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HILBERT MODULES AND TENSOR PRODUCTS OF OPERATOR SPACES

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Abstract. The classical identification of the predual of $\mathrm{B}(\mathcal{H})$ (the algebra of all bounded operators on a Hilbert space $\mathcal{H}$) with the projective operator space tensor product $\overline{\mathcal{H}}\hat{\otimes}\mathcal{H}$ is extended to the context of Hilbert modules over commutative von Neumann algebras. Each bounded module homomorphism b between Hilbert modules over a general C^*-algebra is shown to be completely bounded with $\|b\|_{\mathrm{cb}} = \|b\|$. The so called projective operator tensor product of two operator modules X and Y over an abelian von Neumann algebra C is introduced and if Y is a Hilbert module, this product is shown to coincide with the Haagerup tensor product of X and Y over C.

0. Introduction. Recently the theory of tensor products of operator spaces has evolved considerably (see e.g. [6], [18]). The present paper is an attempt to put a part of this theory in a broader context of operator modules in which the role of the compex field $\mathbb{C}$ is played by a von Neumann algebra. It is well known, for example, that $\mathrm{B}(\mathcal{H})$ (the space of all bounded linear operators on a Hilbert space $\mathcal{H}$) is isometric to the dual of the projective tensor product $\overline{\mathcal{H}}\overset{\wedge}{\otimes}\mathcal{H}$. (In [15] and [2] a more recent improvement of this result can be found and in [12] there is even an extension to general von Neumann algebras instead of $\mathrm{B}(\mathcal{H})$.) Here we shall present a generalization of this classical result to Hilbert modules. To achieve this, we have first to extend some parts of the theory of tensor products of operator spaces to operator modules. We have tried to make this paper accessible to everyone familiar with basic notions of functional analysis and operator algebras (and the definition of algebraic tensor product of vector spaces), so all the necessary background concerning operator spaces, completely bounded mappings and Hilbert modules will be explained below. (For a more complete treatment, however, see [28], [32] and [11] for operator spaces and [23], [27], [30] and [20] for Hilbert modules.)

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In Section 1 we shall review some basic facts concerning operator spaces, completely bounded mappings, the Haagerup tensor product and the projective operator tensor product of operator spaces. In Section 2 we look at the two tensor products in the context of operator modules over von Neumann algebras. In Section 3 Hilbert modules are introduced and it is proved in particular that each bounded homomorphism between two Hilbert modules over a C^*-algebra is automatically completely bounded with the completely bounded norm equal to the usual norm (Theorem 3.3). In the case of Hilbert spaces this result was proved earlier by Effros and Ruan [15]. In Section 4 it is explained how a Hilbert module E over a von Neumann algebra can be represented as a concrete space of operators between two Hilbert spaces so that its weak operator closure is self dual. The fact that E can be embedded into a self dual module has been proved previously by Paschke [27] and Rieffel [30], but here we give a different proof, based on the so called linking algebra of E, a technique that gives some additional information. Finally, in Section 5 the above mentioned Hilbert space duality is extended to Hilbert modules. It is also proved that for an abelian von Neumann algebra C the Haagerup tensor product $X \otimes_C^h E$, where X is an operator C-module and E is a Hilbert C-module, coincides with the projective operator tensor product $X \hat{\otimes}_C E$. In the case $C = \mathbb{C}$ this was proved previously by Effros and Ruan [15].

A few months after the first version of this paper was submitted we received the information from D. P. Blecher that Theorem 3.3 was proved also by the authors of [3] and [5] and independently by L. G. Brown (unpublished) at least for the case of operators that have adjoints. In fact in [3] and [5] the concept of a Hilbert module is generalized to the context of non-self-adjoint operator algebras and it is shown in particular that a Hilbert module over a C^*-algebra is completely determined by its operator space structure. Related questions are studied also in the recent paper [4].

1. Completely bounded operators and tensor products. *An operator space* is a subspace of $\mathrm{B}(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space). For each operator space X and positive integer n the space $\mathrm{M}_n(X)$ of all $n \times n$ matrices with entries in X is a subspace of $\mathrm{M}_n(\mathrm{B}(\mathcal{H})) = \mathrm{B}(\mathcal{H}^n)$ and inherits from $\mathrm{B}(\mathcal{H}^n)$ a norm. A linear mapping $\Phi : X \to Y$ between two operator spaces induces for each n a mapping

$$\Phi_n : \mathrm{M}_n(X) \to \mathrm{M}_n(Y), \quad \Phi_n([x_{ij}]) \stackrel{\mathrm{def}}{=} [\Phi(x_{ij})].$$

Φ is called *completely bounded* iff

$$\|\Phi\|_{\mathrm{cb}} \stackrel{\mathrm{def}}{=} \sup_n \|\Phi_n\| < \infty$$

and the quantity $\|\Phi\|_{\mathrm{cb}}$ is the *completely bounded norm* of Φ. If all the maps Φ_n are isometries, Φ is called a *complete isometry.* The space of all completely bounded mappings from an operator space X to an operator space Y will be denoted by $\mathrm{CB}(X, Y)$ and by $\mathrm{CB}(X)$ if $Y = X$.

Given an operator space X, the norms on various spaces $\mathrm{M}_n(X)$ satisfy the following two conditions:

(i) $$\|\alpha x \beta\| \le \|\alpha\| \|x\| \|\beta\| \quad (\alpha, \beta \in \mathrm{M}_n(\mathbb{C}),\ x \in \mathrm{M}_n(X));$$

(ii) $\quad \| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \| = \max\{\|x\|, \|y\|\} \quad (x \in \mathrm{M}_m(X),\ y \in \mathrm{M}_n(Y),\ m, n$ arbitrary).

Ruan [31] proved that these two conditions characterize operator spaces: if X is a complex vector space such that each of the spaces $\mathrm{M}_n(X)$ is equipped with a norm so that the conditions (i) and (ii) are satisfied, then X is completely isometric to a linear subspace of $\mathrm{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. (A shorter proof of this is in [17].) We shall use the name 'operator space' also for any abstract space X such that each of the spaces $\mathrm{M}_n(X)$ carries a norm, whereby the conditions (i) and (ii) are satisfied. It follows easily from Ruan's theorem that the quotient of an operator space by a closed subspace is again an operator space (see [31]). Given an operator space X, for every positive integers m, n the space of rectangular matrices $\mathrm{M}_{m,n}(X)$ is also equipped with a norm obtained by completing each rectangular matrix to a square matrix by zero entries.

If X and Y are operator spaces, various natural norms can be introduced on the algebraic tensor product $X \otimes Y$ for which $X \otimes Y$ becomes an operator space. Here we shall need only two such norms: the Haagerup and the projective operator space norm (for other norms see [6]). Given $x \in \mathrm{M}_{m,n}(X)$ and $y \in \mathrm{M}_{n,l}(Y)$ the element $x \odot y \in \mathrm{M}_{m,l}(X \otimes Y)$ is defined by

$$x \odot y = [\sum_{j=1}^{n} x_{ij} \otimes y_{jk}].$$

(This resembles formally the usual matrix multiplication.) For each m the *Haagerup norm* on $\mathrm{M}_m(X \otimes Y)$ is defined by

$$\|t\|_h = \inf\{\|x\|\|y\| : \ t = x \odot y,\ x \in \mathrm{M}_{m,n}(X),\ y \in \mathrm{M}_{n,m}(Y),\ n \in \mathbf{N}\}. \tag{1.1}$$

It is not obvious that (1.1) defines a norm on the whole space $\mathrm{M}_m(X \otimes Y)$. The proof of the triangle inequality can be found in [13]. To see that $\|t\|_h = 0$ implies $t = 0$ for $t \in \mathrm{M}_m(X \otimes Y)$, we may suppose that $X \subseteq \mathrm{B}(\mathcal{H})$ and $Y \subseteq \mathrm{B}(\mathcal{K})$ for some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and let $t = x \odot y$, where $x \in \mathrm{M}_{m,n}(X) \subseteq \mathrm{B}(\mathcal{H}^n, \mathcal{H}^m)$ and $y \in \mathrm{M}_{n,m}(Y) \subseteq \mathrm{B}(\mathcal{K}^m, \mathcal{K}^n)$. Denoting by $\tilde{t}$ the map from $\mathrm{B}(\mathcal{K}^n, \mathcal{H}^n)$ to $\mathrm{B}(\mathcal{K}^m, \mathcal{H}^m)$ defined by

$$\tilde{t}(a) = xay \quad (a \in \mathrm{B}(\mathcal{K}^n, \mathcal{H}^n)),$$

we easily have that $\|\tilde{t}\|_{\mathrm{cb}} \leq \|x\|\|y\|$ for any representation of t in the form $t = x \odot y$. Hence

$$\|\tilde{t}\|_{\mathrm{cb}} \leq \|t\|_h,$$

which shows that $\|t\|_h = 0$ implies t=0 (since the map $t \mapsto \tilde{t}$ is easily seen to be one to one). In fact, it follows from [33] that the correspondence $t \mapsto \tilde{t}$ defines a complete isometry from $(X \otimes Y, \| \ \|_h)$ to $\mathrm{CB}(\mathrm{B}(\mathcal{K}, \mathcal{H}))$. The completion $X \otimes^h Y$ of $X \otimes Y$ in the Haagerup norm is called *the Haagerup tensor product* of X and Y. Identifying $\mathrm{M}_m(X \otimes^h Y)$ with the completion of $\mathrm{M}_m(X \otimes Y)$ in the Haagerup norm for each m, $X \otimes^h Y$ is an operator space.

