THE DOUADY–EARLE EXTENSION OF QUASIHOMOGRAPHIES

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Abstract. Quasihomography is a useful notion to represent a sense-preserving automorphism of the unit circle \( T \) which admits a quasiconformal extension to the unit disc. For \( K \geq 1 \) let \( A_T(K) \) denote the family of all \( K \)-quasihomographies of \( T \). With any \( f \in A_T(K) \) we associate the Douady–Earle extension \( E_f \) and give an explicit and asymptotically sharp estimate of the \( L^\infty \) norm of the complex dilatation of \( E_f \).

Introduction. Let \( A_T \) denote the family of all sense-preserving automorphisms of the unit circle \( T \). With any \( f \in A_T \) we associate the Douady–Earle extension \( E_f \) which is a homeomorphic automorphism of the unit disc \( \Delta \) and has a continuous extension to \( f \) on the boundary \( T = \partial \Delta \) (see [DE] and [LP]). If \( z \in \Delta \) and \( f \in A_T \), then \( E_f(z) \) is the unique \( w \in \Delta \) such that

\[
\int_T \left( \frac{f(\zeta) - w}{1 - \overline{w}f(\zeta)} \right) \frac{(1 - |z|^2)}{|z - \zeta|^2} |d\zeta| = 0.
\]

Moreover, the correspondence \( f \mapsto E_f \) is conformally natural in the sense that

\[
E_{h_1 \circ f \circ h_2} = h_1 \circ E_f \circ h_2
\]

holds for any \( f \in A_T \) and all Möbius transformations \( h_1, h_2 \), which map \( \Delta \) onto itself.

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The property that a given \( f \in A_T \) admits a quasiconformal extension to \( \Delta \) is equivalent to the assumption that \( f \) is a quasihomography (see [Z1]). For \( K \geq 1 \), we denote by \( A_T(K) \) the family of all \( f \in A_T \) that are \( K \)-quasihomographies (see Chap. 1).

Starting with an automorphism \( f \) of \( T \), which is the boundary automorphism of a given \( K \)-quasiconformal mapping of \( \Delta \) onto itself, Douady and Earle proved that, given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( K(E_f) \leq 4^{3+\epsilon} \) for \( 1 \leq K \leq 1 + \delta \) (see [DE, Corollary 2]). Their explicit estimate starts from \( 4 \cdot 10^{-8} \cdot 3^5 \), for \( K \) near 1.

Making some refinements and using more subtle tools, Partyka obtained an asymptotically sharp estimate for \( K(E_f) \) (see [P1, Theorem 3.1]), improving the result of Douady and Earle for \( 1 \leq K < 50 \). Using the notion of quasisymmetry for unit circle, introduced by Krzyż [K], he considered also, as the starting point, a given \( \rho \)-quasisymmetric automorphism \( f \) of \( T \).

It is very natural from different points of view if we may extend an automorphism \( f \) of \( T \) that satisfies certain condition on \( T \) only, and next to study how particular properties of such an \( f \) effects the extension.

Rotation, but not conformally invariant notion of quasisymmetry of \( T \), mentioned above, is meaningless in these considerations. This is mostly because neither there exists \( \rho \geq 1 \) such that boundary values of Möbius automorphisms of \( \Delta \) are \( \rho \)-quasisymmetric (see [Z1, Example]), nor \( \rho \)-quasisymmetric automorphisms of \( T \) represent uniformly boundary values of \( K \)-quasiconformal automorphisms of \( \Delta \), for any \( K \geq 1 \).

We assume that a given automorphism \( f \) of \( T \) is a \( K \)-quasihomography \((\equiv 1\text{-dimensional } K\text{-quasiconformal mapping}) \) of \( T \), \( K \geq 1 \). The family \( A_T(K) \), \( K \geq 1 \), representing uniformly \( K \)-quasiconformal mappings, with the same \( K \) of necessity, is conformally invariant under composition and thus very natural with respect to the Douady–Earle extension.

Developing in Sect. 1 the argument of normal families in \( A_T \) in a way related to the Douady–Earle extension and introducing necessary functionals, defined on families of \( K \)-quasihomographies of \( T \), we estimate in Theorem 3 the \( L_\infty \)-norm of the complex dilatation \( \mu_{E_f} \) for the Douady–Earle extension of a given \( K \)-quasihomography \( f \) of \( T \), with \( K \) close to 1. In Corollary 3 we describe an asymptotically sharp estimate of \( K(E_f) \), expressed explicitly by (2.20), for \( K \) close to 1.

