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ON THE GROWTH OF THE RESOLVENT OPERATORS FOR POWER BOUNDED OPERATORS

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Outline. In this paper I discuss some quantitative aspects related to power bounded operators T and to the decay of $T^n(T-1)$. For background I refer to two recent surveys J. Zemánek [1994], C. J. K. Batty [1994]. Here I try to complement these two surveys in two different directions.

First, if the decay of $T^n(T-1)$ is as fast as $O(1/n)$ then quite strong conclusions can be made. The situation can be thought of as a discrete version of analytic semigroups; I try to motivate this in Section 1 by demonstrating the similarity and lack of it between power boundedness of T and uniform boundedness of $e^{t(cT-1)}$ where c is a constant of modulus 1 and $t > 0$. Section 2 then contains the main result in this direction. I became interested in studying the quantitative aspects of the decay of $T^n(T-1)$ since it can be used as a simple model for what happens in the early phase of an iterative method (O. Nevanlinna [1993]).

Secondly, the so called Kreiss matrix theorem relates bounds for the powers to bounds for the resolvent. The estimate is proportional to the dimension of the space and thus has as such no generalization to operators. However, qualitatively such a result holds in Banach spaces e.g. for Riesz operators: if the resolvent satisfies the resolvent condition, then the operator is power bounded (but without an estimate). I introduce in Section 3 a growth function for bounded operators. This allows one to obtain a result of the form: if the resolvent condition holds and if the growth function is finite at 1, then the powers are bounded and can be estimated. In Section 4 in addition to the Kreiss matrix theorem, two other applications of the growth function are given.

1. Power bounded operators. We consider operators T which are *power bounded* (for positive n):

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The paper is in final form and no version of it will be published elsewhere.

$$\|T^n\| \le C \quad \text{for } n \ge 0. \tag{1.1}$$

In numerical analysis one often faces the following situation: we actually have a family of operators $\{T_h\}$ where h could be for example a small discretization parameter. Then one is typically interested in knowing whether

$$\|T_h^n\| \le C$$

holds for all $n \ge 0$ with a constant C independent of h. We see immediately from the power series representation of the resolvent that (1.1) implies

$$\|(\lambda - T)^{-1}\| \le \frac{C}{|\lambda| - 1} \quad \text{for } |\lambda| > 1. \tag{1.2}$$

This is often much easier to check than (1.1) which one really would need to know. It is natural to ask when, if ever, we can conclude an inequality of the form (1.1) based on knowing the constant C in (1.2).

A simple application of the Cauchy integral formula together with (1.2) gives

$$\|T^n\| \le Ce(n+1) \quad \text{for } n \ge 0. \tag{1.3}$$

Example 1.1. The linear growth can really be there. In fact, if X is the space of functions f analytic in the open disc such that f' is in the Hardy space H^1, then f is continuous at the boundary and we can put

$$\|f\| := |f|_\infty + |f'|_1.$$

Let $T := M_z$ denote the multiplication by the variable. Then $\|T^n\| = n + 1$ while the resolvent condition (1.2) holds (see Theorem 1 and Proposition 3 in A. L. Shields [1978], the constant $C = 3/2$ will do).

Differentiating the resolvent $n - 1$ times one concludes that (1.1) implies actually the following stronger condition:

$$\|(\lambda - T)^{-n}\| \le \frac{C}{(|\lambda| - 1)^n} \quad \text{for } n \ge 0. \tag{1.4}$$

Now the "gap" is narrower but it is still there: we do not get the power boundedness back from (1.4) as (1.4) only implies

$$\|T^n\| \le C\sqrt{2\pi(n+1)} \quad \text{for } n \ge 0, \tag{1.5}$$

see Ch. Lubich and O. Nevanlinna [1991], C. A. McCarthy [1971], O. Nevanlinna [1993], A. L. Shields [1978]. For continuous semigroups there is no such gap. In fact, the Hille–Yosida theorem says that

$$\|e^{tA}\| \le C \quad \text{for } t > 0$$

holds if and only if

$$\|(\lambda - A)^{-n}\| \le \frac{C}{(\Re\lambda)^n} \qquad \text{for } n \ge 0,\ \Re\lambda > 0$$

(it suffices for us to consider bounded A only). We can actually see the "origin" of the gap in the discrete case from this characterization. Clearly, (1.1) implies, using the power series expansion, for all complex z,

$$\|e^{zT}\| \le Ce^{|z|}. \tag{1.6}$$

Put $z = te^{i\theta}$ and write (1.6) as

$$\|e^{t(e^{i\theta}T-1)}\| \le C \quad \text{for } t > 0, \text{ and for all } \theta. \tag{1.7}$$

By the Hille–Yosida theorem we see that (1.7) is equivalent with (1.4) and therefore the "gap" is between (1.1) and (1.6). However, this is an "old gap" between the growth of an entire function and the decay of its Taylor coefficients. In fact, we have

$$\frac{1}{n!}T^n = \frac{1}{2\pi i}\int\limits_{|z|=n} z^{-n-1}e^{zT}\,dz$$

and thus (1.6) and Stirling's approximation give again (1.5). To summarize, we have:

Proposition 1.1. *If T is a bounded linear operator in a Banach space, then*

(i) (1.1) *implies both* (1.4) *and* (1.6);
(ii) (1.4) *and* (1.6) *are equivalent*;
(iii) *both* (1.4) *and* (1.6) *imply* (1.5).

Example 1.2. In Ch. Lubich and O. Nevanlinna [1991] it is demonstrated (based on Ph. Brenner, V. Thomée and L. Wahlbin [1975]) that if S denotes the shift in l_∞ and if ϕ is a Möbius mapping taking the unit disc onto itself but which is not just a rotation, then $T := \phi(S)$ satisfies (1.4) but $\|T^n\| > c\sqrt{n}$ for a positive c.

There are results connecting the growth of the entire function to the decay of its Taylor coefficients, where the "gap" is much smaller than $O(\sqrt{n})$. A sufficient condition is $f(r) = M(r)(:= \max_{|z|=r}|f(z)|)$ for all large r, see W. Hayman [1956].

