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THE BOUNDARY SPECTRUM OF LINEAR OPERATORS IN FINITE-DIMENSIONAL SPACES

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1. The basic concepts. Let X be an n-dimensional real or complex vector space, $n<\infty$. Let $A:X\to X$ be a linear operator. It is called *power bounded* if the semigroup $\Pi(A)=\{A^k\}_0^\infty$ of its natural powers is bounded. (This property can be defined in terms of any norm: $\sup\{\|A^k\|:k\in\mathbb{N}\}<\infty$. The choice of the norm does not matter.) An operator A is called *double power bounded* if A is invertible and A, A^{-1} are both power bounded. (Thus, $\sup\{\|A^k\|:k\in\mathbb{Z}\}<\infty$.) We only consider the power bounded operators. Since in this case the spectral radius $r(A)$ does not exceed 1, the spectrum $\sigma(A)$ consists of two parts, the *boundary* (or *peripheral*) *spectrum*

$$\sigma_1(A)=\{\lambda\ :\lambda\in\sigma(A),|\lambda|=1\}$$

and the *interior spectrum*

$$\sigma_0(A)=\{\lambda\ :\lambda\in\sigma(A),|\lambda|<1\}.$$

(Speaking about spectra in the real case we implicitly change A for its natural complex extension $A_{\mathbb{C}}$.) One of these parts of $\sigma(A)$ may be empty; $\sigma_1(A)\neq\emptyset$ iff $r(A)=1$, $\sigma_0(A)=\emptyset$ iff A is double power bounded (being power bounded). According to this partition of the spectrum we have the decomposition of the space X into the direct sum

$$X=X_1\dotplus X_0 \tag{$*$}$$

where X_1 and X_0 are invariant subspaces such that

$$\sigma(A|X_1)=\sigma_1(A),\qquad \sigma(A|X_0)=\sigma_0(A),$$

the so-called *boundary* and *interior* subspaces. This terminology also applies to the parts of the operator A, $A_1=A|X_1$ and $A_0=A|X_0$. The following quite elementary proposition is in fact the starting point of a rather deep theory presented below.

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PROPOSITION 1.1. *The boundary part A_1 is double power bounded. The interior part A_0 is such that* $\lim_{k\to\infty} A_0^k = 0$.

To explain a key role of this statement let us note that $\Pi(A)$ is precompact being a bounded subset of $L(X)$ where $L(X)$ is the space of all linear operators in X, $\dim L(X) = n^2 < \infty$. The closure $\overline{\Pi(A)}$ is a compact semigroup. In this sense A is an *almost periodic operator*. It is a finite-dimensional version of a general statement regarding operators in a Banach space or elements of a Banach algebra (see for example [11, Chapter 4]). Proposition 1.1 corresponds to the so-called *de Leeuw–Glicksberg decomposition* [6] or, in terms of [11], the *boundary spectrum splitting-off theorem*. There are a lot of interesting applications of the general theory of almost periodic operators to Markov chains (e.g. [3], [10], [19]), dynamical systems (e.g. [7], [8], [12], [13], [16], [22]), harmonic analysis and spectral theory (e.g. [6], [9], [11], [21]). Though this development is rather recent, its roots are in an abstract theory of compact semigroups which arose much earlier. A principal concept in this way is the *Sushkevich kernel* (*S.k.*).

2. The Sushkevich kernel. This object can be formally defined as the smallest two-sided ideal of a given semigroup but this may not exist if the semigroup is taken arbitrarily. For example, there is no S.k. in the additive semigroup $\mathbb{N}$. However, the S.k. does exist in any finite semigroup [20] and even in any compact semigroup [17]. Moreover, it has a lot of remarkable properties, among them:

THEOREM 2.1. *For any compact commutative semigroup its S.k. is a compact group* (*whose unit is an idempotent*).

Coming back to our case of a power bounded linear operator A in a space X, $\dim X = n < \infty$, we can introduce the following

DEFINITION 2.2. The S.k. of the semigroup $\overline{\Pi(A)}$ is called the *S.k. of A*.