In order to be in contact with results mentioned above we give, in Theorem 2, a relation between some important families in \( A_T(K) \) and functions \( \rho \)-quasisymmetric on the unit circle.

1. Normal families in \( A_T \). Let \( \Delta \) be the unit disc in the complex plane \( C \) and \( T = \partial \Delta \) be the unit circle. We consider the family \( A_T \) of all sense-preserving automorphisms of \( T \) as a subspace of the Banach space \( C_T \) of all complex-valued continuous functions on \( T \), with the supremum norm. In this section, we first discuss normality of certain subfamilies of \( A_T \). As an application, we shall then show that some subfamilies of \( K \)-quasihomographies on \( T \), which play an important role for our purpose, turn out to be families of \( \rho \)-quasisymmetric functions of \( T \) where \( \rho \) depends on \( K \) only.

For \( f \in A_T \), we denote by \( E_f \) the Douady–Earle extension of \( f \) to \( \Delta \).
Lemma 1. The functional $E_f(0)$ is continuous on $A_T$. ([DE, Prop. 2]).

For every $r$, $0 \leq r < 1$, we denote by $F_T(r)$ the family of all $f \in A_T$ satisfying $|E_f(0)| \leq r$. A family $F$ in $A_T$ is said to be a normal family if $F$ is relatively compact in $A_T$. Thus a family $F$ in $A_T$ is a normal family if and only if for any infinite sequence $\{f_n\}$ in $F$, there exists a subsequence $\{f_{n_k}\}$ which converges to some $f$ in $A_T$.

Lemma 2. Let $F$ be a family in $A_T$. Then $F$ is normal family in $A_T$ if and only if $F$ is equicontinuous on $T$ and there exists $r$, $0 \leq r < 1$, such that $F \subset F_T(r)$, where $F$ is the closure of $F$ in the Banach space $C_T$.

**Proof.** We note that by the Ascoli–Arzela’s theorem, a family $G$ in $C_T$ is a normal family in $C_T$ if and only if $G$ is uniformly bounded and equicontinuous on $T$. Suppose that $F$ is a normal family in $A_T$. By definition, it then follows that $F$ is compact and $F \subset A_T$. Thus, by Lemma 1, there exists some $f_0 \in F$ such that $|E_{f_0}(0)| = \sup_{f \in F} |E_f(0)|$. Then $F$ is equicontinuous and $F \subset F_T(r)$, where $r = |E_{f_0}(0)|$.

On the contrary, suppose that $F$ is equicontinuous on $T$ and that $F \subset F_T(r)$ for some $r$, $0 \leq r < 1$. Then $F$ is a normal family in $C_T$, that is, $F$ is compact in $C_T$. Since $F \subset F_T(r) \subset A_T$, then $F$ is a normal family in $A_T$. q.e.d.

For $K \geq 1$, we denote by $A_T(K)$ the family of all $f \in A_T$ such that

$$
\Phi_{1/K}([z_1, z_2, z_3, z_4]) \leq |f(z_1), f(z_2), f(z_3), f(z_4)| \leq \Phi_K([z_1, z_2, z_3, z_4])
$$

holds for every ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in T$, where

$$
[z_1, z_2, z_3, z_4] = \left\{ \frac{z_4 - z_2}{z_3 - z_1}, \frac{z_4 - z_2}{z_4 - z_1} \right\}^{1/2}
$$

is the real-valued cross-ratio of $\{z_1, z_2, z_3, z_4\}$ (see [Z1]). Moreover, $\Phi_K$ in (1.1) is the Hersch–Pfluger distortion function defined by

$$
\Phi_K(t) = \mu^{-1}\left(\frac{1}{K} \mu(t)\right)
$$

where $\mu(t)$ stands for the conformal modulus of $\Delta \setminus [0; t]$, $0 \leq t < 1$. The function $\mu$ can be expressed in the form:

$$
\mu(t) = \frac{K(\sqrt{1 - t^2})}{K(t)}, \quad 0 < t < 1,
$$

where

$$
K(t) = \int_0^{\pi/2} (1 - t^2 \sin^2 \varphi)^{-1/2} d\varphi
$$

is the elliptic integral of the first kind. Every $f \in A_T(K)$ is called a $K$-quasihomography of $T$.