It is possible to characterize power bounded operators in terms of the resolvent, but it comes close to a tautology. Denote by $Y(\lambda, T)$ the Yosida approximation of T:

$$Y(\lambda, T) := \lambda T(\lambda - T)^{-1}$$

(for λ outside the spectrum). Knowing $Y(\lambda, T)$ is mathematically equivalent of knowing the resolvent as

$$(\lambda - T)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda^2}Y(\lambda, T).$$

Now the following holds:

Proposition 1.2. *The following are equivalent*:

(i) $$\|T^n\| \le C \quad \textit{for } n = 1, 2 \ldots$$

(ii) $$\|Y(\lambda, T)^n\| \le \frac{C}{(1 - 1/|\lambda|)^n} \quad \textit{for } n \ge 1, |\lambda| > 1.$$

(This is Theorem 2.7.1 in O. Nevanlinna [1993]. When writing the book I did not know that essentially the same observation had been published earlier, A. G. Gibson [1972].)

If we replace the power boundedness by a stronger requirement

$$\sum_{n=1}^{\infty}\|T^n\| \le B, \tag{1.8}$$

then it follows trivially that

$$\|(\lambda - T)^{-1}\| \le B + 1 \quad \text{for } |\lambda| \ge 1. \tag{1.9}$$

Unlike the resolvent condition (1.2), (1.9) is robust under small perturbations of T. Consider the operator $T_\theta := \theta M_z$ where $\theta < 1$ and M_z is the operator in Example 1.1. Then, as $\theta \to 1$, (1.9) holds with constant $B = O(1/(1-\theta))$ while the constant in (1.8) is of the form $O(1/(1-\theta)^2)$. Here is a general result (slightly stronger than what was included in Ch. Lubich and O. Nevanlinna [1991]).

PROPOSITION 1.3. *If* (1.9) *holds*, *then*

$$\sum_{1}^{\infty} \|T^n\| \le 6B(B+1). \tag{1.10}$$

Proof. By the "Banach Lemma" $\lambda - T$ is invertible for $|\lambda| > \frac{B}{B+1}$ and we have

$$\|(\lambda - T)^{-1}\| \le \frac{1}{|\lambda| - \frac{B}{B+1}} \quad \text{for } \frac{B}{B+1} < |\lambda| \le 1. \tag{1.11}$$

Since

$$T^n = \frac{1}{2\pi i} \int\limits_{|\lambda|=R} \lambda^n (\lambda - T)^{-1} d\lambda$$

we obtain

$$\|T^n\| \le B + 1$$

for all n ($R = 1$) and in particular for $n \ge B$ (with $R = \frac{B(n+1)}{(B+1)n}$),

$$\|T^n\| \le e(n+1)\left(\frac{B}{B+1}\right)^n \ (^1).$$

Summing these implies (1.10).

We end this introduction with a characterization of the resolvent condition (1.2).

PROPOSITION 1.4. *If the second order Cesàro averages are uniformly bounded*, *for all* n *angles* ϕ:

$$\left\| \frac{2}{(n+1)(n+2)} \sum_{0}^{n} (n+1-k)(e^{i\phi}T)^k \right\| \le C \tag{1.12}$$

then the resolvent condition (1.2) *holds. Conversely*, *if* (1.2) *holds*, *then* (1.12) *follows with the constant* C *replaced by* $5C$.

For a proof see a paper by J. C. Strikwerda and B. A. Wade [1991], which contains a detailed discussion of the Kreiss matrix theorem and other related questions (²).

2. Sublinear decay of $T^n(T-1)$. Here we consider estimates of the form

$$\|T^n(T-1)\| \le \frac{M}{n+1} \quad \text{for } n \ge 0. \tag{2.1}$$

We call this *sublinear decay with exponent* (at least) 1. By studying

$$\sup_{\lambda \in \sigma(T)} |\lambda^n(\lambda - 1)|$$

(¹,²) *Editorial note*: See also the papers by J. A. van Casteren, and J. C. Strikwerda and B. A. Wade in this volume.

one concludes that there exists $\delta > 0$ such that

$$\sigma(T) \cap \Omega_\delta = \emptyset, \tag{2.2}$$

where

$$\Omega_\delta = \{\lambda \mid \lambda \neq 1, |\arg(\lambda - 1)| < \pi/2 + \delta\} \tag{2.3}$$

(see Theorem 4.5.4 in O. Nevanlinna [1993]). Furthermore, if Γ denotes the unit circle, then

$$\sigma(T) \cap \Gamma \subset \{1\} \tag{2.4}$$

is for power bounded operators equivalent with the decay $\|T^n(T-1)\| \to 0$ (Y. Katznelson and L. Tzafriri [1986]).

THEOREM 2.1. *Assume that T is a power bounded linear operator in a Banach space and that* (2.4) *holds. Then the following are equivalent*:

(i) *There exists $M < \infty$ such that* (2.1) *holds.*

(ii) *There exists $K < \infty$ such that*

$$\|(T-1)e^{t(T-1)}\| \le K \frac{1-e^{-t}}{t} \quad \textit{for } t > 0. \tag{2.5}$$

(iii) *There exists $K < \infty$ such that*

$$\|(T-1)(\lambda - T)^{-(n+1)}\| \le \frac{K}{n}\left[\frac{1}{(\lambda-1)^n} - \frac{1}{\lambda^n}\right] \quad \textit{for } n \ge 1,\ \lambda > 1. \tag{2.6}$$

(iv) *There exist $B < \infty$ and $\delta > 0$ such that*

$$\|(\lambda - T)^{-1}\| \le \frac{B}{|\lambda - 1|} \quad \textit{for } \lambda \in \Omega_\delta, \tag{2.7}$$

where Ω_δ is given in (2.3).

Proof. (i) implies (ii): For $t > 0$,

$$\|(T-1)e^{tT}\| \le \sum_0^\infty \|T^n(T-1)\| \frac{t^n}{n!} \le \frac{M}{t} \sum_1^\infty \frac{t^n}{n!} = M\frac{e^t - 1}{t}.$$

Observe in particular that if M and K are the smallest constants, then $K \le M$.