We denote it by $K(A)$. Let us emphasize that $\overline{\Pi(A)}$ is, obviously, commutative, so Theorem 2.1 is applicable. However, we do not need it because the S.k. $K(A)$ can be constructed directly in our context (cf. [4]). For this purpose we consider the projection P connected with the decomposition $(*)$. We call it the *boundary projection* of the operator A and prove

LEMMA 2.3. $P \in \overline{\Pi(A)}$.

Proof. Let $\{k_m\}_0^\infty \subset \mathbb{N}$ be such that $k_{m+1} - k_m \to \infty$ and the limit $U = \lim_{m\to\infty} A_1^{k_m}$ exists. Then

$$U^{-1} = \lim_{m\to\infty} A_1^{-k_m} \quad \text{and} \quad \lim_{m\to\infty} A_1^{k_{m+1}-k_m} = UU^{-1} = \mathrm{id} = P|X_1.$$

On the other hand, $\lim A_0^{k_{m+1}-k_m} = 0 = P|X_0$. Thus,

$$(**) \qquad P = \lim_{m\to\infty} A^{k_{m+1}-k_m}. \ \blacksquare$$

LEMMA 2.4. *For any operator $B \in \overline{\Pi(A)}$ the smallest ideal containing B is closed. Furthermore*, $B\overline{\Pi(A)} = \overline{B\Pi(A)}$.

Proof. This ideal is $B\overline{\Pi(A)}$. It is closed since $\overline{\Pi(A)}$ is compact. Obviously, $\overline{B\Pi(A)} \subset B\overline{\Pi(A)}$. On the other hand, if $C \in B\overline{\Pi(A)}$ then

$$C = B(\lim_{m\to\infty} A^{q_m}) = \lim_{m\to\infty}(BA^{q_m})$$

for a sequence $\{q_m\}_0^\infty$. Passing to a convergent subsequence we obtain $C \in \overline{B\Pi(A)}$. ∎

Corollary 2.5. *Every ideal of the semigroup $\overline{\Pi(A)}$ contains a closed ideal.*

Now we can get $K(A)$ in the following way.

Theorem 2.6. *The ideal of the semigroup $\overline{\Pi(A)}$ generated by the boundary projection P is its S.k.*

$$K(A) = P\overline{\Pi(A)}.$$

It is a compact group with the unit P.

Proof. For every ideal $I \subset \overline{\Pi(A)}$ we have to prove that $P\overline{\Pi(A)} \subset I$. One can assume I to be closed because of Corollary 2.5. Let $B \in I$. Then B is a power bounded operator with the same decomposition $(*)$. By Lemma 2.3, $P \in \overline{\Pi(B)} \subset I$. Hence $P\overline{\Pi(A)} \subset I$. Thus, $P\overline{\Pi(A)}$ is the S.k. of $\overline{\Pi(A)}$, $K(A) = P\overline{\Pi(A)}$.

By Lemma 2.4, $K(A)$ is closed. It is a semigroup being an ideal. The boundary projection P is its unit since $P = P^2 \in P\overline{\Pi(A)}$ by Lemma 2.3, and if $B \in K(A)$ then $B = PC$ with $C \in \overline{\Pi(A)}$, hence $PB = PC = B$. Let

$$B = P(\lim_{m\to\infty} A^{l_m})$$

for a sequence $\{l_m\}_0^\infty$. Coming back to $(**)$ one can assume that $k_{m+1} - k_m - l_m \to \infty$ and the limit

$$\widehat{B} = \lim_{m\to\infty} A^{k_{m+1}-k_m-l_m}$$

exists. Then $B\widehat{B} = P^2 = P$. Thus, $K(A)$ is a group. ∎

Corollary 2.7. *If A is double power bounded then $K(A) = \overline{\Pi(A)}$.*

We have described $K(A)$ algebraically. The next result yields a dynamical description.

