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SPECTRAL PROJECTIONS, SEMIGROUPS OF OPERATORS, AND THE LAPLACE TRANSFORM

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I. Introduction. Spectral projections and semigroups of operators may be united with the concept of a *functional calculus*. Of particular interest is a functional calculus that may be represented as an integral with respect to these spectral projections. In particular, we would like a semigroup of operators to be a Laplace transform of spectral projections.

This paper is meant to be an informal introduction and survey. For space and expository considerations, I will only outline proofs and the more technical parts of definitions. I will give references for these details.

I will focus on functional calculi that directly produce spectral projections. See [1], [5], [6], [25], [17], [4], [7], [10], and their references, for more general functional calculi. Similarly, I will not address spectral projections that do not come from functional calculi; see [1], [16], [20], [25], and their references, for the subject of decomposable operators.

For motivation, we will begin in finite dimensions. Some familiar elementary results will be stated in the language we want to use in infinite dimensions. In Section III, I will give some indication of how diagonalizable matrices may be generalized to infinite dimensions. This may be introduced with Fourier series in $L^p[0,1]$. The nature of their convergence, for different p, will be significant. Section IV will introduce the abstract Cauchy problem and semigroups of operators. Section V will give an integrated form of Widder's theorem, that is valid for Laplace transforms of vector-valued functions. All these ideas will be tied together in Section VI.

Throughout, we will assume that A is a closed, densely defined linear operator, with real spectrum, on a Banach space X. We will write $\mathcal{D}(A)$ for the domain of A, $\sigma(A)$ for the spectrum of A, $\rho(A)$ for the resolvent set of A, $B(X)$ for the Banach space of bounded linear operators from X to itself. We will write $\mathrm{Im}(B)$ for the image of an operator B.

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The paper is in final form and no version of it will be published elsewhere.

II. Spectral projections and semigroups of operators in finite dimensions. In this section, we will consider linear operators in finite dimensions, that is, matrices.

Recall that a system of n linear constant-coefficient first-order initial-value problems may be written

$$\frac{d}{ds}\vec{u}(s) = A(\vec{u}(s)), \qquad \vec{u}(0) = \vec{x}, \tag{2.1}$$

where A is an $n \times n$ matrix, $\vec{u}(s) \in \mathbb{C}^n$. We *guess* what the solution of (2.1) is, by pretending that A is a number, so that we think we recognize the solution as

$$\vec{u}(s) = e^{sA}\vec{x}.$$

Then we must decide what e^{sA} means; it is not immediately clear how one exponentiates a matrix. This is an example of a *functional calculus*—making sense out of $f(A)$, where A is an operator and f is a function; in this case, f is an exponential function, $f(t) \equiv e^{ts}$.

For a polynomial p, it is clear how to define $p(A)$; if $p(t) = \sum_{k=0}^{N} \alpha_k t^k$, then

$$p(A) \equiv \sum_{k=0}^{N} \alpha_k A^k. \tag{2.2}$$

For any bounded linear operator, the power series for the exponential function may be similarly applied:

$$e^{sA} \equiv \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k. \tag{2.3}$$

Equation (2.3) defines a (*norm*) *continuous semigroup*, *generated by* A. The semigroup property means that $e^{(t+s)A} = e^{tA}e^{sA}$, for any $s, t \geq 0$.

Note that the semigroup property corresponds to the fact that $f \mapsto f(A)$ is an algebra homomorphism, since, if $h_s(t) \equiv e^{st}$, then $h_s h_r = h_{s+r}$.

It is not hard to show that $\vec{u}(s) \equiv e^{sA}\vec{x}$ is a solution of (2.1). However, the formula (2.3) is not pleasant; it is not something we hope to calculate directly. We would like a simpler expression for e^{sA}; more generally, let us try to construct $f(A)$, for as many functions f as we reasonably can.