Given operator spaces X and Y and $x \in \mathrm{M}_{m,n}(X)$, $y \in \mathrm{M}_{k,l}(Y)$ (where m, n, k, l are arbitrary positive integers), it is customary to denote by $x \otimes y$ the matrix in $\mathrm{M}_{mk,nl}(X \otimes Y)$ with entries

$$(x \otimes y)_{i,j} = x_{i_1 j_1} \otimes y_{i_2 j_2},$$

where $i = (i_1, i_2)$, $j = (j_1, j_2)$ and the pairs i (and j) are ordered lexicographically. For each $t \in \mathrm{M}_n(X \otimes Y)$ let

$$\|t\|_\wedge = \inf\{\|\alpha\|\|\beta\|\|x\|\|y\| : \ t = \alpha(x \otimes y)\beta\}, \tag{1.2}$$

where the infimum is over all $x \in \mathrm{M}_k(X)$, $y \in \mathrm{M}_l(Y)$, $\alpha \in \mathrm{M}_{n,kl}(\mathbb{C})$ and $\beta \in \mathrm{M}_{kl,n}(\mathbb{C})$ (over all k, l). An elementary proof of the fact that (1.2) defines a norm on $\mathrm{M}_n(X \otimes Y)$ which dominates the Haagerup norm for each n and that in this way $X \otimes Y$ becomes an operator space is in [16]. The completion $X \overset{\wedge}{\otimes} Y$ of $X \otimes Y$ in the norm $\| \ \|_\wedge$ is called the *projective operator tensor product* of X and Y. (By [6] the norm $\| \ \|_\wedge$ is the largest among all operator space cross norms.)

2. The two tensor products of operator modules. A subspace $X \subseteq \mathrm{B}(\mathcal{H})$ is a *right* (respectively, a *left*) *operator module* over a C^*-algebra $R \subseteq \mathrm{B}(\mathcal{H})$ iff $XR \subseteq X$ (respectively, $RX \subseteq X$). Operator modules can be characterized in an abstract (= representation free) way [14].

Given a right R-module X and a left R-module Y, the algebraic tensor product $X \otimes_R Y$ is by definition the quotient of $X \otimes Y$ by the subspace N spanned by all elements of the form $xr \otimes y - x \otimes ry$, where $x \in X$, $y \in Y$ and $r \in R$. If $R \subseteq \mathrm{B}(\mathcal{H})$ is a von Neumann algebra with commutant R' and $X, Y \subseteq \mathrm{B}(\mathcal{H})$, then N is a closed subspace of $X \otimes Y$, where $X \otimes Y$ is equipped with the Haagerup norm. This can be shown by considering the contraction

$$\mu : X \otimes Y \to \mathrm{CB}(R', \mathrm{B}(\mathcal{H})),$$

defined by

$$\mu(\sum_{j=1}^n x_j \otimes y_j)(r') = \sum_{j=1}^n x_j r' y_j \quad (r' \in R').$$

It is not hard to see that the kernel of μ is precisely the subspace N [24], so N is closed. Since N is closed, we can equip the quotient space $X \otimes_R Y = (X \otimes Y)/N$ by the quotient norm. This quotient norm will be called simply the *Haagerup norm* on $X \otimes_R Y$. The completion of $X \otimes_R Y$ in the Haagerup norm is the *Haagerup tensor product* of X and Y over R, denoted by $X \otimes_R^h Y$. Basic properties of the Haagerup tensor product of modules are studied recently also in [5].

It was proved in [24] that the above defined map μ induces an isometry from $X \otimes_R^h Y$ to $\mathrm{CB}(R', \mathrm{B}(\mathcal{H}))$, and it is not hard to deduce that this is in fact a complete isometry (see [25]). Hence we shall from now on regard $X \otimes_R^h Y$ as a subspace of $\mathrm{CB}(R', \mathrm{B}(\mathcal{H}))$ whenever convenient. In particular, for each row matrix $x \in \mathrm{M}_{1,n}(X)$ and each column matrix $y \in \mathrm{M}_{n,1}(Y)$ we denote the equivalence class of $x \odot y$ in $X \otimes_R^h Y$ by $x \odot_R y$ and we regard $x \odot_R y$ as a map in $\mathrm{CB}(R', \mathrm{B}(\mathcal{H}))$, given by

$$(x \odot_R y)(r') = x{r'}^{(n)} y \quad (r' \in R'), \tag{2.1}$$

where ${r'}^{(n)}$ is the direct sum of n copies of r'. It will be convenient to use this notation also for infinite rows and columns. First we choose a sufficiently large cardinal $\mathcal{I}$ such that any family of non-zero operators $\{a_i\}$ in $\mathrm{B}(\mathcal{H})$ for which the sum $\sum_i a_i^* a_i$ is in $\mathrm{B}(\mathcal{H})$ has cardinality at most $\mathcal{I}$. (Here the infinite sum means the least upper bound of the net of all finite subsums.) For example, $\mathcal{I}$ may be the cardinality of an orthonormal basis of $\mathcal{H}$ (or $\mathcal{I}$ is the countably infinite cardinal if $\mathcal{H}$ is finite dimensional). Then we denote by $\mathrm{R}(X)$ the set of all rows x with the entries x_i in X such that the sum $\sum_{i\in\mathcal{I}} x_i x_i^*$ converges to a bounded operator (in the strong operator topology); in particular $\mathrm{R}(\mathrm{B}(\mathcal{H})) = \mathrm{B}(\mathcal{H}^{\mathcal{I}}, \mathcal{H})$. The space of columns, $\mathrm{C}(X)$, is defined similarly (thus $\mathrm{C}(X) \subseteq \mathrm{B}(\mathcal{H}, \mathcal{H}^{\mathcal{I}})$). Now for each $x \in \mathrm{R}(X)$ and $y \in \mathrm{C}(Y)$ we define $x\odot_R y$ to be the element of $\mathrm{CB}(R', \mathrm{B}(\mathcal{H}))$ given by (2.1).

The space of all completely bounded maps of the form $x \odot_R y$ ($x \in \mathrm{R}(X)$, $y \in \mathrm{C}(Y)$) is called the *full Haagerup tensor product* of X and Y over R and it is denoted by $X\overline{\otimes}^h_R Y$.

It is easy to verify that $X\overline{\otimes}^h_R Y$ is a linear subspace of $\mathrm{CB}(R', \mathrm{B}(\mathcal{H}))$. For example, the sum of two elements can be expressed as

$$x \odot_R y + v \odot_R w = [x, v] \odot_R \begin{bmatrix} y \\ w \end{bmatrix},$$

where $[x, v]$ can be regarded as an element of $\mathrm{R}(X)$ since $\mathcal{I}$ is infinite and similarly for the column (y, w).

As customary in the theory of operator spaces, we shall use the notation $(x_1, x_2, \ldots)$ for a column vector and $[x_1, x_2, \ldots]$ for a row vector.

If X and Y are weak* closed subspaces of $\mathrm{B}(\mathcal{H})$ and $R = \mathbb{C}$, the above defined tensor product $X\overline{\otimes}^h_R Y$ coincides with the weak* Haagerup tensor product of Blecher and Smith [7]. The basic result concerning this product is the following theorem. We denote by $\mathrm{NCB}(R', \mathrm{B}(\mathcal{H}))$ the space of all normal (= weak* continuous) completely bounded linear maps from R' to $\mathrm{B}(\mathcal{H})$.

THEOREM 2.1. *If $R \subseteq \mathrm{B}(\mathcal{H})$ is a von Neumann algebra and $X, Y \subseteq \mathrm{B}(\mathcal{H})$ are weak* closed subspaces such that $XR = X$ and $RY = Y$, then $X\overline{\otimes}^h_R Y$ is a subspace of* $\mathrm{NCB}(R', \mathrm{B}(\mathcal{H}))$ *containing $X\otimes^h_R Y$ and the norm of each element $t \in X\overline{\otimes}^h_R Y$ can be expressed as*

$$\|t\|_{\mathrm{cb}} = \inf\{\|x\|\|y\| : \ t = x \odot y, \ x \in \mathrm{R}(X), \ y \in \mathrm{C}(Y)\}.$$

Moreover, a similar conclusion holds for the norm of elements of $\mathrm{M}_n(X\overline{\otimes}^h_R Y)$ for each positive integer n, since $\mathrm{M}_n(X\overline{\otimes}^h_R Y)$ is naturally completely isometrically isomorphic to $\mathrm{M}_{n,1}(X)\overline{\otimes}^h_R \mathrm{M}_{1,n}(Y)$.

In the special case $R = \mathbb{C}$ Theorem 2.1 is proved in [7]. The proof of Theorem 2.1 can be found in [25] (in a more general context of strong operator modules) and it will be sketched also in the appendix to this paper. It was proved by Halpern in [19] that for each von Neumann algebra R there exists a faithful representation π such that the restriction of π to the center C of R is normal and $\pi(R)' = \pi(C)$. If we use this result and the completely isometric inclusion of $\mathrm{B}(\mathcal{H})\otimes^h_R \mathrm{B}(\mathcal{H})$ into $\mathrm{NCB}(R', \mathrm{B}(\mathcal{H}))$, it is not hard to deduce the following theorem (see [24]), which has been proved in a different way by Chatterjee and Smith [10].

THEOREM 2.2 [10]. *Let R be a von Neumann algebra and C the center of R. Then the natural map*

$$\vartheta : R\otimes^h_C R \to \mathrm{CB}(R), \quad \vartheta(x \otimes_C y)(r) = xry$$

is completely isometric.

In case R is a factor, Theorem 2.2 was proved by Chatterjee and Sinclair in [9]. An extension of Theorem 2.2 to the setting of C^*-algebras is obtained in [1].

If $C \subseteq \mathrm{B}(\mathcal{H})$ is a commutative von Neumann algebra, then $C \subseteq C'$, so C' is a C-bimodule. It follows from [24, Theorem 1.2] that each normal completely bounded C-module homomorphism t from C' to $\mathrm{B}(\mathcal{H})$ can be represented in the form $t = x \odot_C y$ for some $x \in \mathrm{R}(C')$ and $y \in \mathrm{C}(C')$. Hence the range of t is contained in C' and we deduce the following.