Theorem 2.8. *The S.k. $K(A)$ coincides with the Ω-limit set $\Omega(A)$ of the semigroup $\Pi(A)$.*

Proof. First, $K(A) \subset \Omega(A)$ since $\Omega(A)$ is an ideal in $\overline{\Pi}(A)$. Indeed, $\Omega(A)$ is nonempty and $\Omega(A)A^k \subset \Omega(A)$ for all $k \in \mathbb{N}$. Since $\Omega(A)$ is closed, we get $\Omega(A)\overline{\Pi(A)} \subset \Omega(A)$. To establish the converse inclusion $\Omega(A) \subset K(A)$ we take $B \in \Omega(A)$, i.e.

$$B = \lim_{m\to\infty} A^{l_m}$$

with a sequence $\{l_m\}_0^\infty, l_m \to \infty$. Then $B|X_0 = 0$ hence, $B = PB \in P\overline{\Pi(A)} = K(A)$. ∎

The intrinsic structure of the group $K(A)$ is completely determined by the boundary spectrum $\sigma_1(A)$. Let $\sigma_1(A) = \{\lambda_1, \ldots, \lambda_s\}$. Since all the eigenvalues lie on the unit circle $\mathbb{T}$, one can consider the ordered set $\sigma_1(A)$ as a point a_1 on the s-dimensional torus $\mathbb{T}^s$ which is a compact group (due to the standard group structure on $\mathbb{T}$).

Theorem 2.9 ([4], Ch. 1, Th. 2.4). *There exists a unique continuous monomorphism* $h: K(A) \to \mathbb{T}^s$ *such that* $h(A_1) = a_1$ *where* A_1 *is the boundary part of* A.

Proof. Let

$$A_1 = \sum_{j=1}^{s} \lambda_j P_j$$

be the spectral decomposition, so P_j are projections in X_1 whose images are the corresponding eigenspaces. Moreover,

$$\sum_{j=1}^{s} P_j = \mathrm{id}, \quad P_{j_1} P_{j_2} = 0 \quad (j_1 \neq j_2).$$

Then

$$A_1^l = \sum_{j=1}^{s} \lambda_j^l P_j, \quad l \in \mathbb{N}.$$

If now $B \in K(A)$ then by Theorem 2.8,

$$B = \lim_{m\to\infty} A^{l_m} = \lim_{m\to\infty} A_1^{l_m} P$$

for a sequence $\{l_m\}_0^\infty$, $l_m \to \infty$. Hence,

$$B = \sum_{j=1}^{s} \lambda_j(B) P_j P \quad \text{where} \quad \lambda_j(B) = \lim_{m\to\infty} \lambda_j^{l_m}, \ 1 \le j \le s.$$

(These limits do not depend on the choice of the sequence because $\lambda_j(B)$ is the unique eigenvalue of B in the subspace $\operatorname{Im} P_j$.) Letting $h(B) = (\lambda_1(B), \ldots, \lambda_s(B))$ we obtain the desired homomorphism. The uniqueness statement is obvious. ∎

Corollary 2.10. *The S.k.* $K(A)$ *is topologically isomorphic to a closed subgroup of the torus* $\mathbb{T}^s$ *where* $s = \mathrm{card}[\sigma_1(A)]$. *This subgroup coincides with the semigroup of* $\mathbb{T}^s$ *topologically generated by the point* $a_1 = \sigma_1(A)$.

We denote this subgroup by $\langle a_1 \rangle$. Every closed subgroup $G \subset \mathbb{T}^s$ is of the form $\mathbb{T}^\rho \times F$ where $0 \le \rho \le s$ and F is a finite group. Indeed, the dual group G^* is isomorphic to a factor group of $(\mathbb{T}^s)^* \approx \mathbb{Z}^s$ so G^* is a commutative group generated by some s elements. Therefore $G^* \approx \mathbb{Z}^\rho \times F$, thus $G \approx G^{**} \approx \mathbb{T}^\rho \times F$.

If G is monothetic then F is cyclic. In particular, we have this information about $K(A)$ in virtue of Corollary 2.10. Now we can describe the parameters ρ and $\mathrm{ord}(F)$ in arithmetical terms concerning the boundary spectrum $\sigma_1(A)$.

Let $\lambda_j = \exp(2\pi i \theta_j)$, $0 \le \theta_j < 2\pi$, $1 \le j \le s$. These numbers can be treated as vectors from the space $\mathbb{R}$ over the rational field $\mathbb{Q}$. Let

$$\rho = \mathrm{rank}_{\mathbb{Q}}\{\theta_0, \theta_1, \ldots, \theta_s\} - 1,$$

where $\theta_0 = 1$. Let $\{\theta_0, \ldots, \theta_\rho\}$ be a maximal linearly independent subsystem of the system $\{\theta_0, \theta_1, \ldots, \theta_s\}$. Then for every $\theta_j, j > \rho$, there exists an integer $m_j \ge 1$ such that $m_j\theta_j$ is a linear combination of $\theta_0, \ldots, \theta_\rho$ with some integer coefficients. Let m_j be minimal possible and let m be the least common multiple of $m_{\rho+1}, \ldots, m_s$. (In the case $\rho = s$ we set $m = 1$.)