We will restrict our attention to the most desirable class of matrices, those that are *diagonalizable*; that is, there exists diagonal D and invertible S so that $A = SDS^{-1}$. What this means is that $\mathbb{C}^n$ may be decomposed into a sum of subspaces on which A behaves like a complex number. In the language of operators, this means the following. There exist projections $\{E_j\}_{j=1}^k$ and numbers $\{a_j\}_{j=1}^k$ (the eigenvalues of A) such that

$$E_j E_i = 0, \quad \forall j \neq i, \quad I = \sum_{j=1}^{k} E_j, \quad \text{and} \quad A = \sum_{j=1}^{k} a_j E_j. \tag{2.4}$$

Semigroups and spectral projections are united by the functional calculus that I will now construct.

For any polynomial p, (2.2) becomes $p(A) = \sum_{j=1}^{k} p(a_j)E_j$.

Thus it seems natural to define, for any function f,

$$f(A) \equiv \sum_{j=1}^{k} f(a_j)E_j. \tag{2.5}$$

The map $f \mapsto f(A)$ is an algebra homomorphism; (2.4) is saying that $f_0(A) = I, f_1(A) = A$, where $f_0(s) \equiv 1, f_1(s) \equiv s$.

For any subset Ω of the complex plane, we may use (2.5) to define a projection: $E(\Omega) \equiv 1_\Omega(A)$. Note that

$$E(\Omega) = \sum_{a_j \in \Omega} E_j, \quad \text{and} \quad AE(\Omega) = \sum_{a_j \in \Omega} a_j E_j.$$

Thus we have the following.

Proposition 2.6. *Suppose A is a diagonalizable matrix. Then*

$$A : \mathrm{Im}(E(\Omega)) \to \mathrm{Im}(E(\Omega)), \quad \textit{and} \quad \sigma(A|_{\mathrm{Im}(E(\Omega))}) = \Omega \cap \sigma(A).$$

The operator $E(\Omega)$ is a *spectral projection*, *corresponding to* Ω, for A. It is desirable thus to decompose A into pieces with specific behaviour.

Of particular interest is to recover our semigroup e^{sA}, defined by (2.3). By choosing $f \equiv h_s$, where $h_s(t) \equiv e^{ts}$, in (2.5), it is not hard to see that, for any s,

$$e^{sA} = h_s(A) \equiv \sum_{j=1}^{k} e^{sa_j} E_j. \tag{2.7}$$

III. Infinite-dimensional analogues of diagonalizable matrices. The unifying theme in Section II was the concept of a functional calculus, as in (2.5).

Definition 3.1. Suppose $\mathcal{F}$ is a Banach algebra of complex-valued functions on a subset of the complex plane containing $f_0(s) \equiv 1$ and $g_\lambda(s) \equiv (\lambda - s)^{-1}$, for some complex λ. An *$\mathcal{F}$-functional calculus for A* (see, for example, [5]) is a continuous algebra homomorphism, $f \mapsto f(A)$, from $\mathcal{F}$ into $B(X)$ such that

(1) $f_0(A) = I$, and
(2) $\lambda \in \varrho(A)$, with $g_\lambda(A) = (\lambda - A)^{-1}$, whenever $g_\lambda \in \mathcal{F}$.

In Section II, we obtained spectral projections by choosing $f \equiv 1_\Omega$, for some closed set Ω, and a semigroup of operators by choosing $h_s(t) \equiv e^{ts}$.

We will be interested in the following two choices of $\mathcal{F}$. We will write $B(\mathbb{R})$ for the space of bounded, Borel measurable complex-valued functions on the real line, with the supremum norm, and $AC(\mathbb{R})$ for the space of absolutely continuous functions on the real line such that

$$\|f\|_{AC(\mathbb{R})} \equiv |f(\infty)| + \int_{-\infty}^{\infty} |f'(t)|\, dt$$

is finite. It is clear how $AC([0,\infty))$ would be similarly defined.

For an infinite-dimensional example, let's first consider the discrete analogue of a diagonal matrix, an operator A that has a total set of eigenvectors. We would like a functional calculus analogous to (2.5), with the finite sum replaced by an infinite sum.