(ii) and (iii) are equivalent (with the same constant K): For $n \ge 1$, $\lambda > 1$ we have

$$(\lambda - T)^{-(n+1)} = \frac{1}{n!}\int_0^\infty t^n e^{-(\lambda-1)t} e^{t(T-1)}\, dt.$$

Multiplying with $T - 1$ and using (2.5) this gives (2.6). Conversely, substituting $\lambda := (n+1)/t$ into

$$\|(T-1)(1 - T/\lambda)^{-(n+1)}\| \le \frac{K\lambda}{n}\left[\frac{1}{(1-1/\lambda)^n} - 1\right]$$

and letting $n \to \infty$ gives (2.5).

(ii) implies (iv): Since T is power bounded, say $\|T^n\| \le C$ for $n \ge 0$, then

$$\|e^{tT}\| \le C \sum \frac{t^n}{n!} = Ce^t$$

and therefore

$$\|e^{t(T-1)}\| \le C \quad \text{for } t > 0.$$

But then we can estimate, with the help of (2.5), $\|e^{z(T-1)}\|$ uniformly inside a sector around the positive axis $t > 0$. This further allows one to change the integration path in

$$(\lambda - T)^{-1} = \int_0^\infty e^{-(\lambda-1)t} e^{t(T-1)}\, dt, \qquad \lambda > 1,$$

to another ray $z = re^{i\theta}$ for small enough θ, eventually leading to an estimate of the form (2.7). This part is exactly the same as in showing the corresponding statement for uniformly bounded analytic semigroups. For details, see the steps: (d) implies (a) implies (b) implies (c) in A. Pazy [1983], pp. 62–63.

(iv) implies (i): This part is given in Theorem 4.5.4 in O. Nevanlinna [1993] and the proof amounts to carrying out a contour integration.

Remark 2.1. It was shown in O. Nevanlinna [1993] that (2.7) and (2.4) imply that T is power bounded. The proof above shows that for the smallest constants we have $K \le M$. In general the constant M depends on the behavior of the spectrum near Γ away from 1 and in particular, it is possible that $K \ll M$ (for the scalar operator $T = -(1-\epsilon)$, $K = 2 - \epsilon$ while $M = O(\frac{1}{\epsilon})$).

Remark 2.2. The constant K (and thus M, too) cannot be arbitrarily small. Consider the Taylor series for

$$f(z) := (T-1)e^{z(T-1)}$$

expanded at $f(t)$, $t > 0$. Since $f^{(n)}(z) = f(\frac{z}{n+1})^{n+1}$ we obtain with the help of Stirling's approximation

$$\|f(z) - f(t)\| \le Ke \sum_1^\infty (Ke|1 - z/t|)^n,$$

which shows that if $Ke < 1$ then $f(z)$ is uniformly bounded on the whole plane. By Liouville's theorem f must therefore be constant. On the other hand, along the positive real axis f tends to 0 by (2.5), which implies that the constant must be 0. Finally, multiplying $f(z)$ by $e^{-z(T-1)}$ shows that $T - 1 = 0$. Summarizing: if (2.5) holds with $K < 1/e$, then $T = 1$.

Remark 2.3. Let us put

$$M_1 := \limsup_{n\to\infty} (n+1)\|T^n(T-1)\|. \tag{2.8}$$

Then, if 1 is an accumulation point of $\sigma(T)$, we always have

$$M_1 \ge 1/e.$$

This is in Theorem 4.5.1 in O. Nevanlinna [1993] and it follows immediately from the spectral mapping theorem. Replacing the limsup in (2.8) by liminf we obtain another constant, say M_2, and $M_2 \le M_1$. By arranging long enough "spectral gaps" (say for a diagonal operator) we can have 1 as an accumulation point of $\sigma(T)$ and thus $M_1 \ge 1$ while $M_2 = 0$.

Remark 2.4. If 1 is not in $\sigma(T)$ (but (2.1) holds), then clearly T^n decays linearly and in particular $M_1 = 0$. Finally, if 1 is an isolated point of $\sigma(T)$, then by spectral projections we can remove the linearly decaying part and assume directly that $\sigma(T) = \{1\}$. But then it is known (M. Berkani [1983]) that if $T - 1$ is quasinilpotent but nonzero, then

$$M_2 = \liminf_{n\to\infty}(n+1)\|T^n(T-1)\| \ge 1/12.$$

Here the proof is based on estimating discs inside which the functions $g(x) = x(1-x)^n$ map univalently. It was conjectured in M. Berkani [1983] that the best constant would again be $1/e$ instead of $1/12$.

Remark 2.5. Assume that all singularities of the resolvent operator $(\lambda - T)^{-1}$ on the unit circle Γ are poles. Then if T is power bounded, these poles must be simple. In fact, if T is power bounded, then the resolvent satisfies a condition of the form

$$\|(\lambda - T)^{-1}\| \le \frac{C}{|\lambda| - 1} \quad \text{for } |\lambda| > 1$$

from which the claim follows. We conclude that if $T^n(T-1)$ tends to 0 then a resolvent condition of the form (2.7) holds, which implies (2.1). However, if P is the Riesz spectral projector for the possible eigenvalue 1, we see that since the pole is simple we actually have $(T-1)P = 0$. But then $T^n(T-1) = T^n(T-1)(1-P)$ and the only contribution comes from the operator $(1-P)T(1-P)$, which however has spectral radius strictly less than 1.

Remark 2.6. What (essentially) remains is the case where $T - 1$ is quasinilpotent, T is power bounded but $T \ne 1$. By the theorem of J. Esterle [1983] we always have $T^n(T-1) \to 0$. Recall that A. Atzmon [1980] has shown that if

$$\limsup_{n\to\infty} n\|(T-1)^n\|^{1/n} = 0$$

then $T = 1$. This means that for a nontrivial operator the resolvent must have a growth at least of first order and positive type. There are a lot of such operators, e.g. $(1+V)^{-1}$ where V is the standard Volterra operator $L_2[0,1]$. We can conclude from Theorem 2.1 that for such operators the decay of $T^n(T-1)$ must be slower than of the form $O(\frac{1}{n+1})$. The conclusion holds actually for all operators with first order growth, independently whether the type is finite or not. We formulate it as a separate theorem.