THEOREM 2.11. *The S.k.* $K(A)$ *is topologically isomorphic to* $\mathbb{T}^\rho \times F$ *where* F *is a cyclic group whose order is a divisor of* m.

P r o o f. The point $a_1 = (\lambda_1, \ldots, \lambda_\rho, \ldots, \lambda_s) \in \mathbb{T}^s$ satisfies some conditions

$$\lambda_j^{m_j} = \prod_{q=1}^{\rho} \lambda_q^{\omega_q j}, \qquad \rho+1 \le j \le s,$$

with integer $\omega_q j$. The same conditions hold for all points $b \in \langle a_1 \rangle$. Consider the canonical projection $r : \mathbb{T}^s \to \mathbb{T}^\rho$ keeping the coordinates with numbers $1, \ldots, \rho$; we get $\mathbb{T}^\rho = \mathrm{Im}(r|\langle a_1 \rangle)$ by the well-known Kronecker theorem. On the other hand, all points $b = (\beta_1, \ldots, \beta_\rho, \ldots, \beta_s)$ from $\Gamma = \ker(r|\langle a_1 \rangle)$ satisfy the conditions

$$\beta_1 = \ldots = \beta_\rho = 1, \qquad \beta_j^{m_j} = 1 \qquad (\rho + 1 \le j \le s).$$

Therefore Γ is finite and $b^m = e$ (e is the unit of $\mathbb{T}^s$) for all $b \in \Gamma$.

We have $\langle a_1 \rangle / \Gamma \approx \mathbb{T}^\rho$ and we know that $\langle a_1 \rangle$ is a direct product of a torus and a cyclic group F. Then this torus must be $\mathbb{T}^\rho$ (up to topological isomorphism) and $F \approx \Gamma$, so $\mathrm{ord}(F)$ is a divisor of m. ∎

The question about the exact value of $\mathrm{ord}(F)$ remains open.

COROLLARY 2.12. $K(A)$ *is infinite if and only if this group contains a subgroup which is topologically isomorphic to* $\mathbb{T}$.

It is just the case $\rho \ge 1$. The opposite case $\rho = 0$ is such that all θ_j, $1 \le j \le s$, are rational or, equivalently, all λ_j, $1 \le j \le s$, are roots of 1. Then we say that the boundary spectrum is *rational*.

COROLLARY 2.13. $K(A)$ *is finite if and only if the boundary spectrum* $\sigma_1(A)$ *is rational.*

This property is of special interest from the dynamical point of view because *every trajectory* $\{A^k x\}_{k=0}^\infty$, $x \in X$, *converges to a limit cycle iff* $K(A)$ *is finite.* By Corollary 2.13 we have a spectral criterion of the cyclic limit behavior: the rationality of the boundary spectrum.

Note that if $\sigma_1(A)$ is rational, so $\lambda_j^{m_j} = 1$, $1 \le j \le s$, and m_j are minimal possible as before then $K(A)$ is a cyclic group of order $\vartheta =$ the least common multiple of m_j. This m is the length of the limit cycles of the trajectories $\{A^k x\}_{k=0}^\infty$ for almost all x.

Below we investigate the problem of rationality of the boundary spectrum relating to the space X endowed with an additional geometric structure.

3. Criteria of the rationality of the boundary spectrum. From now on we only consider the real space X. As a simplest example let us recall the stochastic operators in $\mathbb{R}^n$. The rationality of boundary spectrum in this case was established already by Frobenius. There are at least two independent geometrical reasons for that property, namely, any stochastic operator A in $\mathbb{R}^n$ is: 1) a contraction with respect to the coordinate sup-norm and 2) monotone with respect to the coordinate ordering. Both of them can be included in a more general context.