For every $K \geq 1$ and $r$, $0 \leq r < 1$, we denote by $A_T(K, r)$ the family of all $f \in A_T(K)$ satisfying $|E_f(0)| \leq r$. Obviously, $A_T(K, r) = A_T(K) \cap F_T(r)$. For $a \in \Delta$, we put

$$
h_a(z) = \frac{z - a}{1 - \bar{a}z},
$$
LEMMA 3. Suppose $a_n \in \Delta$ converges to $e^{i\theta} \in T$ as $n$ tends to infinity. Then the function $h_{a_n}(z)$ converges to $-e^{i\theta}$ uniformly on every compact set $S$ in $\overline{\Delta} \setminus \{e^{i\theta}\}$, as $n$ tends to infinity.

Proof. Let $S$ be any compact set in $\overline{\Delta} \setminus \{e^{i\theta}\}$, and let $c_0 = \text{dist}(e^{i\theta}, S)$. For any $\varepsilon$, $0 < \varepsilon < c_0$, there exists $n_0$ such that $|a_n - e^{i\theta}| < \varepsilon/2$, for all $n \geq n_0$. Then, for every $z \in S$, we have

$$|h_{a_n}(z) + e^{i\theta}| \leq \frac{|e^{i\theta} - a_n| + |e^{-i\theta} - \overline{a_n}|}{|1 - \overline{a_n}z|} \leq 2\varepsilon/c_0. \text{q.e.d.}$$

Now we have the following

THEOREM 1. For every $K \geq 1$ and $r$, $0 \leq r < 1$, the family $A_T(K, r)$ is compact in $A_T(K)$.

Proof. Let $A^*_T(K)$ be the family of all $f \in A_T(K)$, $K \geq 1$, normalized by $f(z) = z$ for every $z$ such that $z^3 = 1$. As it is known, $A^*_T(K)$ is compact in $A_T(K)$ (see [Z2]). Let $\{f_n\}$ be an infinite sequence in $A_T(K, r)$. Then there exist $a_n \in \Delta$ and $\varphi_n \in \mathbb{R}$, such that $g_n := e^{i\varphi_n}h_{a_n} \circ f_n$ belongs to $A^*_T(K)$ for every $n$. Taking a subsequence, if necessary, we may assume $g_n \to g \in A^*_T(K)$, $a_n \to a_0 \in \overline{\Delta}$ and $e^{i\varphi_n} \to e^{i\varphi}$ as $n \to \infty$. By Lemma 1, $E_{g_n}(0)$ converges to $E_g(0)$. If $|a_0| = 1$ and $a_0 = e^{i\theta}$ for some $\theta \in \mathbb{R}$, then, since $|E_{f_n}(0)| \leq r$, Lemma 3 and conformal naturality of the Douady–Earle extension imply that $E_{g_n}(0) = e^{i\varphi_n}h_{a_n}(E_{f_n}(0))$ converges to $e^{i(\varphi - \theta)}$ as $n \to \infty$. This contradiction shows that $a_0 \in \Delta$ and that $f_n = h_{-a_n} \circ e^{-i\varphi_n}g_n(z)$ converges to $f_0(z) := h_{-a_0}e^{-i\varphi}g(z) \in A_T(K)$. Hence, by Lemma 1, $f_0 \in A_T(K, r)$, and thus $A_T(K, r)$ is compact in $A_T(K)$.

In view of Lemma 2, we can easily obtain the following:

COROLLARY 1. For every $K \geq 1$ and $r$, $0 \leq r < 1$, the family $A_T(K, r)$ is equicontinuous on $T$.

COROLLARY 2. Let $K \geq 1$ and let $F$ be a family in $A_T(K)$. Then $F$ is a normal family (resp. compact) in $A_T(K)$ if and only if there exists some $r$, $0 \leq r < 1$, such that $F$ is a subfamily (resp. a closed subfamily) of $A_T(K, r)$.

For every $z \in T$ and $f \in A_T(K)$, $K \geq 1$, we denote by $\theta_f(z)$ the angle of the arc on $T$ directed counterclockwise from $f(z)$ to $f(-z)$. In this sense $\theta_f(z) = \arg \frac{f(z)}{f(-z)}$ and we note that $\theta_f(-z) = 2\pi - \theta_f(z)$. By continuity of $f$, there exists $z_f \in T$ such that

$$\theta_f(z_f) = \min_{z \in T} \theta_f(z).$$

For every $r$, $0 \leq r < 1$, we define

$$\theta(K, r) := \inf_{f \in A_T(K, r)} \min_{z \in T} \theta_f(z).$$

LEMMA 4. For every $K \geq 1$ and $r$, $0 \leq r < 1$, there exist $f_0 \in A_T(K, r)$ and $z_0 \in T$ such that $\theta_{f_0}(z_0) = \theta(K, r)$. 

