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THE DOUADY–EARLE EXTENSION OF QUASIHOMOGRAPHIES

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Abstract. Quasihomography is a useful notion to represent a sense-preserving automorphism of the unit circle T which admits a quasiconformal extension to the unit disc. For $K \geq 1$ let $A_T(K)$ denote the family of all K-quasihomographies of T. With any $f \in A_T(K)$ we associate the Douady–Earle extension E_f and give an explicit and asymptotically sharp estimate of the L_∞ norm of the complex dilatation of E_f.

Introduction. Let A_T denote the family of all sense-preserving automorphisms of the unit circle T. With any $f \in A_T$ we associate the Douady–Earle extension E_f which is a homeomorphic automorphism of the unit disc Δ and has a continuous extension to f on the boundary $T = \partial\Delta$ (see [DE] and [LP]). If $z \in \Delta$ and $f \in A_T$, then $E_f(z)$ is the unique $w \in \Delta$ such that

$$\int_T \left(\frac{f(\zeta) - w}{1 - \overline{w} f(\zeta)}\right) \frac{(1 - |z|^2)}{|z - \zeta|^2}\, |d\zeta| = 0. \tag{0.1}$$

Moreover, the correspondence $f \mapsto E_f$ is conformally natural in the sense that

$$E_{h_1 \circ f \circ h_2} = h_1 \circ E_f \circ h_2 \tag{0.2}$$

holds for any $f \in A_T$ and all Möbius transformations h_1, h_2, which map Δ onto itself.

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The property that a given $f \in A_T$ admits a quasiconformal extension to Δ is equivalent to the assumption that f is a quasihomography (see [Z1]). For $K \geq 1$, we denote by $A_T(K)$ the family of all $f \in A_T$ that are K-quasihomographies (see Chap. 1).

Starting with an automorphism f of T, which is the boundary automorphism of a given K-quasiconformal mapping of Δ onto itself, Douady and Earle proved that, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbf{K}(E_f) \leq 4^{3+\varepsilon}$ for $1 \leq K \leq 1+\delta$ (see [DE, Corollary 2]). Their explicit estimate starts from $4 \cdot 10^8 e^{35}$, for K near 1.

Making some refinements and using more subtle tools, Partyka obtained an asymptotically sharp estimate for $\mathbf{K}(E_f)$ (see [P1, Theorem 3.1]), improving the result of Douady and Earle for $1 \leq K < 50$. Using the notion of quasisymmetry for unit circle, introduced by Krzyż [K], he considered also, as the starting point, a given ρ-quasisymmetric automorphism f of T.

It is very natural from different points of view if we may extend an automorphism f of T that satisfies certain condition on T only, and next to study how particular properties of such an f effects the extension.

Rotation, but not conformally invariant notion of quasisymmetry of T, mentioned above, is meaningless in these considerations. This is mostly because neither there exists $\rho \geq 1$ such that boundary values of Möbius automorphisms of Δ are ρ-quasisymmetric (see [Z1, Example]), nor ρ-quasisymmetric automorphisms of T represent uniformly boundary values of K-quasiconformal automorphisms of Δ, for any $K \geq 1$.

We assume that a given automorphism f of T is a K-quasihomography ($\equiv$ 1-dimensional K-quasiconformal mapping) of T, $K \geq 1$. The family $A_T(K)$, $K \geq 1$, representing uniformly K-quasiconformal mappings, with the same K of necessity, is conformally invariant under composition and thus very natural with respect to the Douady–Earle extension.

Developing in Sect. 1 the argument of normal families in A_T in a way related to the Douady–Earle extension and introducing necessary functionals, defined on families of K-quasihomographies of T, we estimate in Theorem 3 the L_∞-norm of the complex dilatation μ_{E_f} for the Douady–Earle extension of a given K-quasihomography f of T, with K close to 1. In Corollary 3 we describe an asymptotically sharp estimate of $\mathbf{K}(E_f)$, expressed explicitly by (2.20), for K close to 1.

In order to be in contact with results mentioned above we give, in Theorem 2, a relation between some important families in $A_T(K)$ and functions ρ-quasisymmetric on the unit circle.

1. Normal families in A_T. Let Δ be the unit disc in the complex plane $\mathbb{C}$ and $T = \partial\Delta$ be the unit circle. We consider the family A_T of all sense-preserving automorphisms of T as a subspace of the Banach space C_T of all complex-valued continuous functions on T, with the supremum norm. In this section, we first discuss normality of certain subfamilies of A_T. As an application, we shall then show that some subfamilies of K-quasihomographies on T, which play an important role for our purpose, turn out to be families of ρ-quasisymmetric functions of T where ρ depends on K only.