Let us define a *convex structure* in X as a pair (X,D) where $D\subset X$ is a convex closed body such that $0\in D$. (The term "body" means that $\operatorname{Int} D\neq\emptyset$.)

A linear operator $T:X\to X$ such that $TD\subset D$ is called an *endomorphism* of the convex structure (or *D-endomorphism*). If, moreover, T is invertible and T^{-1} is also an D-endomorphism then T is called a *D-automorphism*. The set $\operatorname{End}(X,D)$ of all *D-endomorphisms* is a semigroup with the usual multiplication of operators. The identity operator I is the unit of this semigroup. The set $\operatorname{Aut}(X,D)$ of all *D-automorphisms* is just the multiplicative group of the semigroup $\operatorname{End}(X,D)$. Note that $\operatorname{Aut}(X,D)\neq\operatorname{End}(X,D)$ since the zero operator belongs to $\operatorname{End}(X,D)$.

A convex structure (X,D) is called *symmetric* if $-D=D$. It is called *completely nonsymmetric* if $(-D)\cap D=\{0\}$.

EXAMPLE 3.1. Let X be a normed space and let D be the unit ball, $D=\{x:\ \|x\|\leq 1\}$. Then (X,D) is a symmetric convex structure. Its endomprhisms are just contractions, $\|A\|\leq 1$, and its automorphisms are just isometries. An additional property of D in this example is compactness. Note that $\operatorname{End}(X,D)$ and $\operatorname{Aut}(X,D)$ are also compact in this case.

EXAMPLE 3.2. Let X be an ordered space and let D be the nonnegative cone, $D=\{x:\ x\geq 0\}$. If D is solid, i.e. $\operatorname{Int}D\neq\emptyset$, then (X,D) is a convex structure but it is completely nonsymmmetric. The semigroup $\operatorname{End}(X,D)$ consists of all monotone operators. It is noncompact because all homotheties λI, $\lambda>0$, are D-endomorphisms. They are even D-automorphisms, so $\operatorname{Aut}(X,D)$ is also noncompact. This is nonclosed in $\operatorname{End}(X,D)$ since $\lambda I\to 0$ as $\lambda\to 0$ but $\operatorname{End}(X,D)$ is obviously closed. As usual, D-endomorphisms are also called the *nonnegative operators* in this case.

We say that a subspace $Y\neq 0$ is *admissible* if $D_Y=D\cap Y$ is a body in Y. In this case we can define a convex structure (Y,D_Y) called a *substructure* of (X,D). Note that if $0\in\operatorname{Int}D$ then all subspaces $Y\neq 0$ are admissible. We call a subspace $Y\neq 0$ *D-complemented* if there exists a projection $Q\in\operatorname{End}(X,D)$ such that $\operatorname{Im}Q=Y$. In the case of the unit ball D in a normed space X that means $\|Q\|=1$ and $\operatorname{Im}Q=Y$; Q is called an *orthoprojection* onto Y and Y is *orthogonally complemented* if such a Q does exist. In the case of a cone D, we say a *positively complemented* subspace for a *D-complemented* one.

LEMMA 3.3. *Every D-complemented subspace is admissible.*

Proof. If $Y=\operatorname{Im}Q$ where $Q\in\operatorname{End}(X,D)$ is a projection, then $D_Y=QD$. Hence, $\operatorname{Int}_Y D_Y\neq\emptyset$ since any projection is an open mapping onto its image. ∎

We are especially interested in power bounded D-endomorphisms. They form a semigroup denoted by $\operatorname{End}_\infty(X,D)$. Accordingly, $\operatorname{Aut}_\infty(X,D)$ is a group of double power bounded D-automorphisms.

LEMMA 3.4. *Let $A\in\operatorname{End}_\infty(X,D)$ and let P be the boundary projection of A. Then $P\in\operatorname{End}(X,D)$.*

Proof. $\overline{\Pi(A)}\subset\operatorname{End}(X,D)$ and $P\in\overline{\Pi(A)}$ by Lemma 2.3. ∎

COROLLARY 3.5. *If* $A \in \mathrm{End}_\infty(X,D)$ *then its boundary subspace* X_1 *is D-complemented.*

By Lemma 3.3 it is admissible. Now we consider the boundary part $A_1 = A|X_1$ in the substructure (X_1, D_1), where $D_1 = D \cap X_1$.