Proof. By (1.7) and (1.8) there exist \( f_n \in A_T(K, r) \), and \( z_n \in T \) satisfying
\[
\theta(K, r) = \lim_{n \to \infty} \theta_{f_n}(z_n)
\]
and
\[
\theta_{f_n}(z_n) = \min_{z \in T} \theta_{f_n}(z).
\]
By Theorem 1, we may assume that \( f_n \to f_0 \in A_T(K, r) \) in \( A_T(K) \) and that \( z_n \to z_0 \in T \) as \( n \to \infty \). Then
\[
\lim_{n \to \infty} \theta_{f_n}(z_n) = \theta_{f_0}(z_0)
\]
and
\[
\lim_{n \to \infty} \theta_{f_n}(z_{f_n}) = \theta_{f_0}(z_{f_0}).
\]
By (1.10) \( \theta_{f_n}(z_{f_n}) \geq \theta_{f_0}(z_{f_0}) \), then by (1.11) and (1.12) we obtain
\[
\theta_{f_0}(z_{f_0}) \geq \theta_{f_0}(z_0) = \theta(K, r).
\]

Lemma 5. For every \( r, 0 \leq r < 1 \), the correspondence \( K \mapsto \theta(K, r) \) is lower semi-continuous in \( 1 \leq K < \infty \). Moreover, the function \( \theta(K, 0) \) is continuous at \( K = 1 \) and
\[
\lim_{K \to 1} \theta(K, r) = \theta(1, 0) = \pi.
\]

Proof. Let \( \{K_n\} \), \( K_n \geq 1 \), be a sequence converging to \( K_0 \) as \( n \to \infty \). Then, by Lemma 4, there exist \( f_n \in A_T(K_n, r) \) and \( z_n \in T \) such that \( \theta_{f_n}(z_n) = \theta(K_n, r) \). By Theorem 1, we may assume that \( f_n \to f_0 \in A_T(K_0, r) \) and that \( z_n \to z_0 \in T \) as \( n \to \infty \). In a way similar to the proof of Lemma 4, we have
\[
\lim_{n \to \infty} \theta(K_n, r) = \theta_{f_0}(z_0) = \theta_{f_0}(z_{f_0}) \geq \theta(K_0, r).
\]
Therefore, \( \lim_{K \to K_0} \theta(K, r) \geq \theta(K_0, r) \). Next, suppose \( r = 0 \). Then \( A_T(1, 0) = \{ f_\theta : 0 \leq \theta < 2\pi \} \), where \( f_\theta(z) = e^{i\theta}z \). In particular, \( \theta_f(z) = \pi \) for every \( f \in A_T(1, 0) \) and every \( z \in T \). Hence, \( \theta(1, 0) = \pi \) and (1.14) implies that \( \lim_{K \to 1} \theta(K, 0) = \theta(1, 0) = \pi \). q.e.d.

Following Krzyż [K], we say that \( f \in A_T \) is \( \rho \)-quasisymmetric, \( \rho \geq 1 \), if the inequality
\[
1 \rho \leq |f(I_1)|/|f(I_2)| \leq \rho
\]
holds for each pair of open, adjacent arcs \( I_1, I_2 \subset T \) such that \( 0 < |I_1| = |I_2| \leq \pi \), where \( |\cdot| \) denotes the Lebesgue measure on \( T \).

Denote by \( QT(\rho) \) the family of all \( \rho \)-quasisymmetric functions in \( A_T \). It is worth while to mention that \( QT(\rho) \) is not conformally invariant and that quasisymmetric functions of \( T \) represent non-uniformly the boundary values of quasiconformal automorphisms of \( \Delta \) (see [Z2]). This and other properties makes \( \rho \)-quasisymmetry of \( T \) not closely related to quasiconformality of \( \Delta \), and technically similar to \( \rho \)-quasisymmetry of \( \mathbb{R} \) only.