For $f \in A_T$, we denote by E_f the Douady–Earle extension of f to Δ.

LEMMA 1. *The functional $E_f(0)$ is continuous on A_T.* ([DE, Prop. 2]).

For every r, $0 \le r < 1$, we denote by $F_T(r)$ the family of all $f \in A_T$ satisfying $|E_f(0)| \le r$. A family F in A_T is said to be a normal family if F is relatively compact in A_T. Thus a family F in A_T is a normal family if and only if for any infinite sequence $\{f_n\}$ in F, there exists a subsequence $\{f_{n_l}\}$ which converges to some f in A_T.

LEMMA 2. *Let F be a family in A_T. Then F is normal family in A_T if and only if F is equicontinuous on T and there exists r, $0 \le r < 1$, such that $\overline{F} \subset F_T(r)$, where $\overline{F}$ is the closure of F in the Banach space C_T.*

Proof. We note that by the Ascoli–Arzela's theorem, a family G in C_T is a normal family in C_T if and only if G is uniformly bounded and equicontinuous on T. Suppose that F is a normal family in A_T. By definition, it then follows that $\overline{F}$ is compact and $\overline{F} \subset A_T$. Thus, by Lemma 1, there exists some $f_0 \in \overline{F}$ such that $|E_{f_0}(0)| = \sup_{f\in\overline{F}} |E_f(0)|$. Then F is equicontinuous and $\overline{F} \subset F_T(r)$, where $r = |E_{f_0}(0)|$.

On the contrary, suppose that F is equicontinuous on T and that $\overline{F} \subset F_T(r)$ for some r, $0 \le r < 1$. Then F is a normal family in C_T, that is, $\overline{F}$ is compact in C_T. Since $\overline{F} \subset F_T(r) \subset A_T$, then F is a normal family in A_T. q.e.d.

For $K \ge 1$, we denote by $A_T(K)$ the family of all $f \in A_T$ such that

$$\Phi_{1/K}([z_1, z_2, z_3, z_4]) \le [f(z_1), f(z_2), f(z_3), f(z_4)] \le \Phi_K([z_1, z_2, z_3, z_4]) \tag{1.1}$$

holds for every ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in T$, where

$$[z_1, z_2, z_3, z_4] = \left\{\frac{z_3 - z_2}{z_3 - z_1} : \frac{z_4 - z_2}{z_4 - z_1}\right\}^{1/2}$$

is the real-valued cross-ratio of $\{z_1, z_2, z_3, z_4\}$ (see [Z1]). Moreover, Φ_K in (1.1) is the Hersch–Pfluger distortion function defined by

$$\Phi_K(t) = \mu^{-1}\left(\frac{1}{K}\mu(t)\right) \tag{1.2}$$

where $\frac{\pi}{2}\mu(t)$ stands for the conformal modulus of $\Delta \setminus [0; t]$, $0 \le t < 1$. The function μ can be expressed in the form:

$$\mu(t) = \frac{K(\sqrt{1-t^2})}{K(t)}, \qquad 0 < t < 1, \tag{1.3}$$

where

$$K(t) = \int_0^{\pi/2} (1 - t^2 \sin^2\varphi)^{-1/2}\, d\varphi$$

is the elliptic integral of the first kind. Every $f \in A_T(K)$ is called a K-quasihomography of T.

For every $K \ge 1$ and r, $0 \le r < 1$, we denote by $A_T(K, r)$ the family of all $f \in A_T(K)$ satisfying $|E_f(0)| \le r$. Obviously, $A_T(K, r) = A_T(K) \cap F_T(r)$. For $a \in \Delta$, we put

$$h_a(z) = \frac{z - a}{1 - \overline{a}z}. \tag{1.5}$$

LEMMA 3. *Suppose* $a_n \in \Delta$ *converges to* $e^{i\theta} \in T$ *as* n *tends to infinity. Then the function* $h_{a_n}(z)$ *converges to* $-e^{i\theta}$ *uniformly on every compact set* S *in* $\overline{\Delta} \setminus \{e^{i\theta}\}$, *as* n *tends to infinity.*