THEOREM 2.2. *Let T be a power bounded operator which satisfies* (2.1). *Either $T = 1$ or there exists $\omega > 1$ such that*

$$\limsup_{n\to\infty} n^{1/\omega}\|(T-1)^n\|^{1/n} = \infty. \tag{2.9}$$

Proof. If $T - 1$ is not quasinilpotent, then

$$\limsup_{n\to\infty} \|(T-1)^n\|^{1/n} > 0$$

and (2.9) holds for all positive ω. Assume therefore that $T-1$ is quasinilpotent. Consider the function $f(\lambda) := (\lambda - 1)(\lambda - T)^{-1}$. By Theorem 2.1 there exists a sectorial set Ω_δ such that f is there uniformly bounded. Changing the variable $z := 1/(\lambda - 1)$ we obtain a function to which we can apply the Phragmén–Lindelöf technique and conclude that either the function is bounded in the whole z-plane or it is entire of order greater than 1. The latter implies (2.9) while the former implies that f is constant which then gives $T = 1$.

Remark 2.7. A. Atzmon [1980] has shown that there is a correspondence between the growth of the resolvent near 1 and between the growth of the negative powers of T. In particular, the borderline first order growth of the resolvent corresponds to growth of order $e^{\sqrt{n}}$ of the powers T^{-n}.

We are thus led to consider slower sublinear decays than $O(\frac{1}{n+1})$. There is a simple result for operators which are convex combinations of identity and a power bounded operator: the exponent is at least 1/2.

THEOREM 2.3. *Let T be power bounded and put, for $0 < \alpha < 1$, $T_\alpha := 1 - \alpha + \alpha T$. Then T_α is power bounded and*

$$\limsup_{n\to\infty} \sqrt{n+1}\|T_\alpha^n(T_\alpha - 1)\| < \infty.$$

Proof. This is Theorem 4.5.3 in O. Nevanlinna [1993]. The proof follows from asymptotic estimates for the terms in the sum

$$[(1-\alpha)+\alpha]^n = \sum \frac{n!}{j!(n-j)!}\alpha^j(1-\alpha)^{n-j}.$$

Such estimates have been used in passing from binomial to normal distributions, see W. Feller [1968].

3. A growth function for operators. In order to be able to use function theory effectively it is necessary to have tools which allow us to "discard" poles of the resolvent. To that end, let us recall that an isolated point $\lambda_0 \in \sigma(T)$ is a *pole* of the resolvent if and only if there exists a Laurent expansion of the form

$$(\lambda - T)^{-1} = \sum_{k=-m}^{\infty} A_k(\lambda - \lambda_0)^k.$$

Here the "coefficients" are bounded linear operators, and the pole is said to be of *order* m if $A_m \neq 0$.

DEFINITION 3.1. We put

$$\rho_m := \inf r$$

where the infimum is over such $r \geq 0$ for which $(\lambda - T)^{-1}$ is either regular or all singularities are poles for $|\lambda| > r$.

Remark 3.1. If the resolvent only has poles (then there are necessarily only finitely many of them since T is bounded), one can write the resolvent as a "rational function"

$$(\lambda - T)^{-1} = \frac{1}{q(\lambda)}\sum_{j=0}^{n-1} q_{n-1-j}(T)\lambda^j \tag{3.1}$$

where q is any polynomial such that $q(T) = 0$, $q(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ and $q_j(\lambda) := \lambda q_{j-1}(\lambda) + a_j$ for $0 < j < n$. Such operators are called *algebraic*, see Sections 2.10 and 5.7 in O. Nevanlinna [1993]. If $Q(T) \neq 0$ for all polynomials of degree $< n$, then q is called the *minimal polynomial of* T, and we say that T is algebraic of degree n: $\deg T = n$. For example, scalars are of degree 1, proper projections of degree 2, Fourier transform in $L_2(-\infty,\infty)$ of degree 4, and operators in d-dimensional spaces of degree at most d.

Remark 3.2. More generally, the case $\rho_m = 0$ can be easily characterized. In O. Nevanlinna [1993] we defined T to be *almost algebraic* if there exists a sequence $\{a_j\}_1^\infty$ of complex numbers so that

$$\|Q_j(T)\|^{1/j} \to 0 \quad \text{as } j \to \infty \quad \text{where } Q_j(\lambda) := \lambda^j + a_1\lambda^{j-1} + \ldots + a_j. \tag{3.2}$$

Theorem 5.7.2 in O. Nevanlinna [1993] says that $\rho_m = 0$ if and only if T is almost algebraic. For example, compact operators and more generally, Riesz operators are almost algebraic.

Consider now a resolvent $(\lambda - T)^{-1}$ for $|\lambda| > \rho_m$. Given $s > \rho_m$, let P_s be the Riesz spectral projection corresponding to all poles λ_j such that $|\lambda_j| \ge s$. Thus $T_s := TP_s$ is algebraic of degree $\sum_{|\lambda_j|\ge s} m_j$ where m_j is the order of the pole. Furthermore, by construction

$$\lambda[(\lambda - T)^{-1} - (\lambda - T_s)^{-1}]$$

is uniformly bounded for $|\lambda| \ge s$. Thus

$$\deg T_s + \sup_{|\lambda|>r} \|\lambda[(\lambda - T)^{-1} - (\lambda - T_s)^{-1}]\| < \infty \quad \text{for } s \le r. \tag{3.3}$$

Taking the infimum of this over all $s \in (\rho_m, r]$ would give us a "growth function". In this case the meromorphic resolvent would be approximated by a rational resolvent, and these would even commute. In the application we have in mind we do not need the commutation and we take the infimum over a larger set. Put

$$m(r, T, A) := \sup_{|\lambda|>r} \|\lambda[(\lambda - T)^{-1} - (\lambda - A)^{-1}]\| \tag{3.4}$$

and then

$$g_T(r) := \inf_A \{\deg A + m(r, T, A)\}, \tag{3.5}$$

where the infimum is over all algebraic operators A.