PROPOSITION 2.3. *If $C \subseteq \mathrm{B}(\mathcal{H})$ is a commutative von Neumann algebra, then $\mathrm{NCB}_C(C') = C'\overline{\otimes}^h_C C'$, where $\mathrm{NCB}_C(C')$ is the space of all normal completely bounded C-module homomorphisms on C'.*

If X and Y are operator bimodules over a commutative von Neumann algebra C such that X and Y commute with C, then the subspace N of $X \otimes Y$ generated by all elements of the form $xc \otimes y - x \otimes cy$ ($x \in X$, $y \in Y$, $c \in C$) is obviously a C-subbimodule of $X \otimes Y$, hence the quotient $X \otimes_C Y = (X \otimes Y)/N$ is a C-bimodule. We define for each n the *projective operator norm* in $\mathrm{M}_n(X \otimes_C Y)$ by

$$\|t\|_\wedge = \inf\{\|a\|\|b\|\|x\|\|y\| : \ t = a(x \otimes_C y)b\}, \tag{2.2}$$

where the infimum is over all $x \in \mathrm{M}_l(X)$, $y \in \mathrm{M}_k(Y)$, $a \in \mathrm{M}_{n,kl}(C)$ and $b \in \mathrm{M}_{kl,n}(C)$ (with variable k and l). Here the symbol $x \otimes_C y$ is, of course, defined by

$$x \otimes_C y \stackrel{\text{def}}{=} [x_{i_1 j_1} \otimes_C y_{i_2 j_2}] = x \otimes y + \mathrm{M}_{kl}(N).$$

The completion of $X \otimes_C Y$ in the norm $\| \ \|_\wedge$ is called the *projective operator tensor product of X and Y over C* and is denoted by $X \hat{\otimes}_C Y$.

THEOREM 2.4. *If X and Y are operator bimodules over an abelian von Neumann algebra C such that $xc = cx$ and $yc = cy$ for all $c \in C$, $x \in X$ and $y \in Y$, then $X \hat{\otimes}_C Y$ is an operator space and the norm $\| \ \|_\wedge$ on $\mathrm{M}_n(X \otimes_C Y)$ dominates the Haagerup norm for each n.*

Theorem 2.4 is an extension of Theorem 3.1 from [16], where the special case $C = \mathbb{C}$ is proved; since the proof is the same as in this special case, it will be omitted here.

3. Hilbert modules as operator spaces. *A right Hilbert module* over a C^*-algebra A is a right A-module E equipped with a map $\langle \ , \ \rangle : E \times E \to A$, called the *inner product*, such that for all $x, y, z \in E$ and $a \in A$ the following familiar conditions are satisfied:

(i) $\langle x, ya + z\rangle = \langle x, y\rangle a + \langle x, z\rangle$;
(ii) $\langle y, x\rangle = \langle x, y\rangle^*$;
(iii) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ implies that $x = 0$.

Moreover, the norm is introduced on E by $\|x\| \stackrel{\text{def}}{=} \|\langle x, x\rangle\|^{1/2}$ and it is required that E is complete for this norm.

Left Hilbert modules are defined in a similar way (the inner product is now A-linear in the first factor). By a *Hilbert module* we shall always mean a right Hilbert module. Hilbert modules were introduced by Kaplansky [22] in the case A is commutative and later they were studied by Paschke [27] and Rieffel [30] for a general A. Today Hilbert modules are one of the basic tools in operator K-theory [20]. A basic example of a Hilbert A-module is $E = A$ with the inner product $\langle x, y\rangle = x^*y$; another example is any closed right ideal in A with the same A-valued inner product. For a fuller account of the theory of Hilbert modules see [23].

Not every bounded A-module homomorphism $b : E \to F$, where E and F are Hilbert modules, has an adjoint $b^* : F \to E$ in the usual sense that $\langle x, b^*y\rangle = \langle bx, y\rangle$ for all $x \in E$, $y \in F$ (see [27] for a concrete example). The space of all linear maps $b : E \to F$ that have an adjoint is denoted by $\mathrm{L}(E, F)$ (or by $\mathrm{L}(E)$, if $F = E$), while the space of all bounded A-module homomorphisms from E to F is denoted by $\mathrm{B}_A(E, F)$. It is not hard to see that $\mathrm{L}(E, F) \subseteq \mathrm{B}_A(E, F)$ (see [27]). For each $x \in E$ and $y \in F$ there is a 'rank one' operator $[y, x] \in \mathrm{L}(E, F)$ defined by

$$[y, x]z = y\langle x, z\rangle \quad (z \in E).$$

$\mathrm{L}(E)$ is a C^*-algebra and $[\cdot, \cdot]$ is an $\mathrm{L}(E)$-valued inner product on E for which E becomes a left Hilbert module over $\mathrm{L}(E)$.

Now we shall show how a Hilbert A-module E can be always embedded into a certain C^*-algebra $\Lambda(E)$. Denote by $F = E \oplus A$ the Hilbert A-module consisting of columns (x, a) $(x \in E,\ a \in A)$ and equipped with the obvious inner product

$$\langle(x_1, a_1), (x_2, a_2)\rangle = \langle x_1, x_2\rangle + a_1^*a_2.$$

Identify each $x \in X$ with the operator from A to X defined by $a \mapsto xa$ and observe that the adjoint of this operator is given by $x^*(y) = \langle x, y\rangle$ $(y \in X)$. Let $\Lambda(E)$ be the subset of $\mathrm{L}(F)$ consisting of all operators that can be represented by matrices of the form

$$\begin{bmatrix} b & x \\ y^* & a \end{bmatrix} \quad (a \in A,\ x, y \in E,\ b \in \mathrm{L}(E)).$$

It is easy to verify that $\Lambda(E)$ is a C^*-subalgebra of $\mathrm{L}(F)$, called the *linking algebra of* E. This construction was introduced by Brown, Green and Rieffel in [8] (in a slightly different way with $\mathrm{L}(E)$ replaced by a certain subalgebra $\mathrm{K}(E)$; see also [5]). The subset $E^* = \{x^* :\ x \in E\}$ of $\Lambda(E)$ is called the *conjugate module of* E.

$\Lambda(E)$ will enable us to exploit the representation theory of C^*-algebras in studying E. We shall identify A, E and $L \stackrel{\text{def}}{=} \mathrm{L}(E)$ with subsets of $\Lambda(E)$ in the obvious way:

$$A \cong \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \quad E \cong \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}, \quad L \cong \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}.$$

After this identifications, the A-valued inner product $\langle x, y\rangle$ of E becomes simply the product x^*y in $\Lambda(E)$ and the module multiplication $E \times A \to E$ becomes a part of the internal multiplication of $\Lambda(E)$.

Suppose now that $\pi : \Lambda(E) \to \mathrm{B}(\mathcal{H})$ is a representation of $\Lambda(E)$. (By a representation of a C^*-algebra we always mean a *-representation.) Then by restriction π defines two maps $\varphi = \pi|A$ and $\Phi = \pi|E$ which together constitute a representation of the Hilbert module E in the following sense.

A representation of a Hilbert A-module E is a pair of maps (Φ, φ) such that $\varphi : A \to$

$\mathrm{B}(\mathcal{H})$ is a representation of the C^*-algebra A and $\Phi : E \to \mathrm{B}(\mathcal{H})$ is a linear map satisfying

$$\varphi(\langle x, y\rangle) = \Phi(x)^*\Phi(y) \quad (x, y \in E) \tag{3.1}$$

and (consequently)

$$\Phi(xa) = \Phi(x)\varphi(a) \quad (x \in E,\ a \in A). \tag{3.2}$$

Note that if φ is faithful (hence isometric) the identity (3.1) implies that Φ is isometric.

Since $\Lambda(E)$ (as any C^*-algebra) can be faithfully represented on a Hilbert space $\mathcal{H}$ (see [21] or [26]), it follows that E can be isometrically embedded into $\mathrm{B}(\mathcal{H})$, hence E can be given the structure of an operator space.

Given a Hilbert A-module E, the direct sum E^n of n copies of E is of course a Hilbert A-module for the inner product

$$\langle x, y\rangle = \sum_{j=1}^{n} \langle x_j, y_j\rangle \quad (x, y \in E^n).$$

LEMMA 3.1. *For arbitrary Hilbert A-modules E and F the space $\mathrm{B}_A(E, F)$ of all bounded A-module homomorphisms from E to F becomes an* (*abstract*) *operator space by the identification*

$$\mathrm{M}_n(\mathrm{B}_A(E, F)) = \mathrm{B}_A(E^n, F^n)$$

for each positive integer n.