The following familiar example will illustrate some of the difficulties.

EXAMPLE 3.2. Let $A \equiv -id/dx$ on $X \equiv L^p[0,1]$ $(1 \le p < \infty)$, with maximal domain. For any integer k, define $h_k(x) \equiv e^{2\pi ikx}(x \in [0,1])$. Then

$$Ah_k = 2\pi k h_k \quad (k \in \mathbb{Z}),$$

and the span of $\{h_k\}_{k\in\mathbb{Z}}$ is dense.

For any integer k, we have the natural one-dimensional projection E_k, onto the eigenspace corresponding to $2\pi k$:

$$E_k f \equiv \widehat{f}(k) h_k, \tag{3.3}$$

where $\widehat{f}(k)$ is the kth Fourier coefficient for f. Thus our desired representation of a functional calculus looks like

$$(g(A))f \equiv \sum_{k\in Z} g(2\pi k)E_k f \equiv \sum_{k\in Z} g(2\pi k)\widehat{f}(k)h_k \quad (f \in X). \tag{3.4}$$

The behaviour of this functional calculus hinges on the nature of the convergence of the Fourier series for f.

First, consider a very natural choice for g, $g \equiv 1_{[0,\infty)}$. Then $P \equiv 1_{[0,\infty)}(A)$ has the effect of removing all the negative values of k from the Fourier series,

$$Pf \equiv 1_{[0,\infty)}(A)f \equiv \sum_{k=0}^{\infty} \widehat{f}(k)h_k \quad (f \in X). \tag{3.5}$$

For X equal to $L^1[0,1]$, (3.5) is known to define an unbounded operator; if we think of X as $L^1(\partial D)$, where D is the unit disc in the complex plane, then P is equivalent to the projection onto $H^1(D)$, the set of functions in $L^1(\partial D)$ with holomorphic extensions to D.

Thus for X equal to $L^1[0,1]$, we are outside the scope of this paper; other functional calculi, or a notion of regularized projections, are necessary (see [7], [11], [9] or [12]).

In general, for any $\Omega \subseteq \mathbb{R}$, we may use (3.4) to define a projection

$$E(\Omega) \equiv 1_{\Omega}(A). \tag{3.6}$$

At the other extreme, consider $X = L^2[0,1]$. Here E_k is an orthogonal projection, for any integer k, thus

$$\sup\{\|E(\Omega)\| \mid \Omega \subseteq \mathbb{R}\} < \infty. \tag{3.7}$$

It may be shown that (3.7) implies that (3.4) defines a $B(\mathbb{R})$ functional calculus for A (see [14, Proposition 5.3]).

For $p \neq 2$, (3.7) is false. But for $1 < p < \infty$,

$$\sup\{\|E([a,b])\| \mid a,b \in \mathbb{R}\} < \infty. \tag{3.8}$$

This enables us to apply summation by parts to (3.4), to obtain an $AC(\mathbb{R})$ functional calculus,

$$g(A)f \equiv \sum_{j=-\infty}^{\infty} [g(2\pi j) - g(2\pi(j-1))]E([j,\infty))f \quad (f \in X). \tag{3.9}$$

We remark that (3.7) is equivalent to unconditional convergence of the Fourier series, for all $f \in X$, while (3.8) is equivalent to convergence; for $p \neq 2$, we do not have unconditional convergence, thus cannot have a $B(\mathbb{R})$ functional calculus for A.

An entirely different problem arises when we consider $-id/dx$ on $L^p(\mathbb{R})$ $(1 \leq p < \infty)$. Here there are no eigenvalues, so a representation such as (3.4) is impossible; $E(\{a\})$ is trivial, for any number a.

However, spectral projections $E(\Omega)$, as in Proposition 2.6 and (3.6), may be generalized.

