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EQUIVALENT NORMS IN SOME SPACES OF ANALYTIC FUNCTIONS AND THE UNCERTAINTY PRINCIPLE

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Abstract. The main object of this work is to describe such weight functions $w(t)$ that for all elements $f \in L_{p,\Omega}$ the estimate $\|wf\|_p \geq \mathrm{K}(\Omega)\|f\|_p$ is valid with a constant $\mathrm{K}(\Omega)$, which does not depend on f and it grows to infinity when the domain Ω shrinks, i.e. deforms into a lower dimensional convex set Ω_∞. In one-dimensional case means that $\mathrm{K}(\sigma) := \mathrm{K}(\Omega_\sigma) \to \infty$ as $\sigma \to 0$. It should be noted that in the framework of the signal transmission problem such estimates describe a signal's behavior under the influence of detection and amplification. This work contains some results of the above-mentioned type which I presented at the Banach Centre in the Summer of 1994. Some of these results had been published earlier, some are new ones.

Introduction. Uncertainty principle in Fourier analysis asserts that the more a function f is concentrated the more its Fourier transform F will be spread out. The corresponding nontrivial relations between f and F admit adequate physical interpretations, for instance in the framework of the signal transmission problem, in which the Fourier transform $F(\xi)$ of a signal $f(t)$ is interpreted as a bandwidth. From the physical point of view it is very natural to consider signals of $f(t)$ with compact supported bandwidths $F(\xi)$. Then the function $f(t)$ itself can be extended into the complex space $\mathbb{C}^1$ as an entire function of exponential type. And this is exactly the class of functions we deal with in the course of the paper. More exactly, let f be a function on $\mathbb{R}^n$ and F its Fourier transform defined by

$$F(\xi) = (2\pi)^{-n/2} \int f(t) e^{-i<t,\xi>} dt$$

where $t = (t_1, t_2, \ldots, t_n)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ are points of $\mathbb{R}^n$, $< t, \xi >= t_1\xi_1 + \ldots + t_n\xi_n$. For $1 \leq p \leq \infty$ we denote by $\|f\|_p$ the L_p-norm of a function f. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain and let $1 \leq p \leq \infty$. We denote by $L_{p,\Omega}$ the space of all functions f such that the norm $\|f\|_p$ is finite, and the Fourier transforms F of f are supported in Ω. Such functions f vanish at infinity in $\mathbb{R}^n$ and can be extended into the

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complex space $\mathbb{C}^n$ as entire functions of exponential type. In one-dimensional case we assume that $\Omega_\sigma = \{x : -\sigma < x < \sigma\}$ and we denote $L_{p,\Omega} = L_{p,\sigma}$.

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1. One-dimensional case. Let I_N be an *arbitrary* interval of length N. For an arbitrary measurable set M we denote by $| M |$ its Lebesgue measure.

DEFINITION 1. We define the *asymptotical density* $\beta(M)$ of an arbitrary measurable set M as

$$\beta(M) = \overline{\lim_{N\to\infty}} \inf | M \cap I_N | /N$$

DEFINITION 2. We define

$$\widetilde{\gamma}(M) = \inf\{N : \inf | M \cap I_N | = N\beta(M)/2\}.$$

It is obvious that the necessary condition for the estimate under consideration

$$\|wf\|_p \geq \mathrm{K}(\sigma)\|f\|_p, \quad f \in L_{p,\sigma},$$

to be valid with $\mathrm{K}(\sigma) \to \infty$ as $\sigma \to 0$, is $\overline{\lim}_{|t|\to\infty} w(t) = \infty$. Therefore, from now on this condition is assumed to hold.

For an arbitrary continuous function w and for $\tau > 0$ put $M_\tau^w = \{t :| w(t) |> \tau\}$ and denote $\widetilde{\gamma}_w(\tau) = \widetilde{\gamma}(M_\tau^w)$. It is clear that $\overline{\lim}\, \widetilde{\gamma}_w(\tau) = \infty$ as $\tau \to \infty$ because $\overline{\lim}\, w(t) = \infty$.

