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A SPARSITY RESULT ON NONNEGATIVE REAL MATRICES WITH GIVEN SPECTRUM

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Abstract. Let $\sigma = (\lambda_1, \ldots, \lambda_n)$ be the spectrum of a nonnegative real $n \times n$ matrix. It is shown that σ is the spectrum of a nonnegative real $n \times n$ matrix having at most $(n+1)^2/2 - 1$ nonzero entries.

Let $A = (a_{ij})$ be a real $n \times n$ matrix. We say that A is *nonnegative* if all its entries $a_{ij} \geq 0$ and that A is *positive* if all $a_{ij} > 0$. The *nonnegative inverse eigenvalue problem* (NIEP) is the problem of characterizing those lists $\sigma = (\lambda_1, \ldots, \lambda_n)$ of complex numbers λ_i for which there exists a nonnegative matrix A with spectrum $\sigma(A) = \sigma$.

If such an A exists we say that the list σ is *realizable* and we say that A *realizes* σ. While considerable work has been done on the NIEP, the problem is still far from being solved and in terms of n, only in the cases $n = 2$ and $n = 3$ (Johnson, Loewy and London) has the question been completely settled. See for example [1], [5] for references.

For a given list $\sigma = (\lambda_1, \ldots, \lambda_n)$, one can attempt to realize σ by the companion matrix $C(f)$ of the polynomial

$$f(x) := (x - \lambda_1) \cdots (x - \lambda_n) := x^n + p_1 x^{n-1} + \cdots + p_n.$$

In this case $C(f)$ is nonnegative if and only if $p_i \leq 0$ for $i = 1, 2, \ldots, n$.

However this condition is very restrictive—it implies for example that $f(x)$ has only one positive real root (see also [2] for a related discussion)—and one can improve the prospects of success by seeking to realize σ by a matrix of the form $\alpha I_n + C$ where $\alpha \geq 0$ and C is a nonnegative companion matrix. There exist realizable sets σ which are not realizable by matrices of this type. (See Reams' Thesis [6], Chapter 3 for examples with $n = 4$.)

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Matrices of the form $\alpha I_n + C$ are relatively sparse, having at most $3n-1$ nonzero entries. This suggests the problem of determining the "sparsest" $n \times n$ matrix realizing a given list σ and in this paper, we make a contribution to its resolution.

THEOREM. *Suppose $\sigma = (\lambda_1, \ldots, \lambda_n)$ is the spectrum of a nonnegative real matrix B. Then there exists a nonnegative real $n \times n$ matrix A with spectrum σ and such that A has at most $\left[(n+1)^2/2\right] - 1$ nonzero entries (where here $[\,\cdot\,]$ denotes the greatest integer function).*

To prove the theorem, we need the following result.

LEMMA. *Let A be an $n \times n$ real matrix and suppose that A has k real eigenvalues. Then there exists a subspace $\mathcal{S}$ of $M_n(\mathbb{R})$ of dimension $(n^2-2n+k)/2$ such that the spectrum $\sigma(A+W) = \sigma(A)$ for all $W \in \mathcal{S}$.*

Proof. By a well-known result of Schur, we can find a real orthogonal matrix U such that $T := U^{-1}AU$ is in upper block triangular form and where each diagonal block is either a 1×1 matrix (a) or a 2×2 matrix $\begin{pmatrix} b & c \\ -c & b \end{pmatrix}$ for some real numbers a, b, c with $c \neq 0$, (corresponding to the eigenvalues a or $b \pm ic$ of A). Furthermore U can be chosen so that the k real eigenvalues of A are $t_{11}, \ldots, t_{kk}$ where $T = (t_{ij})$.

Let $\mathcal{S}_0$ be the space of all strictly upper-triangular real matrices $B = (b_{ij})$ where if $n > k$, so $n - k = 2h$, say, is even, B also has zeros in the positions occurring in the 2×2 diagonal blocks of T corresponding to nonreal eigenvalues. Thus $b_{ij} = 0$ for all $i \leq j$ and also

$$b_{ij} = 0 \quad \text{for } (i,j) = (k+2l-1, k+2l), \qquad l = 1, 2, \ldots, h.$$

Note that if $B \in \mathcal{S}_0$, then $T + B$ and B have the same block diagonal and thus $\sigma(A) = \sigma(T) = \sigma(T+B)$.