DEFINITION 3.10. Suppose Ω is a Borel subset of the complex plane. Then a projection P is a *spectral projection*, *corresponding to* Ω, for A, if

$$A : \mathcal{D}(A) \cap \mathrm{Im}(P) \to \mathrm{Im}(P) \quad \text{and} \quad \sigma(A|_{\mathrm{Im}(P)}) \subseteq \overline{\Omega} \cap \sigma(A).$$

Note that an eigenspace is the image of a spectral projection corresponding to a set containing a single point (the eigenvalue).

We shall also produce an analogue of (2.5), by replacing the sum with an integral, with respect to a family of spectral projections.

A *projection-valued measure* for A (see [14] or [15]) is a bounded map $\Omega \to E(\Omega)$, from the Borel subsets of the real line into $B(X)$, such that $E(\Omega)$ is a spectral projection, corresponding to Ω, for A, for any Borel set Ω, $\Omega \mapsto E(\Omega)x$ is a countably additive vector-valued measure, for all $x \in X$,

$$E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2), \quad \text{for all Borel sets } \Omega_1, \Omega_2, \quad \text{and } I = E(\mathbb{R}). \tag{3.11}$$

DEFINITION 3.12. An operator A with real spectrum is *scalar* (short for spectral operator of scalar type) if there exists a projection-valued measure E such that

$$Ax = \lim_{N,M\to\infty} \int_{-M}^{N} t\, dE(t)x,$$

with maximal domain.

Note that (3.11) and Definition 3.12 are continuous analogues of (2.4).

A $B(\mathbb{R})$ functional calulus for A is now given by

$$f(A)x \equiv \int_{\mathbb{R}} f(t)\, dE(t)x \qquad (x \in X,\ f \in B(\mathbb{R})). \tag{3.13}$$

The integral (3.13) is a continuous analogue of (2.5).

EXAMPLE 3.14. Let $A \equiv -id/dx$, on $L^p(\mathbb{R})$, and let F be the Fourier transform. Then continuous analogues of (3.3) and (3.4) are given by

$$E(\Omega)f \equiv F^{-1}(1_\Omega Ff), \qquad g(A)f \equiv F^{-1}(gFf).$$

For $p = 2$, this defines, respectively, a projection-valued measure and a $B(\mathbb{R})$ functional calculus for A. For $p \neq 2$, $E(\Omega)$ may not define a bounded operator, for arbitrary closed sets Ω. However, for $1 < p < \infty$, $\{E(-\infty, b]\}_{b\in\mathbb{R}}$ defines a uniformly bounded family of spectral projections.

We will be interested in operators with nonnegative real spectrum. Integrating (3.13) by parts then gives us, when $f \in AC([0,\infty))$,

$$\phi\,(f(A)x) \equiv \phi(f(\infty)x) - \int_0^\infty f'(t)\,(F(t)\phi)\,x\,dt \qquad (\phi \in X^*, x \in X), \tag{3.15}$$

where $F(t) \equiv E([0,t])^*$.

The family of operators $\{F(t)\}_{t\geq 0}$ is called a *decomposition of the identity* for A (see [14]). If A were scalar, then F would be of bounded variation, in some sense. However, we would like to consider the case, as with $A \equiv -id/dx$ on $L^p(\mathbb{R}), 1 < p < \infty, p \neq 2$, where $s \mapsto F(s)$ is merely a *bounded* map into the set of spectral projections corresponding to closed intervals, for A^* (see [14, Definition 15.3], for the definition of a decomposition of the identity for X). We then say that A is well-bounded.

We will use the equivalent definition in terms of functional calculi (see [14, Chapter 15] for A bounded, and see [8] or [23] for A unbounded). See also [3] and its references for information on well-bounded operators.

Definition 3.16. The (possibly unbounded) operator A is *well-bounded* on $[0,\infty)$ if it has an $AC([0,\infty))$ functional calculus.

We remark that, for a large class of Banach spaces X, A is scalar if and only if A has a $B(\mathbb{R})$ functional calculus (see [13]).

Summary 3.17. Let us make the following informal dichotomies between scalar operators and well-bounded operators, with real spectrum.