For \( K \geq 1 \), we recall the distortion function
\[
\lambda(K) := \Phi_K^2(1/\sqrt{2})/\Phi_{1/K}^2(1/\sqrt{2}),
\]
where \( \Phi_K \) is given by (1.2). By Theorem 2.9 from [Z2, Chap. II], (1.7), (1.8) and Lemma 5, we obtain the following:
Theorem 2. For every $K \geq 1$ and $r$, $0 \leq r < 1$, there exists a constant $\rho = \rho(K,r)$ such that $A_T(K,r) \subset Q_T(\rho)$ and $\rho \leq \lambda(K)\cot^2(\theta(K,r)/4)$. In particular, $\lim_{K \to 1} \rho(K,0) = 1$.

2. The maximal dilatation of the Douady–Earle extension of $f \in A_T(K)$.

Let $K \geq 1$ and $f \in A_T(K)$. We note that by (0.1) $f \in A_T(K,0)$ if and only if $f$ satisfies
\[ \int_T |f(\zeta)| d\zeta = 0. \]
If $f \in A_T(K,0)$ then there exist $a = a(f) \in \Delta$ and $\varphi = \varphi(f) \in \mathbb{R}$ such that
\[ e^{i\varphi}h_a \circ f \in A_T^0(K), \]
where $h_a$ is the function defined by (1.5), whereas $a(f)$ and $e^{i\varphi(f)}$ are uniquely determined by (2.1).

Define
\[ C(K) = \sup_{f \in A_T^0(K)} \sup_{\zeta \in T} \frac{|\zeta - E_f(0)|}{|f(\zeta) - E_f(0)|}. \]

Lemma 6. For every $K \geq 1$, there exist $f_K \in A_T^0(K)$ and $\zeta_K \in T$ such that
\[ C(K) = \frac{|\zeta_K - E_{f_K}(0)|}{|f_K(\zeta_K) - E_{f_K}(0)|}. \]
Furthermore, $C(K)$ is increasing and right continuous in $1 \leq K < \infty$. In particular, $C(K)$ tends to 1 as $K \to 1$.

Proof. For $f \in A_T^0(K)$ set
\[ l(f) = \sup_{\zeta \in T} \frac{|\zeta - E_f(0)|}{|f(\zeta) - E_f(0)|}, \]
By the continuity of the correspondence $\zeta \mapsto \frac{|\zeta - E_f(0)|}{|f(\zeta) - E_f(0)|}$, there exists $\zeta = \zeta(f) \in T$ such that the supremum in (2.3) is attained at this point. Hence, by (2.2) there exist $f_n \in A_T^0(K)$ and $\zeta_n = \zeta(f_n)$ satisfying
\[ \lim_{n \to \infty} l(f_n) = C(K) \]
and
\[ l(f_n) = \frac{|\zeta_n - E_{f_n}(0)|}{|f_n(\zeta_n) - E_{f_n}(0)|}. \]
Taking a subsequence, if necessary, we may assume that $\zeta_n \to \zeta_0$, and that $f_n \to f_0 \in A_T^0(K)$ with respect to the supremum norm as $n \to \infty$. Then, by Lemma 1, $E_{f_n}(0)$ tends to $E_{f_0}(0)$ as $n \to \infty$. Hence, by (2.4) and (2.5), we have
\[ C(K) = \frac{|\zeta_0 - E_{f_0}(0)|}{|f_0(\zeta_0) - E_{f_0}(0)|}. \]
By (2.2) the function $C(K)$ is clearly increasing. Let $K_0 \geq 1$ be fixed and let $K_n \searrow K_0$. By (2.6), there exist $\zeta_{K_n} \in T$ and $f_{K_n} \in A_T^0(K_n)$ such that
\[ C(K_n) = \frac{|\zeta_{K_n} - E_{f_{K_n}}(0)|}{|f_{K_n}(\zeta_{K_n}) - E_{f_{K_n}}(0)|}. \]
We may assume that $f_{K_n}$ tends to $f_I \in A^0(T(K_0))$; and $\zeta_{K_n}$ tends to $\zeta_I \in T$ as $n \to \infty$. From (2.7) it follows that
\[
\lim_{n \to \infty} C(K_n) = \frac{|\zeta_I - E_{f_I}(0)|}{|f_I(\zeta_I) - E_{f_I}(0)|} \leq C(K_0).
\]
This implies that $\lim_{n \to \infty} C(K_n) = C(K_0)$. Clearly, $C(1) = 1$ and thus $\lim_{K \to 1} C(K) = 1$.