Definition 3.2. We call g_T the *growth function* of T.

For every bounded T the growth function has the following simple properties:

Proposition 3.1. (i) $g_T(r) < \infty$ *and nonincreasing in* r *for* $r > \rho_m$,

(ii) $\lim_{r\to\infty} g_T(r) = 1$,

(iii) *if* $g_T(r_0) = 1$, *for some* r_0, *then* T *is a scalar multiple of identity and* $g_T(r) = 1$ *for all* $r \ge 0$.

Proof. (i) $g_T(r) < \infty$ follows from (3.3); each $m(r, T, A)$ is nondecreasing in $1/r$ and so is the infimum.

(ii) Since deg $A \ge 1$ for any A, $g_T(r) \ge 1$. With $A = 0$ we have deg $A = 1$ and

$$m(r, T, 0) \sim \|T\|/r \to 0 \quad \text{as } r \to \infty,$$

which implies the claim.

(iii) If $g_T(r_0) = 1$, then there are scalars $\{\alpha_k\}$ such that

$$\inf_k m(r_0, T, \alpha_k) = 0.$$

In particular this implies that α_k's are bounded and thus (passing to a subsequence) we may assume that $\alpha_k \to \alpha$. But then $m(r_0, T, \alpha) = 0$, which is possible only if $T = \alpha$.

PROPOSITION 3.2. *T is algebraic if and only if $g_T(0)$ is finite.*

Proof. If T is algebraic, then

$$g_T(r) \le \deg T + m(r, T, T) = \deg T$$

for all $r \ge 0$. Conversely, if $g_T(0)$ is finite, then there exists an algebraic A such that $m(0, T, A) < \infty$. But then by Liouville's theorem the function

$$\lambda \to \lambda[(\lambda - T)^{-1} - (\lambda - A)^{-1}]$$

must be constant and $T = A$.

For operators which are not algebraic but only almost algebraic (including all compact operators and Riesz operators) the growth function is not bounded at $r = 0$ and it is thus natural to consider the speed of its growth.

DEFINITION 3.3. If T is almost algebraic, then we associate with T a *growth order* ω:

$$g_T(r) = O(1/r^{\omega+\epsilon}) \quad \text{as } r \to 0$$

holds for all $\epsilon > 0$ but for none $\epsilon < 0$.

First we check that the concept "scales" naturally.

PROPOSITION 3.3. (a) *For $c \ne 0$, $g_{cT}(r) = g_T(r/|c|)$.*
(b) *$g_{T^2}(r^2) \le g_T(r)$.*

Proof. Since $\deg(cA) = \deg(A)$ the first claim follows from

$$\lambda[(\lambda - cT)^{-1} - (\lambda - cA)^{-1}] = \frac{\lambda}{c}\left[\left(\frac{\lambda}{c} - T\right)^{-1} - \left(\frac{\lambda}{c} - A\right)^{-1}\right].$$

To obtain the other claim we first show that

$$\deg A^2 \le \deg A. \tag{3.6}$$

To that end take a polynomial q such that $q(A) = 0$ and write it as $q(\lambda) = p_0(\lambda^2) + \lambda p_1(\lambda^2)$ so that $p_0(A^2) = -Ap_1(A^2)$. This gives

$$p_0(A^2)^2 - A^2 p_1(A^2)^2 = 0,$$

which implies (3.6). Next, write

$$\lambda^2(\lambda^2 - T^2)^{-1} = \frac{\lambda}{2}(\lambda - T)^{-1} + \frac{-\lambda}{2}(-\lambda - T)^{-1}$$

and conclude that $m(r^2, T^2, A^2) \le m(r, T, A)$. Together with (3.6) we obtain (b).

The inequality in (b) can be strict. In fact, if $T \ne 0$ but $T^2 = 0$ then $g_{T^2}(r^2) \equiv 1$ while $1 < g_T(r) \le 2$.

In the solution of linear operator equations by iterative methods the following problem shows up: knowing how fast $\|p_n(T)\|$ can decay corresponds to knowing the speed of the iteration such as conjugate gradient method, generalized minimal residual method etc, see O. Nevanlinna [1993]. Here p_n denotes a suitably normalized polynomial of exact degree n.

PROPOSITION 3.4. *Assume that T is almost algebraic with the growth function of order ω. Then there exists for all $\epsilon > 0$ a sequence $\{p_n\}$ of monic polynomials such that as $n \to \infty$,*

$$\|p_n(T)\|^{1/n} = O(1/n^{1/(\omega+\epsilon)}). \tag{3.7}$$

Proof. Given an algebraic operator A with minimal polynomial q of degree d put $p_n(\lambda) := \lambda^{n-d}q(\lambda)$ for $n > d$. Since $p_n(A) = 0$ we have

$$\begin{aligned}\|p_n(T)\| &= \left\| \frac{1}{2\pi i} \int\limits_{|\lambda|=r} \lambda^{n-d-1} q(\lambda)\lambda[(\lambda - T)^{-1} - (\lambda - A)^{-1}]d\lambda \right\| \\ &\le r^{n-d} \sup_{|\lambda|=r} |q(\lambda)| m(r, T, A).\end{aligned}$$

Here we may assume that A is such that $d \le g_T(r)$ and $m(r, T, A) \le g_T(r)$. Also, it is easy to see that zeros of q are bounded by $\|T\|$ and thus

$$\sup_{|\lambda|=r} |q(\lambda)| \le (\|T\| + r)^d.$$

We obtain

$$\|p_n(T)\| \le r^{n-g_T(r)}(\|T\| + r)^{g_T(r)} g_T(r) \tag{3.8}$$

which leads to the result if we choose $r = r_n$ suitably. To that end, let C be such that for all small enough r,

$$g_T(r) \le C\left(\frac{1}{r}\right)^{\omega+\epsilon}.$$

This suggests the choice

$$r_n := \left(\frac{C}{\epsilon n}\right)^{1/(\omega+\epsilon)},$$

which gives in particular $g_T(r_n) \le \epsilon n$. Substituting this into (3.8) completes the proof.