If $A \equiv -id/dx$, on $L^p(\mathbb{R})$ or $L^p[0,1]$, then A is scalar if and only if $p = 2$, while A is well-bounded if and only if $1 < p < \infty$.

A scalar operator has a uniformly bounded family of spectral projections corresponding to arbitrary closed sets. A well-bounded operator has a uniformly bounded family of spectral projections corresponding to arbitrary closed intervals.

A scalar operator has a $B(\mathbb{R})$ functional calculus, while a well-bounded operator has an $AC(\mathbb{R})$ functional calculus.

For an operator with a total set of eigenvectors, being scalar is equivalent to the corresponding eigenvalue expansion (such as the Fourier series) for each $x \in X$ converging unconditionally, while being well-bounded is equivalent to the corresponding eigenvalue expansion converging, for each $x \in X$.

For a scalar operator, the decomposition of the identity is of bounded variation, in some sense, while for a well-bounded operator, the decomposition of the identity is merely bounded.

Finally, in future sections we will obtain the following dichotomy in terms of strongly continuous semigroups (see Definition 4.2), when A has real nonnegative spectrum. If A is scalar or well-bounded, then $-A$ generates a strongly continuous semigroup $\{e^{-sA}\}_{s\geq 0}$. If A is scalar, then $\{e^{-sA}\}_{s\geq 0}$ is the Laplace–Stieltjes transform of a measure of bounded variation, while if A is well-bounded, then $\{e^{-sA}\}_{s\geq 0}$ is the once-integrated Laplace transform of an L^∞ function, in some sense.

IV. Semigroups of operators and the abstract Cauchy problem. An intimate relationship between functional calculi and integro-differential equations is obtained from the infinite-dimensional version of (2.1), known as the *abstract Cauchy problem*:

$$\frac{d}{ds}u(s,x) = A(u(s,x)) \quad (s \geq 0), \quad u(0,x) = x, \tag{4.1}$$

where $s \mapsto u(s,x)$ is a map from $[0,\infty)$ into a Banach space X, $x \in X$ and A is an operator on X.

By a *solution* we will mean a strong solution, that is,

$$s \mapsto u(s,x) \in C^1([0,\infty), X) \cap C([0,\infty), [\mathcal{D}(A)]),$$

where $[\mathcal{D}(A)]$ is given the graph norm.

When A is chosen to be a differential operator, (4.1) represents a partial differential equation. This may be thought of as a limit, as $n \to \infty$, of the system of ordinary differential equations (2.1).

Definition 4.2. A strongly continuous family $\{T(s)\}_{s\geq 0} \subseteq B(X)$ is a *strongly continuous semigroup generated by* A if $T(0) = I, T(s)T(t) = T(s+t)$, for $s,t \geq 0$, and

$$Ax = \lim_{s\to 0^+} \frac{1}{s}(T(s)x - x),$$

with maximal domain; that is, $\mathcal{D}(A)$ equals the set of all x for which the limit exists.

We then write $T(s) \equiv e^{sA}$.

See [18], [22] or any other reference on strongly continuous semigroups, for the following fundamental result.

Proposition 4.3. *The following are equivalent.*

(a) *A generates a strongly continuous semigroup.*

(b) *$\varrho(A)$ is nonempty and* (4.1) *has a unique strong solution, for all $x \in \mathcal{D}(A)$.*

The solution is then given by the semigroup

$$u(s,x) = e^{sA}x \quad (s \geq 0, x \in \mathcal{D}(A)).$$

The terminology e^{sA} suggests taking an exponential via a functional calculus, $e^{sA} \equiv h_s(A)$, $h_s(t) \equiv e^{ts}$. This realization is always possible (see [10]). In the cases we are focussing on, scalar and well-bounded operators, it is particularly straightforward and constructive.

In practice, the generator A is observed, while the semigroup it generates is what we need. Thus a representation of the semigroup is of more value than a representation of its generator.