For $K \geq 1$, define $m(K) = \sup_{f \in A_T(K,0)} |a(f)|$ and
\[
M(K) = \max_{0 \leq t \leq 1} [\Phi_K^2(\sqrt{t}) - t]
\]
where $a(f)$ is defined by (2.1) and $\Phi_K$ is given by (1.2). Introduced by the second author functional $M(K)$ was investigated in relation with certain functionals defined on families of $K$-quasihomographies of the real line and the unit circle $T$ (see [Z1]). Surprisingly to both the authors, the following equality
\[
M(K) = 2\Phi_{\sqrt{K}}(1/\sqrt{2}) - 1
\]
was obtained by Partyka [P3]. This is a one of the truly few final results on special functions in quasiconformal theory, which may have some further consequences.

By Lemma 2.1 from [Z2, Chap. II] we have

**Lemma 7.** For each $K \geq 1$ and $f \in A^0_T(K)$ the following inequality
\[
|f(z) - z| \leq \frac{4}{\sqrt{3}} M(K)
\]
holds for every $z \in T$.

Now we prove

**Lemma 8.** For every $K \geq 1$, we have $m(K) < 1$. Moreover,
\[
m(K) \leq \frac{4}{\sqrt{3}} M(K) C(K).
\]
In particular, $m(K) \to 0$ as $K \to 1$.

**Proof.** If $f \in A_T(K,0)$, then $g := e^{i\varphi(f)} h_{a(f)} \circ f \in A^0_T(K)$. Furthermore, by (0.2), we have $E_g(0) = -a(f) e^{i\varphi(f)}$, and thus $|E_g(0)| = |a(f)|$. Conversely, if $g \in A^0_T(K)$ and $E_g(0) = b$, then $b = g \circ h_{b} \circ g$, and thus, the equality $g = h_{-b} \circ h_{b} \circ g$ implies that $|E_g(0)| = |b| = |-b| = |a(h_{b} \circ g)|$. The above observation shows that
\[
m(K) = \sup_{f \in A^0_T(K)} |E_f(0)|.
\]

By Lemma 1, the correspondence $f \mapsto |E_f(0)|$ is continuous on $A_T$. Since $A^0_T(K)$ is compact in $A_T$, then (2.11) implies that there exists some $f_K \in A^0_T(K)$ such that $m(K) = |E_{f_K}(0)|$. Since $\mu^{-1}(1) = 1/\sqrt{2}$, by (2.8), (2.9), the last equality and Lemma 1, we then see that $m(K) < 1$ and that $m(K)$ tends to 0 as $K \to 1$.

Let $f \in A^0_T(K)$ and put $a = E_f(0)$. We then obtain
\[
\int_T \frac{\zeta - a}{1 - \pi_\zeta} |d\zeta| + \int_T \left( \frac{f(\zeta)}{1 - \pi(f(\zeta))} - \frac{\zeta - a}{1 - \pi_\zeta} \right) |d\zeta| = 0.
\]
Since
\[ \int_T \frac{\zeta - a}{1 - \overline{mc}} \, |d\zeta| = \frac{1}{i} \int_T \frac{\zeta - a}{\zeta (1 - \overline{mc})} \, d\zeta = -2\pi a, \]
it follows from (2.12) that
\[ |a| \leq \frac{1}{2\pi} \int_T \omega(\zeta) \left| \frac{1 - |a|^2}{|\zeta - a|^2} \right| \, |d\zeta|, \]
where \( \omega(\zeta) = \frac{|f(\zeta) - \zeta ||\overline{m} - a|}{|f(\zeta) - \zeta|} \). The right-hand side of (2.13) is equal to \( W(a) \), where \( W(z) \) is a harmonic extension of \( w(\zeta) \) into \( \Delta \). By (2.9) and (2.13), we thus have

\[ |a| \leq \max_{\zeta \in T} |\omega(\zeta)| \leq \max_{\zeta \in T} \left| f(\zeta) - \zeta \right| \left| \frac{\zeta - a}{|f(\zeta) - a|} \right| \leq \frac{4}{\sqrt{3}} M(K) C(K). \]

This, in view of (2.11), gives (2.10). q.e.d.

For \( f \in A_T(K, 0) \), we put

\[ A = A(f) = \frac{1}{2\pi} \int_T \zeta f(\zeta) \, |d\zeta|, \]
\[ B = B(f) = \frac{1}{2\pi} \int_T \zeta f(\zeta) \, |d\zeta|, \]
\[ C = C(f) = \frac{1}{2\pi} \int_T f(\zeta)^2 \, |d\zeta| \]
and
\[ (2.14) \quad S(K) = \frac{4}{\sqrt{3}} M(K) C(K). \]

**Lemma 9.** For each \( K \geq 1 \) and \( f \in A_T(K, 0) \) the following inequalities hold:
\[ |B| \leq S(K), \quad |C| \leq 2S(K) + S(K)^2, \quad |A| \geq 1 - S(K)^2 - S(K). \]

Moreover, the third estimate is essential for \( K \geq 1 \) satisfying \( S(K) < (\sqrt{3} - 1)/2 \).

