. Anal. 6, 172–191.

J. L. Taylor (1972a), *Homology and cohomology for topological algebras*, Adv. in Math. 9, 147–182.

J. L. Taylor (1972b), *A general framework for a multioperator functional calculus*, Adv. in Math. 9, 184–252.

F. H. Vasilescu (1982), *Analytic Functional Calculus and Spectral Decompositions*, Reidel, Dordrecht.