EXAMPLE 3.1. Consider a weighted shift T in l_2:

$$Te_j := \frac{1}{(j+1)^{1/\omega}} e_{j+1}, \quad j \ge 0, \text{ where } \omega > 0. \tag{3.9}$$

Thus,

$$\|T^n\| = \frac{1}{(n!)^{1/\omega}}$$

and $(\lambda - T)^{-1}$ is an entire function in $1/\lambda$ of order ω.

Let A_n be the "truncated" operator as follows: $A_n e_j := Te_j$ for $j < n - 1$ while $A_n e_j = 0$ for $j \ge n - 1$. Clearly, $\deg(A_n) = n$ and it is simple to check using power series representations for the resolvents that if we take $r_n := (Ce/n)^{1/\omega}$ with $C > 1$ then $m(r_n, T, A_n)$ stays uniformly bounded. In particular, for every $C > 1$ there exists $\widehat{C}$ such that for all $r > 0$,

$$g_T(r) \le \frac{Ce}{r^\omega} + \widehat{C}.$$

EXAMPLE 3.2. Let T be now the diagonal operator in l_2 with the same weights as in the previous example. Taking again A_n to be the natural truncated operator of degree n

one obtains that for every $C>1$ there exists a constant $\widehat{C}$ such that for all $r>0$,

$$g_T(r) \le \frac{C}{r^\omega} + \widehat{C}.$$

Here it is further trivial (as all eigenvalues have to be removed by the approximant) that

$$g_T(r) \ge \frac{1}{r^\omega}.$$

R e m a r k 3.3. It is natural to ask for a possible connection with the usual concepts in the theory of entire functions. To that end, let f be a scalar valued entire function. We put

$$G_f(r) := \inf_p \{\deg p + M(r, f-p)\} \tag{3.10}$$

where p is a polynomial and

$$M(r,f) := \sup_{|z|\le r} |f(z)|.$$

We shall see that the order ω of f agrees with the infimum of exponents α such that $G_f(r) = O(r^\alpha)$. We formulate the connection in such a way that we can read also the type τ from G_f.

PROPOSITION 3.5. *If G_f satisfies for all large enough r,*

$$G_f(r) \le C r^\omega, \tag{3.11}$$

then f is of order at most ω and if of exact order ω then the type τ of f satisfies $\tau \le C/e\omega$.

P r o o f. If $f(z) = \sum c_k z^k$, then the inequality

$$|c_k| \le \left(\frac{C}{k}\right)^{k/\omega} \tag{3.12}$$

for all large enough k implies the claim, while if for infinitely many n,

$$|c_n| \ge \left(\frac{C}{n}\right)^{n/\omega}, \tag{3.13}$$

then either the order is larger than ω or the type is at least $C/e\omega$, see e.g. R. P. Boas [1954].

For any n we have

$$G_f(r) \le n + \sum_{n+1}^{\infty} |c_k| r^k. \tag{3.14}$$

If (3.12) holds, take any $A>C$ and smallest integer n such that $n \ge A r^\omega$. Substitute these into (3.14) and obtain

$$G_f(r) \le A r^\omega + \frac{1}{1-(C/A)^{1/\omega}}.$$

Conversely, assume that (3.13) holds. Using Parseval's identity we have for some polynomial p_n such that $n = $deg $p_n \le G_f(r)$,

$$G_f(r) \ge M(r, f-p_n) \ge \left(\sum_{k>G_f(r)}^{\infty} |c_k|^2 r^{2k}\right)^{1/2}. \tag{3.15}$$

Fix $A < C$ and for every k put $r_k := (k/A)^{1/\omega}$. We show that assuming $G_f(r) \le Ar^\omega$ for all large enough r leads to a contradiction. Consider those n for which (3.13) holds. From (3.15) and (3.13) we have

$$n = Ar_n^\omega \ge G_f(r_n) \ge |c_n| r_n^n \ge (C/A)^{n/\omega},$$

which is a contradiction for n large enough. The proof can now be easily completed.

Remark 3.4. The applications of the growth in the next section are all technically of the following form. Take unit vectors x and x^* and put

$$f(\lambda) := \langle x^*, (\lambda - T)^{-1} x \rangle.$$

Then f becomes approximated with a rational function q originating similarly from the algebraic approximation of T. The results therefore hold as such if we replace g_T with the following growth function:

$$G_T(r) := \sup_{\|x\|=\|x^*\|=1} \inf_q [\deg q + \sup_{|\lambda|>r} |\lambda[f(\lambda) - q(\lambda)]|].$$

Clearly, $G_T(r) \le g_T(r)$. We shall explore this approach in more detail elsewhere. In particular, in separable Hilbert spaces and for compact operators K the growth function $G_K(r)$ can be estimated in terms of the singular values of K. If, for example, we assume that K is of trace class and $\|K\|_1$ denotes the sum of the singular values, then $g_T(1)$ in Theorem 4.1 can be replaced by $C_1(\|T\|_1 + 1)$, where C_1 is a universal constant.

4. Applications of the growth function. Our first application concerns the *Kreiss matrix theorem*. The theorem has quite a long history, see e.g. H.-O. Kreiss [1962], R. L. LeVeque and L. N. Trefethen [1984], M. N. Spijker [1991], J. L. M. van Doersselaer, J. F. B. M. Kraaijevanger and M. N. Spijker [1993] and in particular J. C. Strikwerda and B. A. Wade [1991]. The original version of the theorem listed several equivalent conditions for a family of matrices being power bounded with a uniform constant. The relations between the constant in different conditions have been greatly simplified and here J. C. Strikwerda and B. A. Wade [1991] is a good reference ([2]).

As said in the introduction, if T is power bounded, then the resolvent condition (1.2) or

$$\|(\lambda - T)^{-1}\| \le \frac{C}{|\lambda| - 1}, \quad |\lambda| > 1, \tag{4.1}$$

holds. Part of the Kreiss matrix theorem is the reverse implication. In fact, the sharpest form says that if (4.1) holds, then

$$\|T^n\| \le Ced \quad \text{for } n \ge 0 \tag{4.2}$$

where d is the dimension of the space. We generalize this to infinite-dimensional spaces. In the previous section we introduced the growth function $g_T(r)$ and in particular for d-dimensional spaces we always have $g_T(r) \le d$ for all $r \ge 0$.