Suppose A is scalar, with real spectrum, as in Definition 3.12. Then

$$e^{isA}x \equiv \int_{\mathbb{R}} e^{its}\, dE(t)x \quad (s \in \mathbb{R}, x \in X) \tag{4.4}$$

defines a strongly continuous group generated by iA (see (3.13)). If the spectrum of A is positive, then

$$e^{-sA}x \equiv \int_0^\infty e^{-st}\, dE(t)x \quad (s \geq 0, x \in X) \tag{4.5}$$

defines a strongly continuous semigroup generated by $-A$.

If A is well-bounded on $[0,\infty)$, with decomposition of the identity $\{F(t)\}_{t\geq 0}$, then it may be shown (see (3.15)) that

$$\phi(e^{-sA}x) \equiv s\int_0^\infty e^{-st}\,(F(t)\phi)x\, dt \quad (s \geq 0, x \in X, \phi \in X^*) \tag{4.6}$$

defines a strongly continuous semigroup generated by $-A$.

These representations of semigroups may be thought of as continuous analogues of (2.7); a linear combination of spectral projections is replaced by integrals with respect to projection-valued measures or functions. Note that (4.6) follows from (4.5) by integration by parts, since, when A is scalar with nonnegative spectrum, $F(t) \equiv E([0,t])^*$.

V. Vector-valued Laplace transforms. For complex-valued functions, we have the well-known *Widder's theorem* (see [24]).

PROPOSITION 5.1. *Suppose $f : (0,\infty) \to \mathbb{C}$ and $M \geq 0$. The following are equivalent.*

(a) *There exists $F \in L^\infty([0,\infty))$ such that*

$$f(s) = \int_0^\infty e^{-st}F(t)\, dt, \qquad \forall s > 0,$$

and $\|F\|_\infty \leq M$.

(b) *f is infinitely differentiable on $(0,\infty)$ and*

$$\frac{s^{k+1}}{k!}|f^{(k)}(s)| \leq M, \qquad \forall s > 0, k = 0,1,2,\ldots$$

Widder's theorem is not true for vector-valued functions. It is shown in [2, Theorem 1.4] and (independently) [26] that, if X is a Banach space, then Widder's theorem holds for $f : (0,\infty) \to X$ if and only if X has the Radon–Nikodym property.

If we integrate by parts in the Laplace transform, we obtain

$$f(s) = s\int_0^\infty e^{-st}\int_0^t F(r)\, dr\, dt;$$

note that $F \in L^\infty([0,\infty))$ if and only if $G(t) \equiv \int_0^t F(r)\, dr$ is Lipschitz continuous.

It is surprising that the following "integrated version of Widder's theorem" is valid, on *any* Banach space X.

PROPOSITION 5.2 ([2, Theorem 1.1]). *Suppose $f : (0,\infty) \to X$, $M \geq 0$. Then the following are equivalent.*

(a) *There exists $G : [0,\infty) \to X$ such that*

$$f(s) = s\int_0^\infty e^{-st}G(t)\, dt, \qquad \forall s > 0,$$

$G(0) = 0$, and $\|G(t_1) - G(t_2)\| \leq M|t_1 - t_2|$, for all $t_1, t_2 \geq 0$.

(b) *f is infinitely differentiable on* $(0,\infty)$ *and*

$$\frac{s^{k+1}}{k!}\|f^{(k)}(s)\| \le M, \qquad \forall s>0, k=0,1,2,\dots$$

Most of the classical theory of the Laplace transform has a vector-valued "integrated" analogue. See [2] and [21].

VI. Equivalences. For a scalar operator, it is not hard to show that (4.4) provides an equivalence.

PROPOSITION 6.1. *Suppose A has real spectrum. Then the following are equivalent.*

(a) *A is scalar.*

(b) *iA generates a strongly continuous group $\{e^{isA}\}_{s\in\mathbb{R}}$ and there exists a projection-valued measure E such that*

$$e^{isA}x = \int_{\mathbb{R}} e^{its}\, dE(t)x, \qquad \forall x\in X, s\in\mathbb{R}.$$

Or, when A has nonnegative spectrum, we may characterize being scalar in terms of a Laplace–Stieltjes transform, as in (4.5).

