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CARATHÉODORY BALLS AND NORM BALLS IN $H_{p,n} = \{z \in \mathbb{C}^n : \|z\|_p < 1\}$

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Abstract. It is shown that for $n \geq 2$ and $p > 2$, where p is not an even integer, the only balls in the Carathéodory distance on $H_{p,n} = \{z \in \mathbb{C}^n : \|z\|_p < 1\}$ which are balls with respect to the complex ℓ_p norm in $\mathbb{C}^n$ are those centered at the origin.

1. Introduction. Consider the unit ball

$$H = H_{p,n} = \{z \in \mathbb{C}^n : \|z\|_p < 1\}$$

in the complex n-space $\mathbb{C}^n$ with respect to the metric which is induced by the complex ℓ_p norm in $\mathbb{C}^n$

$$\|z\|_p = \Big(\sum_{k=1}^{n} |z_k|^p \Big)^{1/p}, \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \quad p \geq 1.$$

H is a convex bounded domain in $\mathbb{C}^n$. Next consider the Carathéodory distance $C = C_H$ on H

$$C(z, w) = \sup \rho(f(z), f(w)), \qquad z, w \in H,$$

where the supremum is taken over all holomorphic functions f from H into the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Here ρ is the hyperbolic distance on Δ. Note that on $H_{p,n}$, the Carathéodory distance and the Kobayashi distance are the same.

For $n = 1$, and all $p > 0$, $H_{p,1} = \Delta$, and $C(a, b) = \rho(a, b)$, $a, b \in \Delta$, and $\|z\|_p = |z|$, $z \in \Delta$, and since, cf. [Sch, Lemma 2.1],

$$\rho(z, a) = r \Leftrightarrow |z - b| = R, \qquad a, b, z \in \Delta, \tag{1.1}$$

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where

$$b = a\,\frac{1-\alpha^2}{1-\alpha^2|a|^2} \text{ and } R = \alpha\,\frac{1-|a|^2}{1-\alpha^2|a|^2}, \quad \alpha = \tanh r, \tag{1.2}$$

it follows that in the case $n = 1$ every ball (i.e. disk) in the Carathéodory distance on H is a ball with respect to the ℓ_p norm in $\mathbb{C}$ for $p \geq 1$.

Also, every ball in the Carathéodory distance on $H_{p,n}$, $n \geq 1$, $p \geq 1$ which is centered at the origin is a ball in the ℓ_p norm of $\mathbb{C}^n$. (See Lemma 2 in Section 3.)

For $n \geq 2$ and $p = 1$ the following theorem holds (cf. [Sch], [Sr] and [Z]):

Theorem A. *The only balls in $H_{1,n} = \{z \in \mathbb{C}^2 : \|z\|_1 < 1\}$ in the Carathéodory distance on $H_{1,n}$ which are balls in the ℓ_1 norm in $\mathbb{C}^n$ are those which are centered at the origin.*

The proof of this theorem is based on the following proposition for $p = 1$ (cf. [Sch], [Sr]).

Proposition. *Given*

(i) *n circles $\gamma_k = \{z \in \mathbb{C} : |z - a_k| = r_k > 0\}$, $k = 1, \ldots, n$,*

(ii) *n points $z_1, \ldots, z_n$ in motion such that the point z_k moves along γ_k with state equation*

$$z_k(t) = a_k + r_k e^{i(t+\vartheta_k)}, \quad -\infty < t < \infty, \quad k = 1, \ldots, n,$$

where the phases $\vartheta_1, \ldots, \vartheta_n$ are given,

(iii) *n points $b_1, \ldots, b_n$ in $\mathbb{C}$,*

(iv) *n real positive numbers $\lambda_1, \ldots, \lambda_n$ and a real positive number c,*

(v) *a positive real number p which is not an even integer, such that*

$$\sum_{k=1}^{n} \lambda_k |z_k(t) - b_k|^p \equiv c, \qquad -\infty < t < \infty, \tag{1.3}$$

then $a_k = b_k$ for all $k = 1, \ldots, n$.