Proof. Let S be any compact set in $\overline{\Delta} \setminus \{e^{i\theta}\}$, and let $c_0 = \operatorname{dist}(e^{i\theta}, S)$. For any ε, $0 < \varepsilon < c_0$, there exists n_0 such that $|a_n - e^{i\theta}| < \varepsilon/2$, for all $n \geq n_0$. Then, for every $z \in S$, we have

$$|1 - \overline{a}_n z| \geq |1 - e^{-i\theta} z| - |(\overline{a}_n - e^{-i\theta}) z| \geq |e^{i\theta} - z| - |\overline{a}_n - e^{-i\theta}| \geq c_0/2. \tag{1.6}$$

For every $z \in S$ and $n \geq n_0$, it then follows from (1.6) that

$$|h_{a_n}(z) + e^{i\theta}| \leq \frac{|e^{i\theta} - a_n| + |e^{-i\theta} - \overline{a}_n|}{|1 - \overline{a}_n z|} \leq 2\varepsilon/c_0. q.e.d.$$

Now we have the following

THEOREM 1. *For every* $K \geq 1$ *and* r, $0 \leq r < 1$, *the family* $A_T(K, r)$ *is compact in* $A_T(K)$.

Proof. Let $A_T^\circ(K)$ be the family of all $f \in A_T(K)$, $K \geq 1$, normalized by $f(z) = z$ for every z such that $z^3 = 1$. As it is known, $A_T^\circ(K)$ is compact in $A_T(K)$ (see [Z2]). Let $\{f_n\}$ be an infinite sequence in $A_T(K, r)$. Then there exist $a_n \in \Delta$ and $\varphi_n \in \mathbb{R}$, such that $g_n := e^{i\varphi_n} h_{a_n} \circ f_n$ belongs to $A_T^\circ(K)$ for every n. Taking a subsequence, if necessary, we may assume $g_n \to g \in A_T^\circ(K)$, $a_n \to a_0 \in \overline{\Delta}$ and $e^{i\varphi_n} \to e^{i\varphi}$ as $n \to \infty$. By Lemma 1, $E_{g_n}(0)$ converges to $E_g(0)$. If $|a_0| = 1$ and $a_0 = e^{i\theta}$ for some $\theta \in \mathbb{R}$, then, since $|E_{f_n}(0)| \leq r$, Lemma 3 and conformal naturality of the Douady–Earle extension imply that $E_{g_n}(0) = e^{i\varphi_n} h_{a_n}(E_{f_n}(0))$ converges to $e^{i(\varphi - \theta)}$ as $n \to \infty$. This contradiction shows that $a_0 \in \Delta$ and that $f_n = h_{-a_n} \circ e^{-i\varphi_n} g_n(z)$ converges to $f_0(z) := h_{-a_0} e^{-i\varphi} g(z) \in A_T(K)$. Hence, by Lemma 1, $f_0 \in A_T(K, r)$, and thus $A_T(K, r)$ is compact in $A_T(K)$. q.e.d.

In view of Lemma 2, we can easily obtain the following:

COROLLARY 1. *For every* $K \geq 1$ *and* r, $0 \leq r < 1$, *the family* $A_T(K, r)$ *is equicontinuous on* T.

COROLLARY 2. *Let* $K \geq 1$ *and let* F *be a family in* $A_T(K)$. *Then* F *is a normal family* (*resp. compact*) *in* $A_T(K)$ *if and only if there exists some* r, $0 \leq r < 1$, *such that* F *is a subfamily* (*resp. a closed subfamily*) *of* $A_T(K, r)$.

For every $z \in T$ and $f \in A_T(K)$, $K \geq 1$, we denote by $\theta_f(z)$ the angle of the arc on T directed counterclockwise from $f(z)$ to $f(-z)$. In this sense $\theta_f(z) = \arg \frac{f(-z)}{f(z)}$ and we note that $\theta_f(-z) = 2\pi - \theta_f(z)$. By continuity of f, there exists $z_f \in T$ such that

$$\theta_f(z_f) = \min_{z \in T} \theta_f(z). \tag{1.7}$$

For every r, $0 \leq r < 1$, we define

$$\theta(K, r) := \inf_{f \in A_T(K,r)} \min_{z \in T} \theta_f(z). \tag{1.8}$$

LEMMA 4. *For every* $K \geq 1$ *and* r, $0 \leq r < 1$, *there exist* $f_0 \in A_T(K, r)$ *and* $z_0 \in T$ *such that* $\theta_{f_0}(z_0) = \theta(K, r)$.