Proof. We must show that the Ruan conditions (i) and (ii) stated in Section 1 are satisfied. Only the second condition, namely that

$$\|b \oplus c\| = \max\{\|b\|, \|c\|\}$$

for arbitrary $b \in \mathrm{B}_A(E^m, F^m)$ and $c \in \mathrm{B}_A(E^n, F^n)$, is not completely trivial. But, by a result of Paschke [27] each bounded homomorphism d between arbitrary Hilbert A-modules E_0 and F_0 satisfies $\langle dz, dz\rangle \le \|d\|^2\langle z, z\rangle$ for each $z \in E_0$. Applying this to the homomorphisms b and c we find that

$$\langle (b \oplus c)(x, y), (b \oplus c)(x, y)\rangle \le \max\{\|b\|^2, \|c\|^2\}\|(x, y)\|^2$$

for all $x \in E^m$ and $y \in E^n$. This means that $\|b \oplus c\| \le \max\{\|b\|, \|c\|\}$ and the reverse inequality is obvious. ∎

LEMMA 3.2. *Given a Hilbert A-module E, the norm of an arbitrary matrix $x = [x_{ij}] \in \mathrm{M}_n(E)$ satisfies*

$$\|x\| = \sup\{\|\textstyle\sum_{i,j=1}^{n} \langle y_i, x_{ij}\rangle a_j\| :\ y_i \in E,\ a_j \in A,\ \displaystyle\sum_{i=1}^{n} \langle y_i, y_i\rangle \le 1,\ \sum_{j=1}^{n} a_j^* a_j \le 1\}. \tag{3.3}$$

(*Here E is an operator space as a subspace of the C^*-algebra $\Lambda(E)$.*)

Proof. The elements of the Hilbert A-module $E \oplus A$ are columns (x, a) $(x \in E,\ a \in A)$, so the elements of the conjugate module $(E \oplus A)^*$ are considered as rows $[x^*, a^*]$ $(x^* \in E^*,\ a \in A)$. We let $\Lambda(E)$ act on $(E \oplus A)^*$ by the matrix multiplication from the right, which defines a *-antimonomorphism from $\Lambda(E)$ into $\mathrm{L}((E \oplus A)^*)$, hence an isometric embedding. Similarly, for each n the C^*-algebra $\mathrm{M}_n(\Lambda(E))$ acts on $(E \oplus A)^{*n}$ by the matrix multiplication from the right. The lemma follows now by noting that for each $x \in \mathrm{M}_n(E)$ the quantity on the right side of (3.3) is just the norm of the associated operator on $(E \oplus A)^{*n}$. ∎

Let $\mathrm{CB}_A(E,F)$ be the operator space of all completely bounded A-module homomorphisms from a Hilbert A-module E to a Hilbert A-module F, where for each n the norm in $\mathrm{M}_n(\mathrm{CB}_A(E,F))$ is introduced through the identification $\mathrm{M}_n(\mathrm{CB}_A(E,F)) = \mathrm{CB}_A(E,\mathrm{M}_n(F))$. The following theorem was proved in the case $A = \mathbb{C}$ by Effros and Ruan [15].

THEOREM 3.3. *Let E and F be Hilbert modules over a C^*-algebra A. Then for each positive integer n and each $b = [b_{ij}] \in \mathrm{M}_n(\mathrm{B}_A(E,F)) = \mathrm{B}_A(E^n,F^n)$ the map*

$$\tilde{b} : E \to \mathrm{M}_n(F), \quad \tilde{b}(x) \stackrel{\text{def}}{=} [b_{ij}(x)]$$

is completely bounded with $\|\tilde{b}\|_{\mathrm{cb}} = \|b\|$. Thus, the operator spaces $\mathrm{B}_A(E,F)$ and $\mathrm{CB}_A(E,F)$ are the same.

P r o o f. First we shall prove the inequality $\|\tilde{b}\|_{\mathrm{cb}} \le \|b\|$. We may assume that $\|b\| = 1$. Let $x = [x_{kl}] \in \mathrm{M}_m(E)$, $\|x\| \le 1$ (m arbitrary). We have to show that the norm of the matrix $[b_{ij}(x_{kl})] \in \mathrm{M}_{mn}(F)$ is at most 1. By Lemma 3.2 this means that we have to prove the inequality

$$\| \textstyle\sum_{i,j=1}^{n} \sum_{k,l=1}^{m} \langle y_{ik}, b_{ij}(x_{kl})\rangle a_{jl} \| \le 1 \tag{3.4}$$

for all $y \stackrel{\text{def}}{=} (y_{ik}) \in F^{mn}$ and $a \stackrel{\text{def}}{=} (a_{jl}) \in A^{mn}$ satisfying

$$\sum_{i=1}^{n}\sum_{k=1}^{m} \langle y_{ik}, y_{ik}\rangle \le 1 \quad \text{and} \quad \sum_{j=1}^{n}\sum_{l=1}^{m} a_{jl}^* a_{jl} \le 1. \tag{3.5}$$

For all $k,\ j$ put

$$z_{kj} = \sum_{l=1}^{m} x_{kl} a_{jl} \tag{3.6}$$

and let

$$z_k = (z_{k1}, \ldots, z_{kn}) \in E^n \quad \text{and} \quad z = (z_1, \ldots, z_m) \in E^{nm}.$$

Since each b_{ij} is a homomorphism of right A-modules, the left side of (3.4) can be written as $\|\langle y, b^{(m)}(z)\rangle\|$, where $b^{(m)}$ is the direct sum of m copies of b. Hence by the Schwarz inequality for the A-valued inner product (see [23, p. 3] or [27]) the left side of (3.4) is less than or equal to $\|y\|\|b\|\|z\|$, which is at most $\|z\|$ (since $\|b\| = 1$ and $\|y\| \le 1$ by (3.5)). So, it suffices now to prove that $\|z\| \le 1$.

For each $j \in \{1, \ldots, n\}$ put

$$a_j = (a_{j1}, \ldots, a_{jm}) \in A^m \quad \text{and} \quad \tilde{z}_j = (z_{1j}, \ldots, z_{mj}) \in E^m.$$

Then $\tilde{z}_j = x a_j$ for each j by (3.6), hence $\tilde{z} = x^{(n)} a$, where $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n) \in E^{nm}$ and $a = (a_1, \ldots, a_n) \in A^{nm}$. Since z can be obtained from $\tilde{z}$ by rearranging the components, we have $\|z\| = \|\tilde{z}\|$ and it follows easily (by using Lemma 3.2) that $\|z\| \le \|x^{(n)}\|\|a\| \le 1$ (since $\|x\| = 1$ and $\|a\| \le 1$ by the second inequality in (3.5)). This proves (3.4) and therefore $\|\tilde{b}\|_{\mathrm{cb}} \le \|b\|$.

To prove the inequality $\|b\| \le \|\tilde{b}\|_{\mathrm{cb}}$, let $0 < \varepsilon < 1$ and choose $v = (v_1, \ldots, v_n) \in E^n$ so that $\|v\| = 1$ and

$$\|bv\| \ge \|b\| - \varepsilon. \tag{3.7}$$

Note that $\mathrm{R}_n(E) \stackrel{\mathrm{def}}{=} \mathrm{M}_{1,n}(E)$ is a (right) Hilbert module over $\mathrm{M}_n(A)$ with the inner product $\langle t, w\rangle = [\langle t_i, w_j\rangle]$. Let $w = [v_1, \ldots, v_n]$ (= the transpose of v) and put

$$|w| = \langle w, w\rangle^{1/2} \in \mathrm{M}_n(A) \quad \text{and} \quad u = w(|w| + \varepsilon)^{-1} \in \mathrm{R}_n(E).$$

Observe that $\|w\| \le n^{1/2}$. (Indeed, identifying w with the matrix in $\mathrm{M}_n(\Lambda(E))$ which has only the first row non-zero, we have $\|w\|^2 = \|w^*w\| = \|ww^*\| = \|\sum_{j=1}^n v_j v_j^*\| \le \sum_{j=1}^n \|v_j\|^2 \le n\|v\|^2 = n$.) Observe also that $\|u\| \le 1$. (Indeed, $\langle u, u\rangle = (|w| + \varepsilon)^{-1} \times$ $\times \langle w, w\rangle (|w| + \varepsilon)^{-1} \le 1$.) Denote now by a_{ij} the entries of the matrix $|w| + \varepsilon$ and consider the column

$$a = (a_{11}, \ldots, a_{1n}; \ldots; a_{n1}, \ldots, a_{nn}) \in \mathrm{M}_{n^2,1}(A).$$

Since $\|w\| \le n^{1/2}$ and

$$\begin{aligned}\langle a, a\rangle &= \sum_{i,j=1}^n a_{ij}^* a_{ij} = \sum_{i,j=1}^n a_{ji} a_{ij} = \sum_{j=1}^n ((|w| + \varepsilon)^2)_{jj} \\ &= \langle v, v\rangle + 2\varepsilon \sum_{j=1}^n |w|_{jj} + n\varepsilon^2 \le 1 + 2n\varepsilon\|w\| + n\varepsilon^2,\end{aligned}$$

it follows that

$$\|a\| \le 1 + n^2\varepsilon. \tag{3.8}$$

From $w = u(|w| + \varepsilon)$ we have $v_i = \sum_{j=1}^n u_j a_{ji}$ for each i. Now the product of the matrices $[\tilde{b}(u_1), \ldots, \tilde{b}(u_n)] \in \mathrm{M}_{n,n^2}(F)$ (where $\tilde{b}$ is as in the statement of the theorem) and $a \in \mathrm{M}_{n^2,1}(A)$ can be computed as

$$\begin{aligned}[\tilde{b}(u_1), \ldots, \tilde{b}(u_n)]a &= (\textstyle\sum_{i,j=1}^n b_{1i}(u_j)a_{ji}, \ldots, \sum_{i,j=1}^n b_{n,i}(u_j)a_{ji}) \\ &= (\textstyle\sum_{i=1}^n b_{1i}(v_i), \ldots, \sum_{i=1}^n b_{ni}(v_i)) \\ &= b(v).\end{aligned}$$

By using (3.8) and the fact $\|u\| \le 1$ it follows now that

$$\|\tilde{b}\|_{\mathrm{cb}} \ge (1 + n^2\varepsilon)^{-1}\|b(v)\|.$$

By using then (3.7) and letting $\varepsilon \to 0$ it finally follows that $\|\tilde{b}\|_{\mathrm{cb}} \ge \|b\|$. ∎

Now we can easily deduce that any two faithful representations induce the same operator space structure on a Hilbert module. Namely, if (Φ, φ) and (Ψ, ψ) are two faithful representations of a Hilbert A-module E, then we can regard $\Phi(E)$ and $\Psi(E)$ as Hilbert A-modules so that $\Psi\Phi^{-1}$ and $\Phi\Psi^{-1}$ are isometric isomorphisms between them. Theorem 3.3 implies that these two isomorphisms are completely contractive, hence completely isometric since they are inverse to each other.

4. An embedding into a self-dual module through a representation of the linking algebra. A Hilbert module E over a C^*-algebra A is *self-dual* if every bounded A-module homomorphism $\rho : E \to A$ is of the form $\rho(x) = \langle v, x\rangle$ for some $v \in E$. The following result, proved by Rieffel [30] (in a slightly different form), describes typical self-dual modules over von Neumann algebras. To illustrate an application of Theorem 3.3, we shall sketch a simplification of Rieffel's proof.