**Proof.** Let \( f \in A_T(K, 0) \) and let \( g = e^{i\varphi(f)} h_\alpha(f) \circ f, b = -a(f) e^{i\varphi(f)} \). Then, \( g \in A_T(f), E_g(0) = b \), and we see that
\[ e^{i\varphi(f)} f(\zeta) = [g(\zeta) - b]/[1 - \overline{b}g(\zeta)]. \]
As in the proof of Lemma 8, we have
\[ |B| = \frac{1}{2\pi} \left| \int_T \zeta e^{i\varphi(f)} f(\zeta) \, |d\zeta| \right| = \frac{1}{2\pi} \left| \int_T \zeta \left( \frac{g(\zeta) - b}{1 - \overline{b}g(\zeta)} \right) \, |d\zeta| \right| \]
\[ = \frac{1}{2\pi} \left| \int_T \zeta \left( \frac{g(\zeta) - b}{1 - \overline{b}g(\zeta)} - \frac{\zeta - b}{1 - \overline{b}g(\zeta)} \right) \, |d\zeta| \right| \leq \frac{1}{2\pi} \int_T \left| g(\zeta) - \zeta ||1 - \overline{b}g(\zeta)| \left( 1 - |b|^2 \right) |d\zeta| \right| \]
\[ \leq \frac{4}{\sqrt{3}} M(K) C(K) = S(K). \]

Similarly, by Lemma 8, we obtain
\[ |C| = \frac{1}{2\pi} \left| \int_T e^{i2\varphi(f)} f(\zeta)^2 \, |d\zeta| \right| = \frac{1}{2\pi} \left| \int_T \left\{ \left( \frac{g(\zeta) - b}{1 - \overline{b}g(\zeta)} \right)^2 - \left( \frac{\zeta - b}{1 - \overline{b}g(\zeta)} \right)^2 \right\} \, |d\zeta| + 2\pi b^2 \right| \]
\[ \leq |b|^2 + \frac{2}{2\pi} \int_T \left| \frac{g(\zeta) - b}{1 - \overline{b}g(\zeta)} - \frac{\zeta - b}{1 - \overline{b}g(\zeta)} \right| \, |d\zeta| \leq S(K)^2 + 2S(K). \]
Since
\[ \frac{1}{2\pi} \int_T \zeta \left( \frac{\zeta - b}{1 - b\zeta} \right) \, |d\zeta| = \frac{1}{2\pi i} \int_T \frac{\zeta - b}{(1 - b\zeta)^2} \, d\zeta = 1 - |b|^2, \]
we have
\[ |A| = \frac{1}{2\pi} \int_T \left| \zeta e^{i\varphi(f)} f(\zeta) \right| \, |d\zeta| = \frac{1}{2\pi} \int_T \left| \zeta \left( \frac{g(\zeta) - b}{1 - bg(\zeta)} \right) \right| \, |d\zeta| \]
\[ = \frac{1}{2\pi} \int_T \left| \frac{g(\zeta) - b}{1 - bg(\zeta)} - \frac{\zeta - b}{1 - b\zeta} \right| \, |d\zeta| + 2\pi(1 - |b|^2) \geq 1 - |b|^2 - \frac{1}{2\pi} \int_T \left| \frac{g(\zeta) - b}{1 - bg(\zeta)} - \frac{\zeta - b}{1 - b\zeta} \right| \, |d\zeta| \geq 1 - S(K)^2 - S(K). q.e.d. \]

Remark 1. For \( K \geq 1 \) satisfying \( S(K) < (\sqrt{5} - 1)/2 \), we have \( |A| > 0 \).