Note that

$$\begin{aligned} \dim \mathcal{S}_0 &= (n-1) + (n-2) + \ldots + (n-k) + 2\left[(n-k-2) + (n-k-4) + \ldots + 2\right] \\ &= kn - \frac{k(k+1)}{2} + 2h(h-1) = (n^2 - 2n + k)/2. \end{aligned}$$

Defining $\mathcal{S}$ to be $U\mathcal{S}_0U^{-1}$, the desired result follows. ■

Proof of the Theorem. Suppose that σ is realizable and let A be a nonnegative matrix which realizes σ and subject to this has the greatest possible number of zero entries. Let Γ be the set of pairs (i,j) with $a_{ij} \neq 0$. We call Γ the *support* of A. Let $\mathcal{M}$ be the span of the matrices $E_{ij}\big((i,j) \in \Gamma\big)$ (where E_{ij} is the $n \times n$ matrix with 1 in the (i,j) position, zeros elsewhere).

Since A is nonnegative, the Perron–Frobenius theorem implies that A has at least one real eigenvalue, so, by the Lemma, there is a subspace $\mathcal{S}$ of $M_n(\mathbb{R})$ of dimension at least $(n^2-2n+1)/2$ such that $\sigma(A+W) = \sigma(A)$ for all $W \in \mathcal{S}$.

Claim. $\mathcal{S} \cap \mathcal{M} = \{0\}$.

For, if not, let $0 \neq B \in \mathcal{S} \cap \mathcal{M}$. Since the support of B is contained in the support of A, $A + aB$ is nonnegative and has the same support as A for all sufficiently small a and thus we can choose b such that $A + bB$ is nonnegative and such that $(A + bB)$ has its

(i, j) entry 0 for some $(i, j) \in \Gamma$. But since $\sigma(A+bB) = \sigma(A)$, this contradicts our choice of A. So the claim holds.

Now $\mathcal{S} + \mathcal{M}$ is a subspace of $M_n(\mathbb{R})$ so

$$\dim \mathcal{M} \leq n^2 - \dim \mathcal{S} \leq (n^2 + 2n - 1)/2 = (n+1)^2/2 - 1.$$

This proves the Theorem. ∎

Corollary. *Suppose $\sigma = (\lambda_1, \dots, \lambda_n)$ is the spectrum of a nonnegative matrix B. Then σ is the spectrum of an $n \times n$ nonnegative matrix A having at least $n - 1$ of its entries equal to* 0.

Remarks

1. One can show that $(n^2 + n)/2$ is in fact the correct bound in the theorem for $n = 2, 3$. For $n = 4$, the bound in the theorem is 11 but we do not know an example requiring more than 9 nonzero entries.

2. If

$$\sum_{i=1}^{n} \lambda_i = 0,$$

we can replace $\mathcal{M}$ in the above proof by $\mathcal{M}_0 := \mathcal{M} + \operatorname{span}\{E_{11}\}$ and the claim holds because (in the notation of the proof of the theorem)

$$\mathcal{S} \cap \mathcal{M}_0 = \mathcal{S} \cap \mathcal{M}$$

since the elements of $\mathcal{S}$ have trace 0. So in the trace 0 case, the bound can be improved by 1.

3. Suppose that $\sigma = (\lambda_1, \dots, \lambda_n)$ is realizable and that $A = (a_{ij})$ realizes σ. We define

$$s_k := \operatorname{trace}(A^k) = \lambda_1^k + \cdots + \lambda_n^k.$$

If A has exactly m nonzero entries on the diagonal an argument independently constructed by Johnson [3] and Loewy and London [4] shows that

$$m^{k-1} s_k \geq s_1^k.$$

So, in particular, if $(n-1)s_2 < s_1^2$, A must have all its diagonal entries different from 0. Furthermore if the digraph of A has no 2-cycles (that is $a_{ij}a_{ji} \neq 0$ for all i, j with $i \neq j$) then it is an easy exercise to check that $s_1 s_3 \geq s_2^2$. More generally, if $r > 1$ is the smallest integer r for which the digraph of A contains an r-cycle (that is, there exist distinct integers $j_1, j_2, \dots, j_r$ such that $a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_{r-1} j_r} a_{j_r j_1} \neq 0$), then