THEOREM 4.1. *Let $R \subseteq \mathrm{B}(\mathcal{H})$ be a von Neumann algebra, $\varphi : R' \to \mathrm{B}(\mathcal{K})$ a representation of the commutant R' of R on a Hilbert space $\mathcal{K}$ and let*

$$E = \{x \in \mathrm{B}(\mathcal{H}, \mathcal{K}) : \ xr' = \varphi(r')x \quad (\forall r' \in R')\}.$$

Then E is a self-dual Hilbert R-module for the inner product $\langle x, y\rangle = x^*y$ and the module operation xr $(x \in E,\ r \in R)$ the composition of operators. Moreover, if $[E\mathcal{H}] = \mathcal{K}$, then $\mathrm{L}(E)$ can be naturally identified with $\varphi(R')'$.

Sketch of the proof. Since for every $x, y \in E$ the operator $x^*y \in \mathrm{B}(\mathcal{H})$ commutes with all $r' \in R'$, the von Neumann bicommutation theorem shows that $x^*y \in R$. We shall indicate only the proof of the self-duality of E, the fact that E is a Hilbert module is trivial (the completeness follows also from self-duality). Given a bounded R-module homomorphism $\rho : E \to R$, we would like to find an operator $v \in E$ such that $v^*x = \langle v, x\rangle = \rho(x)$ for all $x \in E$. Thus, the adjoint $v^* : \mathcal{K} \to \mathcal{H}$ of v must satisfy

$$v^*\Big(\sum_{j=1}^{n} x_j\xi_j\Big) = \sum_{j=1}^{n} \rho(x_j)\xi_j \tag{4.1}$$

for arbitrary finite subsets $\{x_1, \ldots, x_n\} \subseteq E$ and $\{\xi_1, \ldots, \xi_n\} \subseteq \mathcal{H}$. To prove that there exists a bounded operator v^* satisfying (4.1), we regard $\mathrm{M}_n(E)$ as a (right) Hilbert module over $\mathrm{M}_n(R)$ with the inner product

$$\langle [x_{ij}], [y_{jk}]\rangle = \Big[\sum_{j=1}^{n} \langle x_{ji}, y_{jk}\rangle\Big].$$

Put

$$x = \begin{bmatrix} x_1 & \ldots & x_n \\ 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots \end{bmatrix}.$$

Let $\rho_n : \mathrm{M}_n(E) \to \mathrm{M}_n(R)$ be the map obtained by applying ρ to the entries of matrices. From Theorem 3.3 (with $F = R$ and $n = 1$) it follows that $\|\rho_n\| \le \|\rho\|$; since ρ_n is a homomorphism of $\mathrm{M}_n(R)$-modules, the easier part of Theorem 2.8 from [27] shows that $\langle \rho_n(x), \rho_n(x)\rangle \le \|\rho\|^2 \langle x, x\rangle$ or

$$[\rho(x_i)^*\rho(x_j)] \le \|\rho\|^2 [x_i^* x_j],$$

where both sides are elements of $\mathrm{M}_n(R)$. This means that

$$\|\textstyle\sum_{j=1}^{n} \rho(x_j)\xi_j\|^2 \le \|\rho\|^2 \|\sum_{j=1}^{n} x_j\xi_j\|^2$$

for all $\xi_j \in \mathcal{H}$. Thus, there is a unique bounded operator $v^* : [E\mathcal{H}] \to \mathcal{H}$ satisfying (4.1). Extending v^* by linearity to an operator from $\mathcal{K}$ to $\mathcal{H}$ such that $v^*([E\mathcal{H}]^\perp) = 0$, it is easy to verify that $v \in E$ and $\rho(x) = v^*x = \langle v, x\rangle$ for all $x \in E$.

For each $b \in \varphi(R')'$ define $\tilde{b} \in \mathrm{L}(E)$ simply by $\tilde{b}(x) = bx$ (the composition of operators). It is straightforward to verify that the map $b \to \tilde{b}$ is a *-homomorphism from

$\varphi(R')'$ to $\mathrm{L}(E)$, which is clearly one to one if $[E\mathcal{H}] = \mathcal{K}$. To see that this map is onto, given $\tilde{b} \in \mathrm{L}(E)$, define b first on the dense subspace EH of $\mathcal{K}$ by

$$b(\sum_{j=1}^{n} x_j\xi_j) = \sum_{j=1}^{n} \tilde{b}(x_j)\xi_j \quad (x_j \in E,\ \xi_j \in \mathcal{H}).$$

An argument similar to the one already given in the first part of this proof shows that b is well defined (with $\|b\| \leq \|\tilde{b}\|$) and can be therefore extended to an operator $b \in \mathrm{B}(\mathcal{K})$; moreover $b \in \varphi(R')'$ and $bx = \tilde{b}(x)$ for all $x \in E$. ∎

We shall now describe a special representation of the linking algebra showing in particular that a Hilbert module over a von Neumann algebra can be embedded into a self-dual Hilbert module. This last result was proved in a different way by Paschke [27] and Rieffel [30].

A Hilbert module E over a von Neumann algebra R is called *faithful* iff the ideal $\langle E, E\rangle$ is dense in R in the weak* topology.

Let E be a faithful Hilbert module over a von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$ and put $L = \mathrm{L}(E)$ and $\Lambda = \Lambda(E)$. By a well known result (see [26, 5.5.1]) there exists a Hilbert space $\mathcal{L} \supseteq \mathcal{H}$ and a representation $\pi : \Lambda \to \mathrm{B}(\mathcal{L})$ such that $\pi(r) = r$ for each $r \in R$. Since the subspace

$$\mathcal{K} \stackrel{\text{def}}{=} [\pi(E)\mathcal{H}]$$

of $\mathcal{L}$ is invariant under $\pi(L)$, we have a subrepresentation

$$\psi : L \to \mathrm{B}(\mathcal{K}), \quad \psi(b) = \pi(b)|\mathcal{K}.$$

Denote by p the identity element of R. Since $px = 0$ for all $x \in E$ (the product is computed in Λ), we have $\langle \pi(x)\xi, \eta\rangle = \langle \pi(x)\xi, \pi(p)\eta\rangle = \langle \pi(px)\xi, \eta\rangle = 0$ for all $\xi, \eta \in \mathcal{H}$, hence $\mathcal{K} \perp \mathcal{H}$. Define

$$\Theta : E \to \mathrm{B}(\mathcal{H}, \mathcal{K}) \quad \text{by} \quad \Theta(x) = \pi(x)|\mathcal{H}.$$

Note that

$$[\pi(E)^*\mathcal{K}] = [\pi(E)^*\pi(E)\mathcal{H}] = [\pi(\langle E, E\rangle)\mathcal{H}] = [R\mathcal{H}] = \mathcal{H}$$

since E is faithful over R, hence

$$[\pi(L)\mathcal{H}] = [\pi(L)\pi(E)^*\mathcal{K}] = [\pi(LE^*)\mathcal{K}] = 0$$

since $LE^* = 0$ (the product and * are computed in Λ). Similarly $\pi(R)\mathcal{K} = 0$. It is therefore easy to see that $\mathcal{L}_0 \stackrel{\text{def}}{=} \mathcal{K} \oplus \mathcal{H}$ is an invariant subspace for $\pi(\Lambda)$, so, replacing $\mathcal{L}$ by $\mathcal{L}_0$ and π by $\pi|\mathcal{L}_0$, we may assume that

$$\mathcal{K} \oplus \mathcal{H} = \mathcal{L}.$$

Since π is a representation, we have

$$\Theta(x)^*\Theta(y) = \langle x, y\rangle, \quad \Theta(x)\Theta(y)^* = \psi([x, y]) \quad (x, y \in E) \tag{4.2}$$

and

$$\Theta(bxr) = \psi(b)\Theta(x)r \quad (b \in L,\ r \in R,\ x \in E). \tag{4.3}$$

We claim that Θ and ψ are one-to-one. In fact, the first equality in (4.2) implies that Θ is isometric. If $b \in \ker\psi$, then $\Theta(bx) = 0$ for all $x \in E$ by (4.3), hence $bE = 0$ and $b = 0$.

Since R is weak* closed, the weak* closure of $\pi(\Lambda)$ in $\mathrm{B}(\mathcal{L})$ has the form

$$\overline{\pi(\Lambda)} = \begin{bmatrix} \overline{\psi(L)} & \overline{\Theta(E)} \\ \overline{\Theta(E)}^* & R \end{bmatrix}.$$

We shall show that $\overline{\Theta(E)}$ is a self-dual Hilbert R-module of the form described in Theorem 4.1. To see this, we first compute the commutant of $\pi(\Lambda)$ in $\mathrm{B}(\mathcal{K} \oplus \mathcal{H})$. Evidently each element of $\pi(\Lambda)'$ is of the form

$$\begin{bmatrix} b' & 0 \\ 0 & r' \end{bmatrix},$$

where $b' \in \psi(L)'$, $r' \in R'$ and

$$b'y = yr', \quad b'^*y = yr'^* \ \text{ for all } y \in \Theta(E).$$

Observe now that for each $r' \in R'$ there is a unique element in $\mathrm{B}(\mathcal{K})$, denoted by $\varphi(r')$, such that $\varphi(r')y = yr'$ for all $y \in \Theta(E)$. (Indeed, the uniqueness follows from $\mathcal{K} = [\Theta(E)\mathcal{H}]$. To prove the existence, we may assume that r' is unitary, since in general r' is a linear combination of at most four unitaries, and note that then $\|\sum_{j=1}^n y_j r'\xi_j\|^2 = \|\sum_{j=1}^n y_j\xi_j\|^2$ for arbitrary finite sets $\{y_1, \ldots, y_n\} \subseteq \Theta(E)$ and $\{\xi_1, \ldots, \xi_n\} \subseteq \mathcal{H}$ since $y_i^* y_j \in R$ by (4.2).) It is easy to check that $\varphi(r') \in \psi(L)'$ for all $r' \in R'$ and that φ is a *-homomorphism from R' into $\psi(L)'$. We conclude that

$$\pi(\Lambda)' = \{\begin{bmatrix} \varphi(r') & 0 \\ 0 & r' \end{bmatrix} : \ r' \in R'\}.$$

Moreover, since $[\Theta(E)^*\mathcal{K}] = \mathcal{H}$, a similar argument shows that for each $b' \in \psi(L)'$ there is a unique $\sigma(b') \in R'$ satisfying $\sigma(b')y^* = y^*b'$ for all $y \in \Theta(E)$ and it is easy to verify that σ is inverse to φ, hence φ is a $*$-isomorphism from R' to $\psi(L)'$.