Proof. By (1.7) and (1.8) there exist $f_n \in A_T(K,r)$, and $z_n \in T$ satisfying

$$\theta(K,r) = \lim_{n\to\infty} \theta_{f_n}(z_n) \tag{1.9}$$

and

$$\theta_{f_n}(z_n) = \min_{z\in T} \theta_{f_n}(z). \tag{1.10}$$

By Theorem 1, we may assume that $f_n \to f_0 \in A_T(K,r)$ in $A_T(K)$ and that $z_n \to z_0 \in T$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \theta_{f_n}(z_n) = \theta_{f_0}(z_0) \tag{1.11}$$

and

$$\lim_{n\to\infty} \theta_{f_n}(z_{f_0}) = \theta_{f_0}(z_{f_0}). \tag{1.12}$$

By (1.10) $\theta_{f_n}(z_{f_0}) \ge \theta_{f_n}(z_n)$, then by (1.11) and (1.12) we obtain

$$\theta_{f_0}(z_{f_0}) \ge \theta_{f_0}(z_0). \tag{1.13}$$

By (1.7), (1.13), (1.9) and (1.11) we then have $\theta_{f_0}(z_{f_0}) = \theta_{f_0}(z_0) = \theta(K,r)$. q.e.d.

LEMMA 5. *For every* r, $0 \le r < 1$, *the correspondence* $K \mapsto \theta(K,r)$ *is lower semi-continuous in* $1 \le K < \infty$. *Moreover*, *the function* $\theta(K,0)$ *is continuous at* $K=1$ *and* $\lim_{K\to 1}\theta(K,r) = \theta(1,0) = \pi$.

Proof. Let $\{K_n\}$, $K_n \ge 1$, be a sequence converging to K_0 as $n \to \infty$. Then, by Lemma 4, there exist $f_n \in A_T(K_n,r)$ and $z_n \in T$ such that $\theta_{f_n}(z_n) = \theta(K_n,r)$. By Theorem 1, we may assume that $f_n \to f_0 \in A_T(K_0,r)$ and that $z_n \to z_0 \in T$ as $n\to\infty$. In a way similar to the proof of Lemma 4, we have

$$\lim_{n\to\infty} \theta(K_n,r) = \theta_{f_0}(z_0) = \theta_{f_0}(z_{f_0}) \ge \theta(K_0,r). \tag{1.14}$$

Therefore, $\underline{\lim}_{K\to K_0}\theta(K,r) \ge \theta(K_0,r)$. Next, suppose $r=0$. Then $A_T(1,0) = \{\, f_\theta : 0 \le \theta < 2\pi \,\}$, where $f_\theta(z) = e^{i\theta}z$. In particular, $\theta_f(z) = \pi$ for every $f \in A_T(1,0)$ and every $z \in T$. Hence, $\theta(1,0) = \pi$ and (1.14) implies that $\lim_{K_n\to 1}\theta(K_n,0) = \theta(1,0) = \pi$. q.e.d.

Following Krzyż [K], we say that $f \in A_T$ is ρ-quasisymmetric, $\rho \ge 1$, if the inequality

$$1\rho \le |f(I_1)|/|f(I_2)| \le \rho$$

holds for each pair of open, adjacent arcs I_1, $I_2 \subset T$ such that $0 < |I_1| = |I_2| \le \pi$, where $|\cdot|$ denotes the Lebesgue measure on T.

Denote by $Q_T(\rho)$ the family of all ρ-quasisymmetric functions in A_T. It is worth while to mention that $Q_T(\rho)$ is not conformally invariant and that quasisymmetric functions of T represent non-uniformly the boundary values of quasiconformal automorphisms of Δ (see [Z2]). This and other properties makes ρ-quasisymmetry of T not closely related to quasiconformality of Δ, and technically similar to ρ-quasisymmetry of $\mathbb{R}$ only.

For $K \ge 1$, we recall the distortion function

$$\lambda(K) := \Phi_K^2(1/\sqrt{2})/\Phi_{1/K}^2(1/\sqrt{2}),$$

where Φ_K is given by (1.2). By Theorem 2.9 from [Z2, Chap. II], (1.7), (1.8) and Lemma 5, we obtain the following:

THEOREM 2. *For every $K \geq 1$ and r, $0 \leq r < 1$, there exists a constant $\rho = \rho(K,r)$ such that $A_T(K,r) \subset Q_T(\rho)$ and $\rho \leq \lambda(K)\cot^2(\theta(K,r)/4)$. In particular*, $\lim_{K\to 1}\rho(K,0) = 1$.