PROPOSITION 6.2.*Suppose A has nonnegative real spectrum. Then the following are equivalent.*

(a) *A is scalar.*

(b) *$-A$ generates a strongly continuous semigroup $\{e^{-sA}\}_{s\ge 0}$ and there exists a projection-valued measure E such that*

$$e^{-sA}x = \int_0^\infty e^{-st}\, dE(t)x, \qquad \forall x\in X, s\ge 0.$$

See [19] for recent results regarding scalar operators and the semigroups they generate.

The Laplace transform (4.6) is of more interest to us, because it is easier to characterize being a Laplace transform, than it is to characterize being a Laplace–Stieltjes transform.

In some sense, (4.6) is saying that the operator-valued function $s\mapsto \frac{1}{s}e^{-sA}$ is a Laplace transform. Especially when X is not reflexive, the "sense" is something we have been avoiding, involving as it does the precise definition of the decomposition of the identity $\{F(t)\}_{t\ge 0}$. Formally, (4.6) followed from (4.5) by integrating by parts. If we integrate by parts one more time, letting $G(t)\equiv\int_0^t F(r)\,dr$, so that $\frac{1}{s}e^{-sA}$ is the integrated Laplace transform of $G(t)$, as in Proposition 5.2,

$$\frac{1}{s}e^{-sA} = s\int_0^\infty e^{-st}G(t)\,dt,$$

we shall see that we can avoid the technical problems associated with $\{F(t)\}_{t\ge 0}$. In fact, although $F(t)$ acts on X^*, and may not be the adjoint of an operator on X, $G(t)$ will be a bounded operator on X. The Laplace transform of G will converge in the operator norm, while the Laplace transform of F converged only in a very weak sense.

We may then use Section V to characterize being well-bounded, once we have shown it to be equivalent to $\frac{1}{s}e^{-sA}$ being the appropriate integrated Laplace transform. Since F is bounded, G will be Lipschitz continuous, which is perfect for the vector-valued Widder's theorem, Proposition 5.2. Laplace transform inversion will enable us to construct the decomposition of the identity $\{F(t)\}_{t\geq 0}$.

THEOREM 6.3. *The following are equivalent.*

(a) *A is well-bounded on $[0,\infty)$.*

(b) *$-A$ generates a strongly continuous semigroup $\{e^{-sA}\}_{s\geq 0}$ and there exists Lipschitz continuous $G:[0,\infty)\to B(X)$ such that $G(0)=0$ and*

$$e^{-sA}=s^2\int_0^\infty e^{-st}G(t)\,dt,\qquad \forall s>0.$$

(c) *$-A$ generates a strongly continuous differentiable semigroup $\{e^{-sA}\}_{s\geq 0}$ such that*

$$\{H_n(s)\mid s>0, n=0,1,2,\ldots\}$$

is bounded in $B(X)$, where

$$H_n(s)\equiv\bigg(\sum_{k=0}^{n}\frac{s^kA^k}{k!}\bigg)e^{-sA}\qquad (s>0, n=0,1,2,\ldots).$$

(d) *$-A$ generates a strongly continuous holomorphic semigroup $\{e^{-zA}\}_{\mathrm{Re}(z)>0}$ such that*

$$K(t)\equiv\int_{1+i\mathbb{R}} e^{zt}e^{-zA}\,\frac{dz}{2\pi i z^3},$$

with the integral converging in the operator norm, has a Lipschitz continuous derivative, with $K'(0)=0$.