This proposition is proved in [Sr] in the special case $p = 1$. The generalization for every $p > 0$ which is not an even integer can be obtained by modifying the proof of [Sr]. For completeness, we will present the whole proof of the general case. With the aid of the proposition we establish here the following theorem.

Theorem B. *Let $p > 2$ be a real number which is not an even integer. Then the only balls in $H_{p,n} = \{z \in \mathbb{C}^n : \|z\|_p < 1\}$ in the Carathéodory distance on $H_{p,n}$ which are balls in the ℓ_p norm in $\mathbb{C}^n$ are those which are centered at the origin.*

Remark. Theorem B holds also for $p = 2$, as can be deduced from [R, pp. 29–30].

2. Proof of the proposition. If for some k, $a_k = b_k$ as desired, then the term $|z_k(t) - b_k|^p$ yields a constant contribution to the sum in (1.1), and may be dropped. We thus may assume that $a_k \neq b_k$ for all $k = 1, \ldots, n$. By rotating, translating, rescaling and renaming the constants c, λ_k, b_k and ϑ_k we may assume that for all $k = 1, \ldots, n$,

$a_k = 0$, $r_k = 1$ and b_k is real and positive. Then $|z_k(t)| = 1$, $\arg\, z_k(t) = t + \vartheta_k$, and by the Cosine Theorem, (1.3) becomes

$$\sum_{k=1}^{n} \lambda_k \,(\,1 + b_k^2 - 2b_k \cos(t+\vartheta_k)\,)^{p/2} \equiv c, \qquad -\infty < t < \infty \tag{2.1}$$

or

$$\sum_{k=1}^{n} f_k(t) \equiv c, \qquad -\infty < t < \infty, \tag{2.2}$$

where

$$f_k(t) = \lambda_k' \,(\,A_k - \cos(t+\vartheta_k)\,)^{p/2}, \quad k = 1,\ldots,n, \tag{2.3}$$

and where $\lambda_k' = \lambda_k(2b_k)^{p/2}$ and $A_k = \frac{1}{2}\,(\,b_k + b_k^{-1}\,)$. Since $b_k > 0$, it follows that

$$A_k \geq 1. \tag{2.4}$$

Now let

$$\varphi_k(z) = A_k - \cos(z+\vartheta_k),\; z \in \mathbb{C},\; k = 1,\ldots,n, \tag{2.5}$$

and

$$Z_k = \{z \in \mathbb{C} :\, \varphi_k(z) = 0\}, \qquad k = 1,\ldots,n. \tag{2.6}$$

Then $A_k = 1 \;\Rightarrow\; Z_k = \{w_k + 2m\pi :\, m \in Z\}$ for some real number w_k, and

$$A_k > 1 \;\Rightarrow\; Z_k = \{w_k + 2m\pi :\, m \in Z\} \cup \{\overline{w}_k + 2m\pi :\, m \in Z\}$$

for some non-real number w_k in $\mathbb{C}$.

We may assume that $\varphi_j \neq \varphi_k$ for all $j \neq k$, since otherwise the corresponding terms in (2.2) may be grouped together. Then $A_j \neq A_k$ or $\vartheta_j \neq \vartheta_k (\mathrm{mod}\ 2\pi)$ for $j \neq k$. In the first case $\mathrm{Im}\, w_j \neq \mathrm{Im}\, w_k$, and in the latter case we have $\mathrm{Re}\, w_j \neq \mathrm{Re}\, w_k$ (mod 2π) for any $w_j \in Z_j$ and $w_k \in Z_k$ with $j \neq$ k. Therefore

$$Z_j \cap Z_k = \phi \qquad \text{for all } j \neq k. \tag{2.7}$$

Suppose now that $A_j = 1$. Then $\varphi_j(-\vartheta_j) = 0$ and $\varphi_k(-\vartheta_j) \neq 0$ for all $k \neq j$. Then $f_k(t) = \lambda_k'\varphi_k(t)^{p/2}$ is real analytic at $t = -\vartheta_j$, for all $k, k \neq j$, and by (2.2) so is $\sum_{k=1}^{n} f_k(t)$. Hence, also $f_j(t)$ must be real analytic at $t = -\vartheta_j$. But f_j is not real analytic (it is not even differentiable) at $t = -\vartheta_j$, since

$$f_j(t) = \lambda_j' \,(\,1 - \cos(t+\vartheta_j)\,)^{p/2} = \lambda_j'\, 2^{p/2} |\sin\, \frac{1}{2}(t+\vartheta_j)|^p$$

and p is not a positive even integer. This contradiction shows that $A_k > 1$ for all $k = 1,\ldots,n$.