2. The maximal dilatation of the Douady–Earle extension of $f \in A_T(K)$. Let $K \geq 1$ and $f \in A_T(K)$. We note that by (0.1) $f \in A_T(K,0)$ if and only if f satisfies

$$\int_T f(\zeta)|d\zeta| = 0.$$

If $f \in A_T(K,0)$ then there exist $a = a(f) \in \Delta$ and $\varphi = \varphi(f) \in \mathbb{R}$ such that

$$e^{i\varphi} h_a \circ f \in A_T^0(K), \tag{2.1}$$

where h_a is the function defined by (1.5), whereas $a(f)$ and $e^{i\varphi(f)}$ are uniquely determined by (2.1).

Define

$$C(K) = \sup_{f\in A_T^0(K)} \sup_{\zeta\in T} \frac{|\zeta - E_f(0)|}{|f(\zeta) - E_f(0)|}. \tag{2.2}$$

LEMMA 6. *For every $K \geq 1$, there exist $f_K \in A_T^0(K)$ and $\zeta_K \in T$ such that*

$$C(K) = \frac{|\zeta_K - E_{f_K}(0)|}{|f_K(\zeta_K) - E_{f_K}(0)|}.$$

Furthermore, $C(K)$ *is increasing and right continuous in $1 \leq K < \infty$. In particular*, $C(K)$ *tends to* 1 *as* $K \to 1$.

P r o o f. For $f \in A_T^0(K)$ set

$$l(f) = \sup_{\zeta\in T} \frac{|\zeta - E_f(0)|}{|f(\zeta) - E_f(0)|}. \tag{2.3}$$

By the continuity of the correspondence $\zeta \mapsto \frac{|\zeta - E_f(0)|}{|f(\zeta)-E_f(0)|}$, there exists $\zeta = \zeta(f) \in T$ such that the supremum in (2.3) is attained at this point. Hence, by (2.2) there exist $f_n \in A_T^0(K)$ and $\zeta_n = \zeta(f_n)$ satisfying

$$\lim_{n\to\infty} l(f_n) = C(K) \tag{2.4}$$

and

$$l(f_n) = \frac{|\zeta_n - E_{f_n}(0)|}{|f_n(\zeta_n) - E_{f_n}(0)|}. \tag{2.5}$$

Taking a subsequence, if necessary, we may assume that $\zeta_n \to \zeta_0$, and that $f_n \to f_0 \in A_T^0(K)$ with respect to the supremum norm as $n \to \infty$. Then, by Lemma 1, $E_{f_n}(0)$ tends to $E_{f_0}(0)$ as $n \to \infty$. Hence, by (2.4) and (2.5), we have

$$C(K) = \frac{|\zeta_0 - E_{f_0}(0)|}{|f_0(\zeta_0) - E_{f_o}(0)|}. \tag{2.6}$$

By (2.2) the function $C(K)$ is clearly increasing. Let $K_0 \geq 1$ be fixed and let $K_n \searrow K_0$. By (2.6), there exist $\zeta_{K_n} \in T$ and $f_{K_n} \in A_T^0(K_n)$ such that

$$C(K_n) = \frac{|\zeta_{K_n} - E_{f_{K_n}}(0)|}{|f_{K_n}(\zeta_{K_n}) - E_{f_{K_n}}(0)|}. \tag{2.7}$$

We may assume that f_{K_n} tends to $f_I \in A_T^0(K_0)$, and ζ_{K_n} tends to $\zeta_I \in T$ as $n \to \infty$. From (2.7) it follows that

$$\lim_{n\to\infty} C(K_n) = \frac{|\zeta_I - E_{f_I}(0)|}{|f_I(\zeta_I) - E_{f_I}(0)|} \le C(K_0).$$

This implies that $\lim_{n\to\infty} C(K_n) = C(K_0)$. Clearly, $C(1) = 1$ and thus $\lim_{K\to 1} C(K) = 1$. q.e.d.

For $K \ge 1$, define $m(K) = \sup_{f\in A_T(K,0)} |a(f)|$ and

$$M(K) = \max_{0\le t\le 1} [\Phi_K^2(\sqrt{t}) - t] \tag{2.8}$$

where $a(f)$ is defined by (2.1) and Φ_K is given by (1.2). Introduced by the second author functional $M(K)$ was investigated in relation with certain functionals defined on families of K-quasihomographies of the real line and the unit circle T (see [Z1]). Surprisingly to both the authors, the following equality

$$M(K) = 2\Phi^2_{\sqrt{K}}(1/\sqrt{2}) - 1$$

was obtained by Partyka [P3]. This is a one of the truly few final results on special functions in quasiconformal theory, which may have some further consequences.