Put now

$$F = \{y \in \mathrm{B}(\mathcal{H}, \mathcal{K}) : \ \varphi(r')y = yr' \ \ (\forall r' \in R')\}.$$

From the above description of $\pi(\Lambda)'$ we immediately compute that

$$\pi(\Lambda)'' = \{\begin{bmatrix} b & x \\ y^* & r \end{bmatrix} : \ r \in R, \ b \in \psi(L)'', \ x, y \in F\}.$$

From $\pi(\Lambda)'' = \overline{\pi(\Lambda)}$ it then follows that $\overline{\Theta(E)} = F$. By Theorem 4.1 F is a self-dual Hilbert R-module and we have $\mathrm{L}(F) = \varphi(R')' = \psi(L)'' = \overline{\psi(L)}$. Note that each $y \in F$ defines a bounded module homomorphism from $\Theta(E)$ to R by $x \mapsto y^*x$, hence $F = \Theta(E)$ if E is self-dual. We summarize all these conclusions in the following theorem.

THEOREM 4.2. *Let E be a faithful Hilbert module over a von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$. Then there exist a Hilbert space $\mathcal{K}$ and a faithful representation $\pi : \Lambda(E) \to \mathrm{B}(\mathcal{K} \oplus \mathcal{H})$ such that the restriction of π to R is the identity representation (and $\pi(R)\mathcal{K} = 0$), the restriction of π to $L \overset{\text{def}}{=} \mathrm{L}(E)$ is a faithful representation ψ of L on $\mathcal{K}$ (and $\pi(L)\mathcal{H} = 0$) and the restriction of π to E defines an isometry $\Theta : E \to \mathrm{B}(\mathcal{H}, \mathcal{K})$ such that $[\Theta(E)\mathcal{H}] =$*

*$\mathcal{K}$ and the relations (4.2) and (4.3) are satisfied. Moreover, there exists a (necessarily normal) *-isomorphism $\varphi : R' \to \psi(R')'$ such that the weak* closure $\overline{\Theta(E)}$ of $\Theta(E)$ is the self-dual Hilbert R-module consisting of all $y \in \mathrm{B}(\mathcal{H},\mathcal{K})$ satisfying $\varphi(r')y = yr'$ for all $r' \in R'$, while the weak* closure of $\psi(L)$ coincides with $\varphi(R')' = \mathrm{L}(\overline{\Theta(E)})$. Each bounded module homomorphism from $\Theta(E)$ to R can be represented by a unique element of $\overline{\Theta(E)}$. If E is self-dual, then $\overline{\Theta(E)} = \Theta(E)$ and $\mathrm{L}(E)$ is a W^*-algebra.*

5. Duality. Now it is possible to extend the familiar duality between $\overline{\mathcal{H}}\overset{\wedge}{\otimes}\mathcal{H}$ and $\mathrm{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, to the context of Hilbert modules. We shall first formulate the result in terms of the Haagerup tensor product, then we show that this product coincides with the projective operator tensor product if the second factor is a Hilbert module.

If E is a faithful self-dual Hilbert module over a von Neumann algebra R let $E^*\breve{\otimes}^h_R E$ be the space of all completely bounded mappings from $\mathrm{L}(E)$ to R of the form

$$(x^*\breve{\odot}_R y)(b) = \sum_{i\in\mathcal{I}} \langle x_i, by_i\rangle \quad (b \in \mathrm{L}(E)),$$

where $\mathcal{I}$ is a sufficiently large cardinal so that $\Lambda(E)$ can be represented faithfully and normally on a Hilbert space of dimension $\mathcal{I}$ and $x,\ y$ are from the space $\mathrm{C}(E)$ of all columns of length $\mathcal{I}$ that represent bounded operators and have components x_i and y_i in E.

THEOREM 5.1. *Let R be a von Neumann algebra, C the center of R and E a faithful (right) Hilbert R-module. Then the linear map*

$$\mu : E^*\otimes^h_C E \to \mathrm{CB}(\mathrm{L}(E), R), \quad \mu(x^* \otimes_C y)(b) = \langle x, by\rangle \quad (x, y \in E,\ b \in \mathrm{L}(E))$$

is completely isometric. If $R = C$ (that is, if E is a Hilbert C-module) and E is self-dual, then

$$E^*\breve{\otimes}^h_C E = \mathrm{NCB}_C(\mathrm{L}(E), C),$$

where $\mathrm{NCB}_C(\mathrm{L}(E), C)$ is the space of all normal completely bounded C-module homomorphisms from $\mathrm{L}(E)$ to C.

P r o o f. Let $\mathcal{H}$ be the Hilbert space on which R acts. By Theorem 4.2 there exist a Hilbert space $\mathcal{K}$ and a faithful normal representation $\varphi : R' \to \mathrm{B}(\mathcal{K})$ such that E can be identified with a subspace of $\mathrm{B}(\mathcal{H},\mathcal{K})$ in such a way that its weak* closure $\overline{E}$ consists of all $x \in \mathrm{B}(\mathcal{H},\mathcal{K})$ satisfying $xr' = \varphi(r')x$ for all $r' \in R'$, $\overline{\mathrm{L}(E)} = \varphi(R')' = \mathrm{L}(\overline{E})$ and $\mathcal{K} = [E\mathcal{H}]$. Note that the center $\tilde{C}$ of $\overline{\Lambda(E)}$ is

$$\tilde{C} = \{\begin{bmatrix} \varphi(c) & 0 \\ 0 & c \end{bmatrix} :\ c \in C\},$$

hence $\tilde{C}$ is isomorphic to C. Put $L = \mathrm{L}(E)$ and $\Lambda = \Lambda(E)$. From Theorem 2.2 we have a complete isometry

$$\vartheta : \overline{\Lambda} \otimes^h_{\tilde{C}} \overline{\Lambda} \to \mathrm{CB}(\overline{\Lambda}), \quad \vartheta(x^* \otimes_{\tilde{C}} y)(a) = x^*ay.$$

If $x, y \in E$ and $b \in \Lambda$ is of the form

$$b = \begin{bmatrix} \hat{b} & v \\ w^* & r \end{bmatrix} \quad (v, w \in E,\ r \in R,\ \hat{b} \in L), \tag{5.1}$$

then $\mu(x^* \otimes_C y)(\hat{b}) = \vartheta(x^* \otimes_{\tilde{C}} y)(b)$ by a straightforward computation. By linearity and continuity we then have $\mu(t)(\hat{b}) = \vartheta(t)(b)$ for all $t \in E^* \otimes_C^h E$ and $b \in \Lambda$. Thus $\mu(t)_n(\hat{b}) = \vartheta(t)_n(b)$ for all matrices $b = [b_{ij}] \in \mathrm{M}_n(\Lambda)$ (n arbitrary), where $\hat{b} = [\hat{b}_{ij}]$ and $\hat{b}_{ij} \in L$ is the entry of b_{ij} on the position (1,1). Since $\|\hat{b}\| \le \|b\|$ for each $b \in \mathrm{M}_n(\Lambda)$ and each n, it follows easily that $\|\mu(t)\|_{\mathrm{cb}} = \|\vartheta(t)|\Lambda\|_{\mathrm{cb}}$. By the Kaplansky density theorem the unit ball of $\mathrm{M}_n(\Lambda)$ is strongly dense in the unit ball of $\mathrm{M}_n(\overline{\Lambda})$ for each n; since the operators $\vartheta(t)_n$ are strongly continuous, it follows that $\|\vartheta(t)\|_{\mathrm{cb}} = \|\vartheta(t)|\Lambda\|_{\mathrm{cb}}$ and we conclude that $\|\mu(t)\|_{\mathrm{cb}} = \|\vartheta(t)\|_{\mathrm{cb}} = \|t\|$ for each $t \in E^* \otimes_C^h E$, where $\|t\|$ denotes the Haagerup norm (see Theorem 2.1). This proves that μ is isometric and a similar argument shows that μ is in fact completely isometric.

Suppose now that $R = C$ and E is self-dual, hence $\overline{E} = E$ and $\overline{\Lambda} = \Lambda$. Let $t \in \mathrm{NCB}_C(L, R)$. We have to show that $t = x^* \breve{\odot}_C y$ for suitable $x, y \in \mathrm{C}(E)$. Consider the extension $\tilde{t} \in \mathrm{NCB}_C(\Lambda)$ of t defined by

$$\tilde{t}(b) = t(\hat{b}),$$

where $b \in \Lambda$ is represented in the form (5.1). By Proposition 2.3 there exist $u, z \in \mathrm{C}(\Lambda)$ such that $\tilde{t} = u^* \odot_{\tilde{C}} z$. Denote by u_i and z_i the components of u and z (respectively), let $x_i^* \in E^*$ be the entry of u_i^* on the position (2,1), let $y_i \in E$ be the entry of z_i on the position (1,2) and let $x, y \in \mathrm{C}(E)$ have components x_i and y_i. Then an easy computation shows that $t = x^* \breve{\odot}_C y$. ∎

In the case $C = \mathbb{C}$ the following theorem was proved by Effros and Ruan in [15].