To complete the proof fix j, $1 \leq j \leq n$, and choose a point w such that $\varphi_j(w) = 0$. Since $A_j > 1, w$ is not real, and by (2.7), $\varphi_k(w) \neq 0$ for all $k \neq j$. Choose a real number t_0. Since $A_k > 1$ for all $k = 1,..,n$ it follows that $\varphi_k(t_0) \neq 0$ for all $k = 1,\ldots,n$. We can, therefore, find a simply connected neighborhood U of t_0 such that $\varphi_k(z) \neq 0$ for all $z \in U$ and all $k = 1,\ldots,n$. Next, choose a path γ in $\mathbb{C} \setminus \bigcup_{k=1}^{n} Z_k$ which starts and ends at t_0, winds once around the point w and does not wind around any other point of $\bigcup_{k=1}^{n} Z_k$. Now, for $k = 1,\ldots,n$, let $F_k(z)$ be an analytic branch of

$\lambda'_k\varphi_k(z)^{p/2}$ such that $F_k(t) = f_k(t)$ for all real numbers t in U, (see (2.3) and (2.5)). Such branches exist since $\varphi_k(z) \neq 0$ for all $z \in U$ and $k = 1, \ldots, n$, and since U is simply connected. Finally, set $F_0(z) = \sum_{k=1}^n f_k(z)$. Then $F_0(z)$ is analytic in U, and $F_0(t) = \sum_{k=1}^n f_k(t)$ for all real numbers t in U. Therefore, by (2.2),

$$F_0(z) \equiv c \text{ in } U. \tag{2.8}$$

For $k = 0, 1, \ldots, n$, let $G_k(z)$ denote the analytic function in U which is obtained by continuing $F_k(z)$ along γ.

The point w is a simple zero of $\varphi_j(z)$, and γ winds once around w and does not wind around any other zero of $\varphi_j(z)$. Therefore $G_j(z) = \omega F_j(z)$, where $\omega = e^{ip\pi}$. Then $\omega \neq 0$, and since p is not an even integer, also $\omega \neq 1$. For any other k, $0 < k \neq j$, γ does not wind around any zero of $\varphi_k(z)$. Hence $G_k(z) = F_k(z)$ for all $0 < k \neq j$. In view of (2.8), $G_0(z) \equiv c$ in U, and by the Permanence Theorem, $G_0(z) = \sum_{k=1}^n G_k(z)$. Hence

$$(1-\omega)F_j(z) = (\,F_0(z) - G_0(z)\,) \equiv 0 \text{ in } U.$$

Then $f_j(t) = 0$ for all real numbers t in U, and thus $\lambda_j = 0$, contradicting assumption (iv) of the proposition. This completes the proof. ■

3. Proof of Theorem B. In the sequel $\|a\|_p$ will stand for the ℓ_p norm of

$$a = (a_1, \ldots, a_n) \in \mathbb{C}^n : \|a\|_p = \Big(\sum_{k=1}^n |a_k|^p\Big)^{1/p}$$

and in the meantime we assume that $p \geq 1$.

For the proof we shall also use the following two lemmas.

Lemma 1. *Let $\zeta \in \mathbb{C}$ and $a = (a_1, \ldots, a_n) \in H$ such that*

$$(\,a_1, \ldots, a_{j-1},\ \zeta,\ a_{j+1}, \ldots, a_n\,) \in H.$$

Then

$$C\left((a_1, \ldots, a_{j-1},\ \zeta,\ a_{j+1}, \ldots, a_n), (a_1, \ldots, a_n)\right) = \rho\left(u_j^{-1}\zeta,\ u_j^{-1}a_j\right) \tag{3.1}$$

Here ρ is the hyperbolic distance in Δ and

$$u_j = \Big(1 - \sum_{1\leq k\leq n, k\neq j} |a_k|^p\Big)^{1/p}. \tag{3.2}$$

The proof of this lemma follows from Theorem 1 of [JPZ]. The details will be given at the end of the proof of Theorem B.