By Lemma 2.1 from [Z2, Chap. II] we have

LEMMA 7. *For each $K \ge 1$ and $f \in A_T^0(K)$ the following inequality*

$$|f(z) - z| \le \frac{4}{\sqrt{3}} M(K) \tag{2.9}$$

holds for every $z \in T$.

Now we prove

LEMMA 8. *For every $K \ge 1$, we have $m(K) < 1$. Moreover,*

$$m(K) \le \frac{4}{\sqrt{3}} M(K) C(K). \tag{2.10}$$

In particular, $m(K) \to 0$ as $K \to 1$.

Proof. If $f \in A_T(K,0)$, then $g := e^{i\varphi(f)} h_{a(f)} \circ f \in A_T^0(K)$. Furthermore, by (0.2), we have $E_g(0) = -a(f)e^{i\varphi(f)}$, and thus $|E_g(0)| = |a(f)|$. Conversely, if $g \in A_T^0(K)$ and $E_g(0) = b$, then $h_b \circ g \in A_T(K,0)$. Thus, the equality $g = h_{-b} \circ h_b \circ g$ implies that $|E_g(0)| = |b| = |-b| = |a(h_b \circ g)|$. The above observation shows that

$$m(K) = \sup_{f\in A_T^0(K)} |E_f(0)|. \tag{2.11}$$

By Lemma 1, the correspondence $f \mapsto |E_f(0)|$ is continuous on A_T. Since $A_T^0(K)$ is compact in A_T, then (2.11) implies that there exists some $f_K \in A_T^0(K)$ such that $m(K) = |E_{f_K}(0)|$. Since $\mu^{-1}(1) = 1/\sqrt{2}$, by (2.8), (2.9), the last equality and Lemma 1, we then see that $m(K) < 1$ and that $m(K)$ tends to 0 as $K \to 1$.

Let $f \in A_T^0(K)$ and put $a = E_f(0)$. We then obtain

$$\int_T \frac{\zeta - a}{1 - \overline{a}\zeta} |d\zeta| + \int_T \left(\frac{f(\zeta) - a}{1 - \overline{a} f(\zeta)} - \frac{\zeta - a}{1 - \overline{a}\zeta} \right) |d\zeta| = 0. \tag{2.12}$$

Since

$$\int_T \frac{\zeta - a}{1-\overline{a}\zeta}\,|d\zeta| = \frac{1}{i}\int_T \frac{\zeta - a}{\zeta(1-\overline{a}\zeta)}\,d\zeta = -2\pi a,$$

it follows from (2.12) that

$$|a| \le \frac{1}{2\pi}\int_T \omega(\zeta)\frac{(1-|a|^2)}{|\zeta - a|^2}\,|d\zeta|, \tag{2.13}$$

where $\omega(\zeta) = \frac{|f(\zeta)-\zeta||\zeta-a|}{|f(\zeta)-a|}$. The right-hand side of (2.13) is equal to $W(a)$, where $W(z)$ is a harmonic extension of $w(\zeta)$ into Δ. By (2.9) and (2.13), we thus have

$$|a| \le \max_{\zeta\in T}|\omega(\zeta)| = \max_{\zeta\in T}|f(\zeta)-\zeta|\frac{|\zeta - a|}{|f(\zeta)-a|} \le \frac{4}{\sqrt{3}}M(K)C(K).$$

This, in view of (2.11), gives (2.10). q.e.d.

For $f \in A_T(K,0)$, we put

$$A = A(f) = \frac{1}{2\pi}\int_T \overline{\zeta} f(\zeta)\,|d\zeta|,$$

$$B = B(f) = \frac{1}{2\pi}\int_T \zeta f(\zeta)\,|d\zeta|,$$

$$C = C(f) = \frac{1}{2\pi}\int_T f(\zeta)^2\,|d\zeta|$$

and

$$S(K) = \frac{4}{\sqrt{3}}M(K)C(K). \tag{2.14}$$

LEMMA 9. *For each $K \ge 1$ and $f \in A_T(K,0)$ the following inequalities hold*;

$$|B| \le S(K), \quad |C| \le 2S(K) + S(K)^2, \quad |A| \ge 1 - S(K)^2 - S(K). \tag{2.15}$$

Moreover, the third estimate is essential for $K \ge 1$ satisfying $S(K) < (\sqrt{5}-1)/2$.