THEOREM 4.1. *If T is a bounded operator in a Banach space such that the resolvent condition* (4.1) *holds, and such that its growth function g_T is finite at $r = 1$, then T is*

([2]) *Editorial note*: See also the paper by J. C. Strikwerda and B. A. Wade in this volume.

power bounded and in fact

$$\|T^n\| \le (Ce+1)g_T(1) \quad \text{for } n \ge 0. \tag{4.3}$$

R e m a r k 4.1. The assertion holds e.g. for trace class operators in the form

$$\|T^n\| \le (Ce+1)C_1[\|T\|_1 + 1] \quad \text{for } n \ge 0$$

with a universal constant C_1, see Remark 3.4. For trace class operators a quantitative result has recently been given in A. Pokrzywa [1994] but there the right hand side grows exponentially with $\|T\|_1$.

P r o o f. Fix $\epsilon > 0$. Then there exists an algebraic A such that

$$\deg(A) + m(1,T,A) \le g_T(1) + \epsilon. \tag{4.4}$$

But then

$$\begin{aligned}\|T^n\| \le \frac{1}{2\pi}\Big\| &\int\limits_{|\lambda|=1+1/n} \lambda^n(\lambda - A)^{-1}\,d\lambda\Big\| \\ &+ \frac{1}{2\pi}\Big\| \int\limits_{|\lambda|=1+1/n} \lambda^n[(\lambda - T)^{-1} - (\lambda - A)^{-1}]\,d\lambda\Big\|.\end{aligned}$$

But (4.1) implies that along $|\lambda| = r > 1$ we have

$$\|(\lambda - A)^{-1}\| \le \frac{C}{r-1} + m(1,T,A)/r. \tag{4.5}$$

Following the proof of Theorem 2.8.14 in O. Nevanlinna [1993] (which is a modification of the "LeVeque–Trefethen–Spijker proof" of (4.2) for algebraic operators) we conclude from (4.5) with $r = 1 + 1/n$,

$$\begin{aligned}\frac{1}{2\pi}\Big\| \int\limits_{|\lambda|=1+1/n} \lambda^n(\lambda - A)^{-1}\,d\lambda\Big\| &\le \frac{e}{n}\deg(A) \sup_{|\lambda|=1+1/n} \|(\lambda - A)^{-1}\| \\ &\le e\deg(A)\Big[C + \frac{1}{n+1}m(1,T,A)\Big].\end{aligned}$$

Since the integrand is analytic for $|\lambda| > 1$ we may integrate

$$\frac{1}{2\pi}\Big\| \int\limits_{|\lambda|=1+1/n} \lambda^n[(\lambda - T)^{-1} - (\lambda - A)^{-1}]d\lambda\Big\|$$

along any circle with $|\lambda| = r > 1$ and conclude that it can be bounded by $m(1,T,A)$. Using (1.3) we may assume that $n + 1 \ge g_T(1) + \epsilon$. Then

$$\begin{aligned}\|T^n\| &\le d\Big[Ce + \frac{e}{n+1}m\Big] + m \le Ceg - Cem + \frac{edm}{g} + m \\ &= Ceg - \Big[Ce - \frac{d}{g}e - 1\Big]m \le Ceg + m \le (Ce+1)g.\end{aligned}$$

Here we denoted $d := \deg(A)$, $m := m(1,T,A)$ and $g := g_T(1) + \epsilon$.

As another application we prove a quantitative version of the following result (see Theorem 8 in J. Zemánek [1994]): if T is a Riesz operator such that its spectrum is in the closed unit disc, then

$$\sup_{n\geq 1}\left\|\frac{1}{n}\sum_{0}^{n-1}T^j\right\|<\infty$$

if and only if $\operatorname{ascent}(1-T)\leq 1$ and $\operatorname{ascent}(\lambda-T)\leq 2$ for any other λ of unit modulus.

To that end we consider the following inequalities:

$$\left\|\frac{1}{n}\sum_{0}^{n-1}T^j\right\|\leq M \quad \text{for } n\geq 1 \tag{4.6}$$

and

$$\|(\lambda-T)^{-1}\|\leq K\frac{|\lambda-1|}{(|\lambda|-1)^2} \quad \text{for } |\lambda|>1. \tag{4.7}$$

Clearly, for any T we have $M\geq 1$, $K\geq 1$.

THEOREM 4.2. *If T satisfies* (4.6), *then it satisfies* (4.7) *with $K:=4M$. Conversely, if T is such that its growth function is finite at* 1 *and* (4.7) *holds, then* (4.6) *holds with $M:=(Ke+1)(g_T(1)+1)$.*

Proof. Assuming (4.6), for $|\lambda|>1$ put

$$f(\lambda):=\sum_{n=1}^{\infty}\left[\frac{1}{n}\sum_{0}^{n-1}T^j\right]\lambda^{-n}$$

so that

$$(\lambda-T)^{-1}=(1-\lambda)f'(\lambda). \tag{4.8}$$

Our assumption (4.6) implies

$$\|f(\lambda)\|\leq\frac{M}{|\lambda|-1} \quad \text{for } |\lambda|>1. \tag{4.9}$$

The claim now follows from (4.8) and (4.9) using the Cauchy inequality

$$\|f'(\lambda)\|\leq\frac{1}{\rho}\max_{|z-\lambda|\leq\rho}\|f(z)\|$$

with $\rho:=(|\lambda|-1)/2$.