Lemma 2. *Let $a \in H$ and $\zeta \in \mathbb{C}$ such that $\zeta a \in H$. Then*

$$C(\zeta a, a) = \rho\,(\,\|a\|_p,\ \|a\|_p\zeta\,)\,. \tag{3.3}$$

See [D, p. 95].

For $a \in H$ let

$$B_c(a, r) = \{z \in H : C(z, a) < r\} \text{ and } B_N(a, r) = \{z \in H : \|z - a\|_p < r\}$$

denote the Carathéodory and the ℓ_p norm balls respectively, of radius r centered at a. To prove the theorem suppose that, contrary to its statement, there are points $a \in H\backslash\{0\}$ and $a^N \in H$ and real numbers $0 < \alpha < 1$ and $r_N > 0$ such that

$$B_N(a^N, r_N) = B_c(a, r) \subset H, \tag{3.4}$$

where

$$r = \tanh^{-1} \alpha. \tag{3.5}$$

Then $\partial B_N(a^N, r_N) = \partial B_c(a, r) \subset H$ where the inclusion follows from the fact that H is bounded and convex cf. [D, p. 88]. We will show that this assumption leads to a contradiction, for $p > 2$, by considering certain one dimensional subsets of $\partial B_c(a, r)$ which correspond to the following subsets of $\mathbb{C}$:

$$A_j = \{\zeta \in \mathbb{C} : (a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n) \in \partial B_c(a, r)\}, \quad j = 1, \ldots n, \tag{3.6}$$

and

$$B = \{\zeta \in C : \zeta a \in \partial B_c(a, r)\}, \tag{3.7}$$

where r is given in (3.5).

First note that for $\zeta \in A_j, j = 1, \ldots, n$,

$$C\left((a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n), (a_1, \ldots, a_n)\right) = r.$$

This and Lemma 1 imply

$$\rho\left(u_j^{-1}\zeta, u_j^{-1}a_j\right) = r = \tanh^{-1} \alpha, \qquad u_j = \Big(1 - \sum_{1\le k\le n, k\ne j} |a_k|^p\Big)^{1/p}.$$

Therefore all points $u_j^{-1}\zeta$, $\zeta \in A_j$, lie on a hyperbolic circle in Δ, hyperbolically centered at the point $u_j^{-1}a_j$, which is also a Euclidean circle, whose center and radius can be computed by (1.1) and (1.2). Hence A_j is a Euclidean circle too which is given by

$$A_j = \Big\{\zeta : \zeta = \frac{(1-\alpha^2)u_j^2}{u_j^2 - \alpha^2|a_j|^2}\, a_j + \alpha u_j \frac{u_j^2 - |a_j|^2}{u_j^2 - \alpha^2|a_j|^2}\, e^{i\varphi},\ 0 \le \varphi \le 2\pi\Big\}, \tag{3.8}$$

where u_j is given in (3.2).

Suppose, as above, that $\partial B_c(a, r) = \partial B_N(a^N, r_N)$ for some $a^N \in H$ and $r_N > 0$. Then for all $\zeta \in A_j$

$$\begin{aligned} r_N^p &= \|(a_1, \ldots, a_{j-1}, \zeta, a_{j+1}, \ldots, a_n) - a^N\|_p^p \\ &= |\zeta - a_j^N|^p + \sum_{1\le k\le n, k\ne j} |a_k - a_k^N|^p. \end{aligned} \tag{3.9}$$