Proof. Let $f \in A_T(K,0)$ and let $g = e^{i\varphi(f)}h_{a(f)} \circ f$, $b = -a(f)e^{i\varphi(f)}$. Then, $g \in A_T^\circ(K)$, $E_g(0) = b$, and we see that

$$e^{i\varphi(f)}f(\zeta) = [g(\zeta) - b]/[1-\overline{b}g(\zeta)].$$

As in the proof of Lemma 8, we have

$$|B| = \frac{1}{2\pi}\left|\int_T \zeta e^{i\varphi(f)}f(\zeta)\,|d\zeta|\right| = \frac{1}{2\pi}\left|\int_T \zeta\left(\frac{g(\zeta)-b}{1-\overline{b}g(\zeta)}\right)|d\zeta|\right|$$

$$= \frac{1}{2\pi}\left|\int_T \zeta\left(\frac{g(\zeta)-b}{1-\overline{b}g(\zeta)} - \frac{\zeta-b}{1-\overline{b}\zeta}\right)|d\zeta|\right| \le \frac{1}{2\pi}\int_T \frac{|g(\zeta)-\zeta||1-b\overline{\zeta}|}{|1-\overline{b}g(\zeta)|}\cdot\frac{(1-|b|^2)}{|\zeta-b|^2}\,|d\zeta|$$

$$\le \frac{4}{\sqrt{3}}M(K)C(K) = S(K).$$

Similarly, by Lemma 8, we obtain

$$|C| = \frac{1}{2\pi}\left|\int_T e^{i2\varphi(f)}f(\zeta)^2\,|d\zeta|\right| = \frac{1}{2\pi}\left|\int_T\left\{\left(\frac{g(\zeta)-b}{1-\overline{b}g(\zeta)}\right)^2 - \left(\frac{\zeta-b}{1-\overline{b}\zeta}\right)^2\right\}|d\zeta| + 2\pi b^2\right|$$

$$\le |b|^2 + \frac{2}{2\pi}\int_T\left|\frac{g(\zeta)-b}{1-\overline{b}g(\zeta)} - \frac{\zeta-b}{1-\overline{b}\zeta}\right||d\zeta| \le S(K)^2 + 2S(K).$$

Since

$$\frac{1}{2\pi}\int_T \overline{\zeta}\left(\frac{\zeta-b}{1-\overline{b}\zeta}\right)|d\zeta| = \frac{1}{2\pi i}\int_T \frac{\zeta-b}{(1-\overline{b}\zeta)\zeta^2}\,d\zeta = 1-|b|^2,$$

we have

$$\begin{aligned}|A| &= \frac{1}{2\pi}\left|\int_T \overline{\zeta}e^{i\varphi(f)}f(\zeta)\,|d\zeta|\right| = \frac{1}{2\pi}\left|\int_T \overline{\zeta}\left(\frac{g(\zeta)-b}{1-\overline{b}g(\zeta)}\right)|d\zeta|\right| \\ &= \frac{1}{2\pi}\left|\int_T \overline{\zeta}\left(\frac{g(\zeta)-b}{1-\overline{b}g(\zeta)} - \frac{\zeta-b}{1-\overline{b}\zeta}\right)|d\zeta| + 2\pi(1-|b|^2)\right| \\ &\geq 1-|b|^2 - \frac{1}{2\pi}\int_T \left|\frac{g(\zeta)-b}{1-\overline{b}g(\zeta)} - \frac{\zeta-b}{1-\overline{b}\zeta}\right||d\zeta| \geq 1 - S(K)^2 - S(K). q.e.d.\end{aligned}$$

Remark 1. For $K \geq 1$ satisfying $S(K) < (\sqrt{5}-1)/2$, we have $|A| > 0$.

3. An estimation of the dilatation. For $K \geq 1$ we define

$$k^*(K) = \sup_{f\in A_T(K,0)} I(f),$$

where

$$I(f) = \left\{\frac{2|B(f)| + |C(f)|^2(|A(f)|-|B(f)|)}{|A(f)|+|B(f)|}\right\}^{1/2}.$$

Since $f \mapsto I(f)$ is continuous on A_T and $A_T(K,0)$ is a compact in $A_T(K)$ hence by Theorem 1, we infer that there exists some $f_K \in A_T(K,0)$ such that $k^*(K) = I(f_K)$. Moreover, $|A(f)| > |B(f)|$ holds for every $f \in A_T$; because f is sense-preserving (see [DE, Lemma 3]). We thus see that $k^*(K) < 1$.