Conversely, fix an $\epsilon>0$. Then there exists an algebraic operator A such that

$$m(1,T,A)\leq g_T(1)+\epsilon-\deg(A)=:C. \tag{4.10}$$

Now (4.7) and (4.10) give

$$\|(\lambda-A)^{-1}\|\leq K\frac{|\lambda-1|}{(|\lambda|-1)^2}+\frac{C}{|\lambda|} \quad \text{for } |\lambda|>1.$$

Write

$$\frac{1}{n}\sum_{0}^{n-1}T^j=I_1+I_2$$

where

$$I_1=\frac{1}{2\pi i}\int_{|\lambda|=R}\left[\frac{1}{n}\sum_{0}^{n-1}\lambda^j\right](\lambda-A)^{-1}\,d\lambda$$

and

$$I_2 = \frac{1}{2\pi i} \int\limits_{|\lambda|=R} \left[\frac{1}{n}\sum_0^{n-1} \lambda^j\right] [(\lambda - T)^{-1} - (\lambda - A)^{-1}]\, d\lambda$$

($R > 1$). We now estimate I_1 using a modification of the LeVeque–Trefethen–Spijker technique. Take any pair of unit vectors, $x \in X$ and $x^* \in X^*$, the dual of X and observe that $\|I_1\| = \sup |\langle x^*, I_1 x\rangle|$. Denote by r the following rational function:

$$r(\lambda) := \frac{1}{\lambda - 1}\langle x^*, (\lambda - A)^{-1} x\rangle.$$

If d :=deg(A), then r is of exact degree $d+1$, see the previous section, or Section 2.10 in O. Nevanlinna [1993]. Furthermore, near ∞, r behaves like $O(1/\lambda^2)$, which means that

$$\frac{1}{2\pi i} \int\limits_{|\lambda|=R} r(\lambda) d\lambda = 0.$$

Thus

$$\begin{aligned}\langle x^*, I_1 x\rangle &= \frac{1}{2\pi i} \int\limits_{|\lambda|=R} \frac{1}{n}(\lambda^n - 1) r(\lambda)\, d\lambda = \frac{1}{2\pi i} \int\limits_{|\lambda|=R} \frac{1}{n}\lambda^n r(\lambda) d\lambda \\ &= \frac{-1}{2\pi i} \int\limits_{|\lambda|=R} \frac{1}{n(n+1)} \lambda^{n+1} r'(\lambda) d\lambda.\end{aligned}$$

By Spijker's lemma (M. N. Spijker [1991], Lemma 2.8.5 in O. Nevanlinna [1993]) we have

$$\frac{1}{2\pi} \int\limits_{|\lambda|=R} |r'(\lambda)||d\lambda| \le (d+1) \sup_{|\lambda|=R} |r(\lambda)|.$$

Taking $R = 1 + 1/n$ we have

$$|r|_\infty \le K n^2 + C\frac{n^2}{n+1},$$

which then gives us

$$|\langle x^*, I_1 x\rangle| \le (d+1)e\left[K + \frac{C}{n+1}\right].$$

To estimate the second integral we may let $R \to 1$ and obtain $\|I_2\| \le C$. Combining these estimates gives the bound for $n \ge g_T(1) + \epsilon$. For small n we can use the simple estimate

$$\left\|\frac{1}{n}\sum_0^{n-1} T^j\right\| \le Ke(n+1).$$

Finally, the third application concerns the estimate (2.1). Recall that the different characterizations in Theorem 2.1 contain constants which are not explicitly given in terms of the others.

Theorem 4.3. *If T satisfies*

$$\|T^n(T-1)\| \le \frac{M}{n+1} \quad \textit{for } n \ge 0, \tag{4.11}$$

then

$$\|(\lambda - T)^{-1} - (\lambda - 1)(\lambda - T)^{-2}\| \le \frac{M}{|\lambda|(|\lambda| - 1)} \quad \textit{for } |\lambda| > 1. \tag{4.12}$$

Conversely, assume that (4.12) *holds and that the growth function* g_T *of* T *is finite for some* $\rho < 1$. *Then*

$$\|T^n(T-1)\| \le \frac{C}{n+1} \quad \text{for } n \ge 0, \tag{4.13}$$

where $C := 2eg_T(\rho)[M + e/(\rho-\rho^2) + 1/\rho]$.

Proof. If (4.11) holds, then the series

$$F(\lambda) := (1-T)\bigg(\frac{1}{\lambda^2} + 2T\frac{1}{\lambda^3} + 3T^2\frac{1}{\lambda^4} + \bigg)$$

converges for $|\lambda| > 1$ and we have

$$\|F(\lambda)\| \le \frac{M}{|\lambda|(|\lambda|-1)}.$$

This, however, is exactly (4.12).

The reverse direction follows the similar lines as the earlier proofs. Take, as before, unit vectors x and x^* and put $f(\lambda) := \langle x^*, (\lambda - T)^{-1}x\rangle$. Clearly,

$$\langle x^*, F(\lambda)x\rangle = f(\lambda) + (\lambda-1)f'(\lambda).$$

If you now take an algebraic approximation of T and form the corresponding rational function, say q which approximates f, then we have (after partial integration)

$$\begin{aligned}\langle x^*, T^n(1-T)x\rangle = {} & \frac{1}{n+1}\frac{1}{2\pi i}\int \lambda^{n+1}(q(\lambda) + (\lambda-1)q'(\lambda))\,d\lambda \\ & + \frac{1}{n+1}\frac{1}{2\pi i}\int \lambda^{n+1}[(f-q)(\lambda) + (\lambda-1)(f-q)'(\lambda)]\,d\lambda\end{aligned}$$

Here $q + (\lambda-1)q'$ is rational of degree at most $2\deg(A)$, and the corresponding term can be estimated by performing still another partial integration and using the Spijker's lemma as before. This gives the estimate for large integers n while for small integers the estimate follows without partial integration and without Spijker's lemma.

The second integral which has the flavor of an error term in the approximation, can be estimated if $f - q$ and its derivative have been estimated. But for $|\lambda| > \rho$ we have

$$|(f-q)(\lambda)| \le \frac{1}{\rho} m(\rho, T, A)$$

while for $|\lambda| \ge 1$,

$$|(f-q)'(\lambda)| \le \frac{1}{1-\rho}\sup_{|\lambda|>\rho} |(f-q)(\lambda)|.$$

This allows one to estimate the second term as well and (4.13) follows. We leave the details for the interested reader.

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