Since $\zeta \in A_j$ and A_j is a circle, and since r_N, a_k and a_k^N are all constants it follows that a_j^N must coincide with the center of A_j. Therefore, by (3.8),

$$a_j^N = \frac{(1-\alpha^2)u_j^2}{u_j^2 - \alpha^2|a_j|^2}\, a_j, \quad j = 1, \ldots, n, \tag{3.10}$$

where u_j is given by (3.2). As a corollary we get

$$a_j = 0 \text{ if and only if } a_j^N = 0. \tag{3.11}$$

Next, consider the set B of (3.7). Then, by Lemma 2 and (3.5). $\zeta \in B$ if and only if

$$\tanh^{-1} \alpha = C(\zeta a, a) = \rho(\|a\|_p \zeta, \|a\|_p).$$

Consequently, the points $\|a\|_p\zeta$, $\zeta \in B$, lie on a hyperbolic circle in Δ, hyperbolically centered at the point $\|a\|_p$. By (1.1) and (1. 2) this is a Euclidean circle. Hence B is a circle which is given by

$$B = \{\zeta = \lambda + Re^{i\varphi} : 0 \le \varphi \le 2\pi\}, \tag{3.12}$$

where

$$\lambda = \frac{1-\alpha^2}{1-\alpha^2\|a\|_p^2} \quad \text{and} \quad R = \frac{\alpha}{\|a\|_p}\,\frac{1-\|a\|_p^2}{1-\alpha^2\|a\|_p^2}\,. \tag{3.13}$$

Thus, for all $\zeta \in B$

$$\zeta a = (\lambda + Re^{i\varphi})\,a \text{ and } \zeta a_k = \lambda a_k + R_k e^{i(\varphi+\psi_k)}, \quad 0 \le \varphi \le 2\pi, \tag{3.14}$$

where

$$R_k = \alpha\,\frac{|a_k|\,(1-\|a\|_p^2)}{\|a\|_p\,(1-\alpha^2\|a\|_p^2)} \text{ and } \psi_k = \arg\ a_k, \quad k = 1,\dots n. \tag{3.15}$$

Suppose again that $\partial B_c(a,r) = \partial B_N(a^N, r_N)$. Then for $\zeta \in B, \zeta a \in \partial B_c(a,r)$ and by (3.12), (3.13) and (3.14)

$$r_N = \|\zeta a - a^N\|_p = \Big(\sum_{k=1}^{n} \Big|\lambda a_k + R_k e^{i(\varphi+\psi_k)} - a_k^N\Big|^p\Big)^{1/p}, 0 \le \varphi \le 2\pi.$$

Then, by the proposition,

$$a_k^N = \lambda a_k, k = 1,\dots n \text{ and } a^N = \lambda a, \tag{3.16}$$

where λ is given by (3.13).

We now consider two cases:

Case 1. $a_j \neq 0$ and $a_k \neq 0$ for some $1 \le j < k \le$ n.
Case 2. $a_j \neq 0$ for some $1 \le j \le n$ and $a_k = 0$ for any other $k \neq$ j.

Suppose that we are in case 1. With no loss of generality we may assume $a_1 \neq 0$, and that $a_k \neq 0$ for some $2 \le k \le n$ Using (3.10), (3.16) and (3.13) for a_1^N we get

$$\frac{(1-\alpha^2)}{u_1^2-\alpha^2|a_1|^2}\,u_1^2 a_1 = a_1^N = \frac{(1-\alpha^2)}{1-\alpha^2\|a\|_p^2}\,a_1, \tag{3.17}$$

where $u_1 = (1-\sum_{k=2}^n |a_k|^p)^{1/p}$ is as in (3.2). Since $a_1 \neq 0$ and $0<\alpha<1$, (3.17) gives

$$u_1^2 - \alpha^2 u_1^2\|a\|_p^2 = u_1^2 - \alpha^2|a_1|^2$$

or $u_1\|a\|_p = |a_1|$. This is equivalent to $u_1^p\|a\|_p^p = |a_1|^p$. By (3.2), this is

$$\left(1-\sum_{k=2}^{n}|a_k|^p\right)\sum_{k=1}^{n}|a_k|^p = |a_1|^p.$$

Hence

$$\sum_{k=1}^{n} |a_k|^p - \sum_{k=1}^{n} |a_k|^p \sum_{k=2}^{n} |a_k|^p = |a_1|^p \quad \text{or} \quad \Big(\sum_{k=2}^{n} |a_k|^p\Big)\Big(1 - \sum_{k=1}^{n} |a_k|^p\Big) = 0.$$

$\|a\|_p < 1$ and $a_k \neq 0$ for some $2 \leq k \leq n$ imply that each factor $\neq 0$, thus leading to a contradiction. (Note that for this case we have assumed only $p \geq 1$, and p is not an even integer.)