A decomposition of the identity for A is then given by

$$\begin{aligned}(F(t)\phi)x&=\frac{d}{dt}(\phi(G(t)x))=\bigg(\frac{d}{dt}\bigg)^2(\phi(K(t)x))\\&=\lim_{n\to\infty}\phi\bigg(H_n\bigg(\frac{n}{t}\bigg)x\bigg),\qquad \forall\phi\in X^*, x\in X, \textit{almost all } t.\end{aligned}$$

Outline of Proof (see [8]). (a)$\Rightarrow$(b) follows from (4.6).

For (b)$\Rightarrow$(a), we may represent $p((1+A)^{-1})$, for any polynomial p, as an appropriate integral as in (3.15), by using the Laplace transform in (b) and the Laplace transform of the semigroup,

$$(1+A)^{-n}x=\frac{1}{(n-1)!}\int_0^\infty s^{n-1}e^{-s}e^{-sA}x\,ds\ (n\in\mathbb{N},\ x\in X).$$

Since the polynomials are dense in $AC[0,1]$, this implies that $(1+A)^{-1}$ is well-bounded on $[0,1]$, which may be shown to be equivalent to A being well-bounded on $[0,\infty)$.

A calculation shows that, for $f(s)\equiv\frac{1}{s}e^{-sA}$,

$$\frac{s^{k+1}}{k!}f^{(k)}(s)=(-1)^kH_k(s),\qquad \forall s>0, k=0,1,2,\ldots$$

Thus the equivalence of (b) and (c) follows from the "integrated version of Widder's theorem," Proposition 5.2.

Finally, the equivalence of (b) and (d), and the representations of the decomposition of the identity, follow from (vector-valued or classical) Laplace transform inversion theorems. ∎

EXAMPLE 6.4. Let $A \equiv \triangle$, the Laplacian, on $L^1(\mathbb{R})$ or $C_0(\mathbb{R})$. Then it may be shown that, for f of compact support, $t \geq 0$,

$$\left(\frac{d}{dt}\right)^2 (K(t)f) = P_t * f,$$

where $K(t)$ is from Theorem 6.3(d), and

$$P_t(x) \equiv \frac{1}{\pi x} \sin(x\sqrt{t}) \qquad (t \geq 0, x \in \mathbb{R}).$$

Since $\|P_t\|_1$ is infinite, for all $t > 0$, our representation of a decomposition of the identity, in Theorem 6.3, implies that A cannot be well-bounded on $[0, \infty)$.

Remarks 6.5. In [8, Theorem 2.4], it is shown that the equivalent conditions of Theorem 6.3 are equivalent to the resolvent of A being a modified Stieltjes transform.

One of the referees has proposed the following interesting open problem. Given $k \in C^\infty[0, \infty)$ which is Laplace transformable for positive λ, is it the case that A has a functional calculus for functions of the form $t \mapsto \int_0^t k'(t-s)f(s)\,ds$, for some $f \in L^1[0, \infty)$, if and only if $-A$ generates a strongly continuous semigroup of the form

$$e^{-sA} = k(s) \int_0^\infty e^{-st}\,dG(t), \qquad \forall s > 0,$$

for G as in Theorem 6.3(b)? Note that Theorem 6.3(a)⇔(b) is the case $k(t) \equiv t$.

For $k(t) = t^n$, essentially the same argument gives us this equivalence; note that, for this choice of k, this class of functions becomes those whose $(n-1)$th derivative is in $L^1([0, \infty))$.

One can also ask for a resolvent equivalence.

Although the Laplacian on $L^1(\mathbb{R})$ or $C_0(\mathbb{R})$ is not well-bounded, it does have a functional calculus for Banach algebras similar to $AC([0, \infty))$. More generally, if A is the Laplacian on $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ or $C_0(\mathbb{R}^n)$, for $n \in \mathbb{N}$, then it is shown in [17] that A has a functional calculus for functions f such that

$$\int_0^\infty f^{(\nu)}(x)\, x^\nu\, dx < \infty,$$

where ν is a nonnegative number that depends on both n and p. In [11] it is shown that A has a $(1-A)^{-m}$-regularized $BC^k([0, \infty))$ functional calculus, where m and k are nonnegative integers that depend on both n and p.

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