3. An estimation of the dilatation. For \( K \geq 1 \) we define
\[ k^*(K) = \sup_{f \in A_T(K, 0)} I(f), \]
where
\[ I(f) = \left\{ \frac{2|B(f)| + |C(f)|^2(|A(f)| - |B(f)|)}{|A(f)| + |B(f)|} \right\}^{1/2}. \]
Since \( f \mapsto I(f) \) is continuous on \( A_T \) and \( A_T(K, 0) \) is a compact in \( A_T(K) \) by Theorem 1, we infer that there exists some \( f_K \in A_T(K, 0) \) such that \( k^*(K) = I(f_K) \). Moreover, \( |A(f)| > |B(f)| \) holds for every \( f \in A_T \); because \( f \) is sense-preserving (see [DE, Lemma 3]). We thus see that \( k^*(K) < 1 \).

Theorem 3. For each \( K \geq 1 \) and \( f \in A_T(K) \) the Douady–Earle extension \( E_f \) is quasiconformal and its complex dilatation \( \mu_{E_f} \) satisfies \( \|\mu_{E_f}\|_{\infty} \leq k^*(K) \). Moreover, if \( K \geq 1 \) is as close to 1, so that \( S(K) < (\sqrt{5} - 1)/2 \) holds, then the following estimate
\[ (2.16) \quad k^*(K) \leq \left\{ \frac{2S(K)}{1 - S(K)^2} + (2S(K) + S(K)^2)^2 \right\}^{1/2} \]
holds, where \( S(K) \) is the number defined by means of (2.2), (2.8) and (2.14). In particular, \( \|\mu_{E_f}\|_{\infty} \to 0 \) as \( K \to 1 \).

Proof. Take any \( z_0 \in \Delta \) and let \( w_0 = E_f(z_0) \). Put \( \tilde{f} = h_{w_0} \circ f \circ h_{-z_0} \), where \( h_\eta(\zeta) = \frac{\zeta - \eta}{1 - \eta\zeta} \). By (0.2) we have \( \tilde{E}_f = h_{w_0} \circ E_f \circ h_{-z_0} \) and therefore \( E_f(0) = 0 \), by which \( \tilde{f} \in A_T(K, 0) \). Moreover, we easily have
\[ (2.17) \quad |\mu_{E_f}(z_0)| = |\mu_{E_f}(0)|. \]
Let \( k_0 = \sup |\mu_{E_f}(0)| \), where the supremum is taken over all \( g \in A_T(K, 0) \). By (2.17) it suffices to show that \( k_0 \leq k^*(K) \).

Take any \( g \in A_T(K, 0) \). Then, as in [DE], we have
\[ (2.18) \quad |\mu_{E_g}(0)| = |AC + B|/|A + CB|, \]
where \( A = A(g), B = B(g) \) and \( C = C(g) \). By (2.18), we obtain
\[ 1 - |\mu_{E_g}(0)|^2 = \frac{(1 - |C|^2)(|A|^2 - |B|^2)}{|A + CB|^2} \geq (1 - |C|^2) \frac{|A| - |B|}{|A| + |B|}. \]
\[ I(g)^4 \leq \left\{ \frac{2|B(g)||A(g)|}{1 + |B(g)||A(g)|} + |C(f)|^2 \right\}^{1/2} \leq \left\{ \frac{2S(K)/(1 - S(K)^2 - S(K))}{1 + S(K)/(1 - S(K)^2 - S(K))} + (2S(K) + S(K)^2)^2 \right\}^{1/2} \]

Thus, \( |\mu_{E_f}(0)| \leq I(g) \leq K^*(K) \) and hence \( k_0 \leq k^*(K) \).

Next we show the latter part of the theorem. Let \( K \geq 1 \) satisfy \( S(K) < (\sqrt{5} - 1)/2 \) which is equivalent to \( 1 - S(K)^2 - S(K) > 0 \). If \( g \in A_T(K, 0) \), then, by (2.15) in Lemma 9, we see that

\[
(2S(K) + S(K)^2)^2 \leq \left\{ \frac{2S(K)}{1 - S(K)^2} + (2S(K) + S(K)^2)^2 \right\}^{1/2}.
\]

q.e.d.

**Corollary 3.** Under the hypotheses of Theorem 3, suppose that \( K \geq 1 \), is so close to 1 that the following inequality

\[
(2S(K) + S(K)^2)^2 \leq \frac{2S(K)}{1 - S(K)^2} < \frac{1}{2}
\]

holds, i.e. if \( 0 \leq S(K) < \sqrt{5} - 2 \). Then the maximal dilatation \( K(E_f) \) of \( E_f \) satisfies

\[
K(E_f) \leq 1 + S(K)^{1/2}g(S(K)) \quad \text{where} \quad g(S) = \left( \frac{4}{1 - S^2} \right)^{1/2}.
\]

**References**