Suppose now that we are in case 2. With no loss of generality we may assume that $a_2 \neq 0$ and that $a_k = 0$ for all other $k \neq 2$. Then $a = (0, a_2, 0, \dots, 0)$. By (3.2) $u_2 = 1$ and by (3.10)

$$a_2^N = \frac{1-\alpha^2}{1-\alpha^2|a_2|^2} a_2 \ \text{ and } \ a_k^N = 0 \text{ for all } k \neq 2. \tag{3.18}$$

Consider the set A_j of (3.6) with $j = 1$ for $a = (0, a_2, 0 \dots, 0)$. Then, by (3.2), $u_1 = (1 - |a_2|^p)^{1/p}$ and by (3.8) $\zeta \in A_1$, i.e. $(\zeta, a_2, 0, \dots, 0) \in \partial B_c(a, r)$ if and only if

$$\zeta = \alpha (1 - |a_2|^p)^{1/p} e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi. \tag{3.19}$$

Assuming $\partial B_c(a, r) = \partial B_N(a^N, r_N)$, it follows that

$$r_N^p = \|a^N - (\zeta, a_2, 0, \dots, 0)\|_p^p = |\zeta|^p + |a_2^N - a_2|^p,$$

and by (3.19) and by (3.18) we get

$$r_N^p = \alpha^p (1 - |a_2|^p) + \alpha^{2p}|a_2|^p \Big(\frac{1-|a_2|^2}{1-\alpha^2|a_2|^2} \Big)^p. \tag{3.20}$$

We now compute r_N^p by considering the set B of (3.7). If $\zeta \in B$, then $\zeta a \in \partial B_c(a, r) = \partial B_N(a^N, r_N)$. Here $a = (0, a_2, 0, \dots, 0)$, $\zeta a = (0, \zeta a_2, 0, \dots, 0)$ and by (3.10) and (3.11), $a^N = (0, a_2^N, 0, \dots, 0)$. Therefore, in view of (3.14),

$$r_N^p = \|\zeta a - a^N\|_p^p = |\zeta a_2 - a_2^N|^p = \left|\lambda a_2 + R_2 e^{i(\varphi+\psi_2)} - a_2^N\right|^p, \ 0 \leq \varphi \leq 2\pi.$$

Hence $\lambda a_2 = a_2^N$ and, consequently, $r_N = R_2$. Then, by (3.15)

$$r_N^p = \alpha^p \Big(\frac{1-|a_2|^2}{1-\alpha^2|a_2|^2} \Big)^p. \tag{3.21}$$

Now subtracting the expression for r_N^p in (3.21) from the expression for r_N^p in (3.20), we get

$$\alpha^p (1 - |a_2|^p) + \alpha^p \Big(\frac{1-|a_2|^2}{1-\alpha^2|a_2|^2}\Big)^p (\alpha^p|a_2|^p - 1) = g(\alpha, |a_2|). \tag{3.22}$$

It follows that

$$\frac{g(\alpha, |a_2|)}{\alpha^p} (1 - \alpha^2|a_2|^2)^p = h(\alpha, |a_2|),$$

where

$$h(\alpha, |a_2|) = (1 - |a_2|^p)(1 - \alpha^2|a_2|^2)^p + (1 - |a_2|^2)^p (\alpha^p|a_2|^p - 1). \tag{3.23}$$

To obtain a contradiction we have to show that, for all α and $|a_2|$ in $(0, 1)$

$$h(\alpha, |a_2|) \neq 0, \quad 0 < \alpha < 1, \quad 0 < |a_2| < 1. \tag{3.24}$$

We are going to show that the inequality (3.24) holds for any p, $p > 2$. (Note that our proof does not apply for $1 < p < 2$.)