$$s_k = a_{11}^k + \cdots + a_{nn}^k \qquad \text{for } k = 1, 2, \dots, r-1$$

and

$$s_r > a_{11}^r + \cdots + a_{nn}^r$$

and thus

$$s_k s_l \geq s_p s_q$$

for all positive integers k, l, p, q with

$$k \leq p \leq q \leq l \quad \text{and} \quad k + l = p + q \leq r.$$

Thus we may use inequalities between the s_i to get lower bounds on the size of the support of A. In this way, one can construct examples to show that the bound in the theorem is of the right order of magnitude for $n = 2, 3$ and 4. For large values of n, however, this author does not know any example of a realizable spectrum of size n which is not realizable by a matrix with at most $4n + 1$ nonzero entries. So it is tempting to conjecture that the best possible bound in the theorem is linear rather than quadratic in n.

4. We say that $\sigma = (\lambda_1, \dots, \lambda_n)$ is an *extreme* spectrum if σ is realizable but for all $\alpha > 0$, $(\lambda_1 - \alpha, \dots, \lambda_n - \alpha)$ is not realizable. One can show that solving the NIEP is equivalent to characterizing extreme spectra. Suppose $A \geq 0$ realizes an extreme spectrum σ. If A is reducible under permutation similarity, that is, there exists a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where A_{11} is $r \times r$, A_{22} is $(n-r) \times (n-r)$ for some r with $1 \leq r < n$, then $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$ and A_{11} and A_{22} have smaller size and an inductive argument can be invoked.

So we can assume A is irreducible.

In this case, the Perron–Frobenius theorem states that A has an eigenvector v corresponding to the positive eigenvalue ρ which that v has all its entries positive. Replacing A by $D^{-1}AD$ for a positive diagonal matrix, we can assume $v = j$, the vector of all 'ones'. Suppose A has a column with all entries positive. We assume column one of A has strictly positive entries. Choose $\epsilon_1 < 0$, $\epsilon_2 > 0, \dots, \epsilon_n > 0$ with $\sum_{i=1}^{n} \epsilon_i = 0$ and such that

$$A_\epsilon = A + (\epsilon_1 j \; \epsilon_2 j \cdots \epsilon_n j) > 0.$$

But A_ϵ has the same spectrum as A.

Since σ is extreme, this is impossible.

Hence A has at least one zero entry in each column and similarly in each row. Suppose now that $\mathrm{trace}(A) > 0$. Let $\Gamma = \mathrm{supp}(A)$, the support of A, and let $X = (x_{ik})$ have zero entries off Γ and indeterminate entries $x_{ik} (i, k) \in \Gamma$. Consider the system of equations

$$\mathrm{trace}(A^r X) = n\rho^r \qquad \text{for } r = 0, 1, 2, \dots, (n-1). \tag{$*$}$$

If this system is solvable, then

$$\mathrm{trace}\big(A^r(J - X)\big) = 0 \qquad \text{for } r = 0, 1, 2, \dots, n-1$$

(where J is the matrix with all entries equal to 1). Thus if A is nonderogatory (so in particular, if A has distinct eigenvalues), then

$$J - X = [A, T] = AT - TA$$

for some matrix T.

Consider for small $\epsilon > 0$,

$$A_\epsilon = (I + \epsilon T)^{-1} A (1 + \epsilon T) = A + \epsilon[A, T] + O(\epsilon^2) = A + \epsilon(J - X) + O(\epsilon^2).$$

Since $\mathrm{supp}(X) \subseteq \mathrm{supp}(A)$, $A + \epsilon(J - X)$ has positive entries for small $\epsilon > 0$. So $A_\epsilon > 0$ for all sufficiently small $\epsilon > 0$. But $\sigma(A_\epsilon) = \sigma(A)$ is extreme. This is a contradiction.

So the system $(*)$ is inconsistent.

Since $(*)$ contains as many indeterminates as the support of A, this inconsistency is particularly restrictive if A is not relatively sparse. The argument shows that the only "generic" class of extreme spectra is the class of those with $s_1 = 0$, that is, the spectra of trace zero nonnegative matrices.

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