LEMMA 3.6. *If* $A \in \mathrm{End}_\infty(X,D)$ *then* $K(A_1) \subset \mathrm{Aut}_\infty(X_1,D_1)$. *In particular*, $A_1 \in \mathrm{Aut}(X_1,D_1)$.

Proof. First, A_1 is a double power bounded operator. Obviously, $\overline{\Pi(A_1)} \subset \mathrm{End}(X_1,D_1)$. By Corollary 2.7, $K(A_1) \subset \mathrm{End}(X_1,D_1)$. ■

Everything is prepared to prove the following basic result.

THEOREM 3.7. *For a convex structure* (X,D) *the following properties are equivalent.*

1) *The boundary spectrum of every power bounded D-endomorphism is rational.*

2) *For every D-complemented subspace* Y *the group of* D_Y*-automorphisms does not contain any infinite compact subgroup.*

3) *For every D-complemented subspace* Y *the group of* D_Y*-automorphisms does not contain a subgroup which is topologically isomorphic to* $\mathbb{T}$.

Proof. 1)⇒2). Let $\Gamma \subset \mathrm{Aut}(Y,D_Y)$ be a compact subgroup and let Q be a projection onto Y, $Q \in \mathrm{Aut}(X,D_X)$. We consider the subset of $\mathbb{T}$ defined as

$$\sigma(\Gamma) = \bigcup_{V\in\Gamma} \sigma(V).$$

First of all, this "united spectrum" is rational because for every $V \in \Gamma$ we have $VQ \in \mathrm{End}_\infty(X,D)$ and $\sigma(V) = \sigma_1(VQ)$.

Secondly, $\lambda \in \sigma(\Gamma) \Rightarrow \lambda^k \in \sigma(\Gamma)$ for all integers k because of $V \in \Gamma \Rightarrow V^k \in \Gamma$. Finally, the set $\sigma(\Gamma)$ is closed because Γ is compact.

With these properties the subset $\sigma(\Gamma) \subset \mathbb{T}$ must be finite. Hence, there exists an integer $q \ge 1$ such that $\lambda^q = 1$ for all $\lambda \in \sigma(\Gamma)$. Then $V^q = \mathrm{id}$ for all $V \in \Gamma$. Letting $R = V - \mathrm{id}$ we obtain

$$\sum_{j=1}^{q} \binom{q}{j} R^j = 0,$$

which implies $R = 0$ if R is small enough. Thus, the unit element is isolated in Γ, which means that Γ is discrete. Being compact, the group Γ is finite.

2)⇒3) trivially.

3)⇒1). Let $A \in \mathrm{End}_\infty(X,D)$ and let X_1 be its boundary subspace, $D_1 = D \cap X_1$ and $A_1 = A|X_1$ as usual. By Corollary 3.5, X_1 is D-complemented. By Lemma 3.6 and Corollary 2.12 condition 3) implies that $K(A_1)$ is finite. Therefore $\sigma(A_1) = \sigma_1(A)$ is rational. ■

For the unit balls the equivalence 1)⇔2) was obtained in [9] in the same way as above. For the cones it was done in [21]. In the first of those cases the group $\mathrm{Aut}(Y,D_Y)$ is automatically compact, so 2) only means that this group is finite as stated in [9]. However, the main results of those papers are some purely geometric criteria of rationality of the boundary spectrum.

THEOREM 3.8 [9]. *Let X be a normed space. Then the boundary spectrum of every contraction in X is rational if and only if there is no orthogonally complemented 2-dimensional subspace $Y \subset X$ such that the disk D_Y is Euclidean.*

The last property means that there exists an inner product $(\,,)$ in Y such that

$$D_Y = \{y : y \in Y,\ (y,y) \le 1\}.$$

One can say that Y is a *Euclidean plane.*

COROLLARY 3.9 [9]. *The boundary spectrum of every contraction in l_p^n, $1 \le p \le \infty$, $p \ne 2$ is rational.*

The point is that there are no Euclidean planes in l_p^n except for p which are even integers. In the last case a Euclidean plane may exist. This depends on n, the criterion of existence is $n \ge p/2$ [15, 18], so the simplest example is l_4^3 (given in [9]). However, there are no orthogonally complemented Euclidean planes in l_p^n for any $p \ne 2$ and an n [9].