To simplify notation we write instead of $|a_2|$ the letter x. So we have to show that

$$h(\alpha, x) \neq 0,\ 0 < \alpha < 1,\ 0 < x < 1. \tag{3.24'}$$

This is the same as

$$(1 - x^p)(1 - \alpha^2 x^2)^p \neq (1 - \alpha^p x^p)(1 - x^2)^p \tag{3.25}$$

or

$$\frac{(1-\alpha^2x^2)^p}{1-\alpha^p x^p} \neq \frac{(1-x^2)^p}{(1-x^p)}, \qquad 0 < \alpha, x < 1. \tag{3.26}$$

We define

$$t_p(x) = t(x) = \frac{(1-x^2)^p}{(1-x^p)}, \qquad p \geq 1,\ 0 \leq x \leq 1. \tag{3.27}$$

Note that for $p > 1$

$$t(0) = 1, \quad t(1) = 0. \tag{3.27'}$$

We are going to show that for $p > 2, t(x)$ is strictly decreasing in (0,1]. This will imply (3.26) and hence prove the Theorem for $p > 2$ ($\neq 4, 6\ldots$). To prove this statement we compute $t'(x)$

$$t'(x) = \frac{-2xp(1-x^2)^{p-1}(1-x^p) + px^{p-1}(1-x^2)^p}{(1-x^p)^2} \tag{3.28}$$

or

$$t'(x) = \frac{xp(1-x^2)^{p-1}}{(1-x^p)^2} \cdot N_p(x) \tag{3.29}$$

where

$$N_p(x) = -2(1 - x^p) + x^{p-2}(1 - x^2) = -2 + x^p + x^{p-2} \tag{3.29'}$$

Let now $p > 2$. Then

$$N(0) = -2, \quad N(1) = 0. \tag{3.30}$$

Furthermore,

$$N'(x) = px^{p-1} + (p-2)x^{p-3} = px^{p-3}\Big[\, x^2 + \frac{p-2}{p} \,\Big]. \tag{3.31}$$

Hence as $p > 2$, $N'(x) > 0$ in $0 < x \leq 1$ and using (3.30) we obtain $N(x) < 0$ in (0,1) and it follows by (3.29) that $t(x)$ is strictly decreasing.

To complete the proof of the theorem we give now the proof of Lemma 1.

Proof of Lemma 1. It seems convenient to change slightly the notation, so we state Lemma 1 in the following form.

Lemma 1 (restated). *Let* $\widetilde{z} = (z_1, b_2, \ldots, b_n)$ *and* $\widetilde{w} = (w_1, b_2, \ldots, b_n)$ *be points in* $H_{p,n}$. *Set* $b = (b_2, \ldots, b_n)$ *and denote*

$$u = (1 - \|b\|_p^p)^{1/p}. \tag{3.2*}$$

Then

$$C(\widetilde{z}, \widetilde{w}) \;=\; \rho\,(\,u^{-1}z_1,\; u^{-1}w_1\,)\,. \tag{3.1*}$$

As stated this follows from a result on the geodesics of convex complex ellipsoids [JPZ, Theorem 1] cf. also [JP]. In the notation of this theorem we set $p_1 \;= \ldots =\; p_n \;=\; q$ and set $2q =$ p. We also set $s \;=\; 1$ and $\alpha_0 \;= \ldots =\; \alpha_n \;=\; 0$. We define the constants a_j of their theorem as follows: $a_1 \;=\; u, a_j \;=\; b_j, j \;=\; 2, \ldots,$n. The linear complex geodesic $\varphi(\lambda) \;=\; (\,\varphi_1(\lambda), \ldots, \varphi_n(\lambda)\,) \;=\; (\,z_1, \ldots, z_n\,)$ is thus of the form $z_1 \;=\; \varphi_1(\lambda) \;=\; u\lambda$, $z_j \;=\; \varphi_j(\lambda) \;=\; b_j$, $j = 2, \ldots, n$, and φ maps Δ into $H_{p,n}$. By the definition of a complex geodesic we have

$$C(\varphi(\lambda'),\; \varphi(\lambda'')) \;=\; \rho(\lambda', \lambda'')$$

for all points $\lambda', \lambda'' \;\in\; \Delta$. Choosing $\lambda' \;=\; u^{-1}z_1,\; \lambda'' \;=\; u^{-1}w_1$, we obtain (3.1)*. This proves the lemma and completes the proof of Theorem B. ∎

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