Theorem 5.2. *If C is an abelian von Neumann algebra, X an operator C-bimodule such that $cx = xc$ for all $x \in X$ and $c \in C$, and E is a right Hilbert C-module considered also as a left C-module by $cy = yc$ ($c \in C$, $y \in E$), then*

$$X \overset{\wedge}{\otimes}_C E = X \otimes_C^h E.$$

To prove this theorem, we need an auxiliary result. In a self-dual Hilbert module over a von Neumann algebra there is an analogue of orthonormal basis in Hilbert spaces (see [27] and [30]), but here we consider modules that are not necessarily self-dual. We shall need the following lemma only in the case R is abelian.

Lemma 5.3. *Let R be a countably decomposable or abelian von Neumann algebra and E a Hilbert R-module. Then for arbitrary elements $y_1, \ldots, y_n$ in E there exist orthogonal elements $v_1, \ldots, v_n$ in E and a unitary matrix $[r_{ij}] \in \mathrm{M}_n(R)$ such that*

$$y_i = \sum_{j=1}^{n} v_j r_{ji} \quad (i = 1, \ldots, n).$$

Proof. Let $g = [\langle y_i, y_j \rangle] \in \mathrm{M}_n(R)$. There exists a unitary matrix $u = [r_{ij}] \in \mathrm{M}_n(R)$ such that the matrix $d \overset{\mathrm{def}}{=} ugu^*$ is diagonal (if R is commutative this follows from [29], if R is countably decomposable see [21, 6.9.35]). Put

$$v_i = \sum_{j=1}^{n} y_j r_{ij}^* \quad (i = 1, \ldots, n).$$

Then

$$\sum_{j=1}^{n} v_j r_{ji} = \sum_{j,k=1}^{n} y_k r_{jk}^* r_{ji} = \sum_{k=1}^{n} y_k \delta_{ik} = y_i$$

and

$$\langle v_i, v_j \rangle = \sum_{k,l=1}^{n} r_{ik} \langle y_k, y_l \rangle r_{jl}^* = \sum_{k,l=1}^{n} r_{ik} g_{kl} r_{jl}^* = d_{ij}$$

for all $i, j = 1, \ldots, n$. In particular, $\langle v_i, v_j \rangle = 0$ for $i \neq j$. ■

P r o o f o f T h e o r e m 5.2. Since for each positive integer m the operator projective norm dominates the Haagerup norm on $\mathrm{M}_m(X \otimes_C E)$ by Theorem 2.4, it suffices to show that $\|t\|_\wedge \leq \|t\|_h$ for each $t \in \mathrm{M}_m(X \otimes_C E)$. We may assume that $\|t\|_h < 1$, which means that $t = x \odot_C y$ for some $x \in \mathrm{M}_{m,k}(X)$ and $y \in \mathrm{M}_{k,m}(E)$ satisfying $\|x\| < 1$ and $\|y\| < 1$. Let $n = mk$. By Lemma 5.3 there exist orthogonal elements $v_1, \ldots, v_n$ in E and elements $c_{ij}^l \in C$ such that

$$y_{ij} = \sum_{l=1}^{n} v_l c_{ij}^l \quad (\text{for all } i = 1, \ldots, k;\ j = 1, \ldots, m). \tag{5.2}$$

We may assume that $\Lambda(E)$ is represented as in Theorem 4.2; in particular we assume that $E \subseteq \mathrm{B}(\mathcal{H}, \mathcal{K})$ for some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Consider the polar decomposition, $v_l = u_l |v_l|$, of v_l in the weak* closure of $\Lambda(E)$. Observe that $u_l |v_l|^s \in E$ for each real $s > 0$. (Indeed, approximating $|v|^s$ by polynomials in non-zero powers of $|v|$ we see that $u_l |v_l|^s \in \Lambda(E)$. Since the only nonzero entry of $u_l |v_l|^s$ can be on the position (1,2), it follows that $u_l |v_l|^s \in E$.) For $0 \leq s \leq 1$ put

$$u_l(s) = u_l |v_l|^s \quad \text{and} \quad d_{ij}^l(s) = |v_l|^{1-s} c_{ij}^l.$$

Then from (5.2) we have

$$y_{ij} = \sum_{l=1}^{n} u_l(s) d_{ij}^l(s). \tag{5.3}$$

Note that $u_l(0) = u_l$, observe that the ranges of u_l are mutually orthogonal (since the elements v_l are orthogonal in E) and u_l is isometric on the range of $d_{ij}^l(0)$.

C l a i m. The norm of the matrix $d = [d_{ij}^l(0)] \in \mathrm{M}_{kn,m}(C)$ (where (i, l) is the row index and j the column index) is equal to the norm of $y = [y_{ij}] \in \mathrm{M}_{k,m}(E)$.

Indeed, for an arbitrary $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{H}^m$ we have (by (5.3) with $s = 0$)

$$\begin{aligned}\|y\xi\|^2 &= \sum_{i=1}^{k} \big\| \textstyle\sum_{j=1}^{m} y_{ij} \xi_j \big\|^2 = \displaystyle\sum_{i=1}^{k} \big\| \textstyle\sum_{j=1}^{m} \sum_{l=1}^{n} u_l d_{ij}^l(0) \xi_j \big\|^2 \\ &= \sum_{i=1}^{k} \sum_{l=1}^{n} \big\| \textstyle\sum_{j=1}^{m} d_{ij}^l(0) \xi_j \big\|^2 = \|d\xi\|^2.\end{aligned}$$

This proves the Claim.

Since $d_{ij}^l(s)$ converges in norm to $d_{ij}^l(0) = d_{ij}^l$ as $s \to 0$ for all i, j, l, it follows by the above Claim that $\|d(s)\| < 1$ for all sufficiently small $s > 0$, where $d(s) = [d_{ij}^l(s)] \in$

$\mathrm{M}_{kn,m}(C)$. Since the ranges of $u_1(s),\ldots,u_n(s)$ are mutually orthogonal, the norm of the matrix $u(s)\stackrel{\text{def}}{=}[u_1(s),\ldots,u_n(s)]\in \mathrm{M}_{1,n}(E)$ satisfies

$$\|u(s)\|^2 = \|[u_i^*(s)u_l(s)]\| = \max_{1\le l\le n}\||v_l|^{2s}\|$$

and it follows that $\|u(s)\|\to 1$ as $s\to 0$ (except in the case all v_l are 0, but then $y=0$ and the proof is trivial).

Finally, observe (using (5.3)) that

$$t = x\odot_C y = 1_m(x\otimes_C u(s))d(s),$$

where 1_m is the identity matrix in $\mathrm{M}_m(C)$, hence $\|t\|_\wedge \le \|x\|\|u(s)\|\|d(s)\|$. It follows that $\|t\|_\wedge < 1$ for all sufficiently small positive s. Thus, $\|t\|_h < 1$ implies that $\|t\|_\wedge < 1$ for an arbitrary t, which proves that $\|t\|_\wedge \le \|t\|_h$. ■

6. Appendix. In this appendix we shall sketch the proof of Theorem 2.1 following [25], where a slightly more general situation is considered. Throughout this section $R\subseteq \mathrm{B}(\mathcal{H})$ will be a von Neumann algebra and $X,Y\subseteq \mathrm{B}(\mathcal{H})$ two (weak* closed) subspaces such that $XR\subseteq X$ and $RY\subseteq Y$. As in Section 2 we denote by $\mathcal{I}$ a sufficiently large cardinal (for example, $\mathcal{I}=\dim\mathcal{H}$) and by $\mathrm{M}(R)$ the set of all $\mathcal{I}\times\mathcal{I}$ matrices with entries in R that represent bounded operators on $\mathcal{H}^{\mathcal{I}}$; also $\mathrm{R}(X)$ and $\mathrm{C}(Y)$ have the same meaning as in Section 2. Further, we use the notation $\mathrm{R}_n(X)=\mathrm{M}_{1,n}(X)$ and $\mathrm{C}_n(Y)=\mathrm{M}_{n,1}(Y)$. Given two operator matrices $x\in\mathrm{M}_{m,n}(\mathrm{B}(\mathcal{H}))$ and $y\in\mathrm{M}_{n,p}(\mathrm{B}(\mathcal{H}))$, $x\odot_R y$ is the matrix $[x_i\odot_R y_j]$ of completely bounded maps from R' to $\mathrm{B}(\mathcal{H})$, where x_i is a row of x, y_j is a column of y and $x_i\odot_R y_j$ is defined by (2.1). (We use this notation also for infinite matrices.) Note that $\mathrm{R}(X)$ and $\mathrm{C}(Y)$ are regarded as subspaces of $\mathrm{M}(\mathrm{B}(\mathcal{H}))=\mathrm{B}(\mathcal{H}^{\mathcal{I}})$ by embedding into the first row and column, respectively. Thus, given $x\in\mathrm{R}(X)$ and $y\in\mathrm{C}(Y)$ we have by definition $(x\otimes_{\mathrm{M}(R)}y)(r'^{(\mathcal{I})}) = xr'^{(\mathcal{I})}y$ for all $r'^{(\mathcal{I})}\in R'^{(\mathcal{I})}=\mathrm{M}(R)'$.

In each of the spaces $\mathrm{M}_n(X\overline{\otimes}_R^h Y)$ (n an integer) we introduce a new norm by

$$\|\vartheta\| = \inf\{\|x\|\|y\|:\ \vartheta = x\odot_R y,\ x\in\mathrm{M}_{n,\mathcal{I}}(X),\ y\in\mathrm{M}_{\mathcal{I},n}(Y)\}$$

and we denote the space $X\overline{\otimes}_R^h Y$ equipped with this new matricial norm structure by $X\tilde{\otimes}_R^h Y$. (This is only a temporal notation, we shall see that the new norm is the same as the old one, which is essentially the content of Theorem 2.1.) The norm of an element $\vartheta\in\mathrm{M}_n(X\overline{\otimes}_R^h Y)$ regarded as a completely bounded map will be denoted by $\|\vartheta\|_{\mathrm{cb}}$. It can be proved (see [25]) that $\|\cdot\|$ is an everywhere defined norm and that $X\tilde{\otimes}_R^h Y$ is an (abstract) operator space.