As a sufficient condition the absence of Euclidean planes was first established in [5]. This yields Corollary 3.9 except for $p = 4, 6, \dots$ Moreover, this also provides

COROLLARY 3.10. *If X is a polyhedral normed space then the boundary spectrum of every contraction in X is rational.*

A similar theory can be developed for cones [1], [5], [21]. A final result is the following.

THEOREM 3.11 [21]. *Let X be ordered by a solid cone D. Then the boundary spectrum of every power bounded nonnegative operator in X is rational if and only if there is no positively complemented 3-dimensional subspace Y such that the cone D_Y is Euclidean.*

The last property means that there exists a direct decomposition

$$Y = Y_1 \dot{+} Y_2, \qquad \dim Y_1 = 1,\ \dim Y_2 = 2,$$

such that

$$D_Y = \{y :\ y = y_1 + y_2,\ y_1 \in Y_1,\ y_2 = Y_2,\ y_1 \ge \sqrt{(y_2, y_2)}\}$$

where Y_1 is identified with $\mathbb{R}$ and $(\,,)$ is an inner product in Y_2. (Another name for such a cone is "*Lorentzian*".)

COROLLARY 3.12. *Let X be ordered by a solid polyhedral cone. Then the boundary spectrum of every power bounded nonnegative operator in X is rational.*

The Euclidean cone construction can be generalized in the following way (cf. [2], [14]). Let Z be normed space and let $X = \mathbb{R} \oplus Z$ be the outer direct sum, so

$$X = \left\{x :\ x = \begin{pmatrix} \xi \\ z \end{pmatrix},\ \xi \in \mathbb{R},\ z \in Z\right\}.$$

A natural cone in this 1-dimensional extension can be introduced as

$$D = \left\{\begin{pmatrix} \xi \\ z \end{pmatrix} :\ \xi \ge \|z\|\right\}.$$

We call it the *hyperbolic cone over the space Z*. (The reason for this name comes from the Euclidean case, because if $\|z\|^2 = (z,z)$ then the inequality $\xi \ge \|z\|$ is equivalent to $\xi^2 - (z,z) \ge 0$ (with the restriction $\xi \ge 0$) and this quadratic form is hyperbolic.)

Corollary 3.13 [21]. *Let the space* $X = \mathbb{R} \oplus l_p^{n-1}$, $1 \le p \le \infty$, $p \ne 2$, *be provided with the hyperbolic cone. Then the boundary spectrum of every power bounded nonnegative operator in* X *is rational.*

It is interesting to extend the previous theory to general convex structures. At present the author has the following preliminary results.

Theorem 3.14. *Let* (X, D) *be a convex structure such that* $0 \in \operatorname{Int} D$. *If there are no* 2*-dimensional Euclidean substructures of this structure then the boundary spectrum of every power bounded* D*-endomorphism is rational.*

We say that an admissible 3-dimensional subspace Y is *semi-Euclidean* if D_Y is a rotation body which means that $Y = Y_1 + Y_2$, $\dim Y_1 = 1$, $\dim Y_2 = 2$ and D_Y is the set of $y = y_1 + y_2$ with $y_1 \in Y_1$, $y_2 \in Y_2$ satisfying the inequality

$$\sqrt{(y_2, y_2)} \le d(y_1)$$

where d is a concave nonnegative function on a closed finite or infinite interval of $Y_1 \equiv \mathbb{R}$. In some extreme cases this function may take the value $+\infty$ which yields the space Y or some of its half-spaces or a layer in Y as extreme examples. If the domain of d is the whole axis $\mathbb{R}$ then d is constant and D_Y is a cylinder.

Theorem 3.15. *Let* (X, D) *be a convex structure such that* $0 \in \partial D$. *If there is no* 3*-dimensional semi-Euclidean substructure of this structure then the boundary spectrum of every power bounded* D*-endomorphism is rational.*

In conclusion let us formulate the following

Problem. Let the boundary spectrum of a D-endomorphism A be rational. How can one characterize the order of the group $K(A)$ in geometrical terms?

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