THEOREM 3. *For each $K \geq 1$ and $f \in A_T(K)$ the Douady–Earle extension E_f is quasiconformal and its complex dilatation μ_{E_f} satisfies $\|\mu_{E_f}\|_\infty \leq k^*(K)$. Moreover, if $K \geq 1$ is as close to* 1, *so that $S(K) < (\sqrt{5}-1)/2$ holds, then the following estimate*

$$k^*(K) \leq \left\{\frac{2S(K)}{1-S(K)^2} + (2S(K)+S(K)^2)^2\right\}^{1/2} \tag{2.16}$$

holds, where $S(K)$ is the number defined by means of (2.2), (2.8) *and* (2.14). *In particular, $\|\mu_{E_f}\|_\infty \to 0$ as $K \to 1$.*

Proof. Take any $z_0 \in \Delta$ and let $w_0 = E_f(z_0)$. Put $\tilde{f} = h_{w_0} \circ f \circ h_{-z_0}$, where $h_\eta(\zeta) = \frac{\zeta-\eta}{1-\overline{\eta}\zeta}$. By (0.2) we have $E_{\tilde{f}} = h_{w_0} \circ E_f \circ h_{-z_0}$ and therefore $E_{\tilde{f}}(0) = 0$, by which $\tilde{f} \in A_T(K,0)$. Moreover, we easily have

$$|\mu_{E_f}(z_0)| = |\mu_{E_{\tilde{f}}}(0)|. \tag{2.17}$$

Let $k_0 = \sup|\mu_{E_g}(0)|$, where the supremum is taken over all $g \in A_T(K,0)$. By (2.17) it suffices to show that $k_0 \leq k^*(K)$.

Take any $g \in A_T(K,0)$. Then, as in [DE], we have

$$|\mu_{E_g}(0)| = |A\overline{C} + \overline{B}|/|A + C\overline{B}|, \tag{2.18}$$

where $A = A(g)$, $B = B(g)$ and $C = C(g)$. By (2.18), we obtain

$$1 - |\mu_{E_g}(0)|^2 = \frac{(1-|C|^2)(|A|^2-|B|^2)}{|A+C\overline{B}|^2} \geq (1-|C|^2)\frac{|A|-|B|}{|A|+|B|}$$

$$= 1 - \left(\frac{2|B| + |C|^2(|A| - |B|)}{|A| + |B|} \right).$$

Thus, $|\mu_{E_g}(0)| \le I(g) \le k^*(K)$ and hence $k_0 \le k^*(K)$.

Next we show the latter part of the theorem. Let $K \ge 1$ satisfy $S(K) < (\sqrt{5} - 1)/2$ which is equivalent to $1 - S(K)^2 - S(K) > 0$. If $g \in A_T(K, 0)$, then, by (2.15) in Lemma 9, we see that

$$I(g)4 \le \left\{ \frac{2|B(g)|/|A(g)|}{1 + |B(g)|/|A(g)|} + |C(f)|^2 \right\}^{1/2} \le \left\{ \frac{2S(K)/(1 - S(K)^2 - S(K))}{1 + S(K)/(1 - S(K)^2 - S(K))} + (2S(K) + S(K)^2)^2 \right\}^{1/2} = \left\{ \frac{2S(K)}{1 - S(K)^2} + (2S(K) + S(K)^2)^2 \right\}^{1/2}.$$

q.e.d.

Corollary 3. *Under the hypotheses of Theorem 3, suppose that* $K \ge 1$, *is so close to* 1 *that the following inequality*

$$(2S(K) + S(K)^2)^2 \le \frac{2S(K)}{1 - S(K)^2} < \frac{1}{2} \tag{2.19}$$

holds, i.e. if $0 \le S(K) < \sqrt{5} - 2$. *Then the maximal dilatation* $\mathbf{K}(E_f)$ *of* E_f *satisfies*

$$\mathbf{K}(E_f) \le \frac{1 + S(K)^{1/2} g(S(K))}{1 - S(K)^{1/2} g(S(K))}, \qquad \text{where } g(S) = \left(\frac{4}{1 - S^2} \right)^{1/2}. \tag{2.20}$$

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