The following lemma is motivated by a simple observation in [24, p. 332] and for the proof see [25, Lemma 3.4].

LEMMA 6.1. *The map*

$$\Phi: X\tilde{\otimes}_R^h Y\to \mathrm{R}(X)\tilde{\otimes}_{\mathrm{M}(R)}^h \mathrm{C}(Y),\quad \Phi(x\odot_R y) = x\otimes_{\mathrm{M}(R)} y$$

is a completely isometric isomorphism.

LEMMA 6.2. *Let n be a positive integer and regard* $\mathrm{R}_n(\mathrm{B}(\mathcal{H}))$ *and* $\mathrm{C}_n(\mathrm{B}(\mathcal{H}))$ *as subspaces in* $\mathrm{M}_n(\mathrm{B}(\mathcal{H}))=\mathrm{B}(\mathcal{H}^n)$ *by embedding into the first row and column, respectively.*

Then every element $\vartheta \in \mathrm{M}_n(X\tilde{\otimes}^h_R Y)$ *can be represented as*

$$\vartheta = x\odot_R y$$

for suitable $x \in \mathrm{M}_{n,\mathcal{I}}(X)$ *and* $y \in \mathrm{M}_{\mathcal{I},n}(Y)$. *Moreover,*

$$\mathrm{M}_n(X\tilde{\otimes}^h_R Y) = \mathrm{C}_n(X)\tilde{\otimes}^h_{R^{(n)}}\mathrm{R}_n(Y) \text{ and } \mathrm{M}_n(X\overline{\otimes}^h_R Y) = \mathrm{C}_n(X)\overline{\otimes}^h_{R^{(n)}}\mathrm{R}_n(Y)$$

completely isometrically.

Proof. We shall prove here only the last identity, the proof of the rest of the lemma can be found in [25, Lemma 3.1]. We shall define a map

$$\Omega : \mathrm{M}_n(X\overline{\otimes}^h_R Y) \to \mathrm{C}_n(X)\overline{\otimes}^h_{R^{(n)}}\mathrm{R}_n(Y).$$

Given $v \in \mathrm{C}_n(X)$ we denote by $\tilde{v} \in \mathrm{M}_n(X)$ the matrix with first column v and the remaining columns 0. Similarly, for $w \in \mathrm{R}_n(Y)$ let $\tilde{w} \in \mathrm{M}_n(Y)$ be the matrix with first row w and the remaining rows 0. Further, given a matrix $x \in \mathrm{M}_{n,\mathcal{I}}(X)$ with columns $x^i \in \mathrm{C}_n(X)$, let $\tilde{x} \in \mathrm{R}_{\mathcal{I}}(\mathrm{M}_n(Y))$ have the components $\tilde{x^i}$. Similarly, we define $\tilde{y} \in \mathrm{C}_{\mathcal{I}}(\mathrm{M}_n(Y))$ for each $y \in \mathrm{M}_{\mathcal{I},n}(Y)$. Finally, to each $\vartheta \in \mathrm{M}_n(X\overline{\otimes}^h_R Y)$ we associate an element $\tilde{\vartheta} \in \mathrm{C}_n(X)\overline{\otimes}^h_{R^{(n)}}\mathrm{R}_n(Y)$ as follows. Choose $x \in \mathrm{M}_{n,\mathcal{I}}(X) = \mathrm{R}_{\mathcal{I}}(\mathrm{C}_n(X))$ and $y \in \mathrm{M}_{\mathcal{I},n}(Y) = \mathrm{C}_{\mathcal{I}}(\mathrm{R}_n(Y))$ such that $\vartheta = x \odot_R y$ (for the verification that this can be done see the proof of Lemma 3.1 in [25]) and put $\tilde{\vartheta} = \tilde{x} \odot_{R^{(n)}} \tilde{y}$.

It is not hard to verify that $\tilde{\vartheta}([r'_{ij}]) = [\vartheta_{ij}(r'_{11})]$ for each $\vartheta = [\vartheta_{ij}] \in \mathrm{M}_n(X\overline{\otimes}^h_R Y)$ and each $[r'_{ij}] \in \mathrm{M}_n(R')$, from which one can deduce in particular that the correspondence $\Omega : \vartheta \mapsto \tilde{\vartheta}$ is a well defined linear bijection from $\mathrm{M}_n(X\overline{\otimes}^h_R Y)$ to $\mathrm{C}_n(X)\overline{\otimes}^h_{R^{(n)}}\mathrm{R}_n(Y)$. Thus, denoting by $\rho : \mathrm{M}_n(R') \to R'$ the projection onto the (1,1) position and by $\kappa : R' \to \mathrm{M}_n(R')$ the embedding into the (1,1) position and regarding ϑ as a completely bounded map from R' to $\mathrm{M}_n(\mathrm{B}(\mathcal{H}))$, we have that $\tilde{\vartheta} = \vartheta\rho$ and $\vartheta = \tilde{\vartheta}\kappa$. Since ρ and κ are complete contractions, it follows that $\|\tilde{\vartheta}\|_{\mathrm{cb}} = \|\vartheta\|_{\mathrm{cb}}$, hence Ω is a complete isometry. ∎

The following lemma is just Lemma 2.1 from [24] (slightly simplified), so we omit the proof (see also formulas (2.4) and (2.4') in [24]).

LEMMA 6.3. *Given* $x, y, z, w \in \mathrm{B}(\mathcal{H})$, *the identity*

$$zr'w = xr'y$$

holds for all $r' \in R'$ *if and only if there exist two commuting projections* $f, f' \in R$ *satisfying* $zf' = z$ *and* $fw = w$, *two decreasing sequences of* (*commuting*) *projections* e_k *and* e'_k *in* R *such that* $\lim \|xe_k\| = 0$ *and* $\lim \|e'_k y\| = 0$, *and two sequences* u_k *and* v_k *in* R *such that* $u_k v_k = 1 - e_k - e'_k$, $\|xu_k\| \le \|z\|$ *and* $\|v_k y\| \le \|w\|$ *for all* k, *and the sequences* xu_k *and* $v_k y$ *converge to* zf *and* $f'w$, *respectively, in the strong operator topology.*

Proof of Theorem 2.1. By definition $X\tilde{\otimes}^h_R Y$ and $X\overline{\otimes}^h_R Y$ are the same set. It suffices to prove that the two norms $\| \ \|$ and $\| \ \|_{\mathrm{cb}}$ agree on $X\overline{\otimes}^h_R Y$ for arbitrary X, Y and R, for then we shall have for each n by Lemma 6.2

$$\mathrm{M}_n(X\tilde{\otimes}^h_R Y) = \mathrm{C}_n(X)\tilde{\otimes}^h_{R^{(n)}}\mathrm{R}_n(Y) = \mathrm{C}_n(X)\overline{\otimes}^h_{R^{(n)}}\mathrm{R}_n(Y) = \mathrm{M}_n(X\overline{\otimes}^h_R Y)$$

isometrically.

We have already mentioned that the inequality $\|\vartheta\|_{\mathrm{cb}} \le \|\vartheta\|$, $(\vartheta \in X\overline{\otimes}^h_R Y)$ is easy (as in the classical case). To prove the reverse inequality, suppose that $\|\vartheta\|_{\mathrm{cb}} < 1$; we shall prove that then $\|\vartheta\| < 1$ (this suffices to conclude the proof). Since $\vartheta \in X\overline{\otimes}^h_R Y$, $\vartheta = x\odot_R y$ for some $x \in \mathrm{R}(X)$ and $y \in \mathrm{C}(Y)$. On the other hand, by the representation theorem for normal completely bounded linear mappings [32] there exist $z \in \mathrm{R}(\mathrm{B}(\mathcal{H}))$ and $w \in \mathrm{C}(\mathrm{B}(\mathcal{H}))$ such that $\|z\|, \|w\| < 1$ and $z\odot_R w = x\odot_R y$. By Lemma 6.3 (applied in $\mathrm{M}(\mathrm{B}(\mathcal{H})) = \mathrm{B}(\mathcal{H}^{\mathcal{I}})$) there exist four sequences (u_k), (v_k), (e_k) and (e'_k) in $\mathrm{M}(R)$ such that $e_k + e'_k + u_k v_k = 1$, $\|xe_k\| \to 0$, $\|e'_k y\| \to 0$, $\|xu_k\| \le \|z\|$ and $\|v_k y\| \le \|w\|$. Note that

$$\begin{aligned}\|x \otimes_{\mathrm{M}(R)} y - xu_k \otimes_{\mathrm{M}(R)} v_k y\| &= \|x \otimes_{\mathrm{M}(R)} (1 - u_k v_k)y\| = \|x \otimes_{\mathrm{M}(R)} (e_k + e'_k)y\| \\ &= \|xe_k \otimes_{\mathrm{M}(R)} y + x \otimes_{\mathrm{M}(R)} e'_k y\| \le \|xe_k\|\|y\| + \|x\|\|e'_k y\| \longrightarrow 0.\end{aligned}$$

By Lemma 6.1 it follows that $\lim_{k\to\infty} \|x\odot_R y - xu_k\odot_R v_k y\| = 0$. Since $\|xu_k\| \le \|z\| < 1$ and $\|v_k y\| \le \|w\| < 1$ (and $xu_k \in \mathrm{R}(X)$, $v_k y \in \mathrm{C}(Y)$ since X and Y are weak* closed), we see that $\|\vartheta\| = \|x\odot_R y\| < 1$. ■

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Added in proof. The author has been recently informed that the fact that bounded module maps between Hilbert C^*-modules are completely bounded (which follows from Theorem 3.3) was proved already by G. Wittstock in *Extension of completely bounded C^*-module homomorphisms*, Operator Algebras and Group Representations, Volume II, Monographs Stud. Math. 18, Pitman, 1984, 238–250.