DEFINITION 3. Let $\gamma_w(\tau)$ be the least left semicontinuous nondecreasing majorant for $\widetilde{\gamma}_w(\tau)$. We define the nondecreasing function

$$\mathit{\Gamma}_w(\lambda) = \inf\{\tau : \gamma_w(\tau) \geq \lambda\}.$$

It is obvious that for and arbitrary $\lambda > 0$ we have $(\gamma_w)\circ\mathit{\Gamma}_w)(\lambda) \leq \lambda$ and $(\gamma_w\circ\mathit{\Gamma}_w)(\lambda) = \lambda$ if the function $\mathit{\Gamma}_w$ is continuous and increases at the point λ. One can regard the function $\mathit{\Gamma}_w$ as the right inverse function to γ_w.

E x a m p l e. Let $w : [0, \infty) \to [0, \infty)$ be an increasing function for which $w(\infty) = \infty$. Then $\beta(M_\tau^w) = 1$ for every $\tau > 0$ and $\gamma_w(\tau) = 2w^{-1}(\tau)$. (Here and later G^{-1} denotes the inverse function to G). Thus, in this case $\mathit{\Gamma}_w(\lambda) = w(\lambda/2)$.

Now we can formulate one of the main results of this work.

THEOREM 1. *Let w be an arbitrary continuous function such that*

$$\beta(M_\tau^w) \geq \beta_0 > 0$$

for all sufficiently large τ. Then there is a constant $c > 0$ which does not depend on f or on σ such that for all p, $1 \leq p \leq \infty$, the estimate

$$\|wf\|_p \geq c\Gamma_w(\sigma^{-1})\|f\|_p, \qquad f \in L_{p,\sigma} \tag{1}$$

is valid.

Corollary 1. *If $\Psi(t)$ is an increasing function and $\gamma_w(t) \le \Psi(t)$ for all sufficiently large $t > t_0$ then the estimate*

$$\|wf\|_p \ge c\Psi^{-1}(\sigma^{-1})\|f\|_p, \qquad f \in L_{p,\sigma} \tag{2}$$

holds. Here a constant c does not depend either on σ or f.

Our next result is related to *sharp* estimates of the described type. Let us start with a definition.

Definition 4. We say that the estimate

$$\|wf\| \ge c\mathrm{K}(\sigma)\|f\|_p, \qquad f \in L_{p.\sigma}$$

is *sharp* (as $\sigma \to 0$) if any other estimate of the same type

$$\|wf\| \ge c\mathrm{K}_1(\sigma)\|f\|_p, \qquad f \in L_{p,\sigma}$$

implies the inequality $\mathrm{K}_1(\sigma) \le c\mathrm{K}(\sigma)$ for all $\sigma > 0$.

One of the typical estimates of this kind is the well-known Hardy's inequality

$$\|tf\|_2 \ge (2\sigma)^{-1}\|f\|_2, \qquad f \in L_{2,\sigma}.$$

Theorem 2. *Assume that for all sufficiently large t and λ, $\quad t \ge A, \quad \lambda \ge B$, there is such a number N that*

$$w(t\lambda) \le \Gamma_w(\lambda)t^N.$$

Then the estimate (1) is sharp.

Corollary 2. *If for a function ψ from Corollary 1 the double inequality $c\Psi(t) \le \gamma_w(t) \le \Psi(t)$ holds for all sufficiently large $t > t_0$ with a constant c, which does not depend on t, then the estimate (2) is sharp.*

Example. Let P be an arbitrary nonzero complex valued polynomial of the degree $N, \quad \alpha \ge 0, \quad \beta \ge 1, \quad 1 \le p \le \infty$. Then, according to Theorem 1,

$$\|P^\alpha(t)\sin(t^\beta)f(t)\|_p \ge c\sigma^{-\alpha N}\|f\|_p, \quad f \in L_{p,\sigma}$$

and this estimate is sharp by Theorem 2.

The following proposition allows us to obtain new weight functions w of the considered type if one such function is already available.

Theorem 3. *Assume that for all t larger than some constant $T \ge 0$, and for an arbitrary constant $B > 0$ the function ϕ satisfies the conditions*

$$\phi\prime(t) \ge r > 0, \quad 0 < r_1(B) \le \phi\prime(t+B)/\phi\prime(B) \le r_2(B) < \infty$$

If the function w satisfies the condition of Theorem 1 then the estimate

$$\|(w \circ \phi)f\|_p > c(\Gamma_w \circ \phi)(\sigma^{-1})\|f\|_p, \quad f \in L_{p,\sigma}$$

is valid.

Let us note that the condition $\phi\prime(t) \ge r > 0$ is essential. For instance if

$$w = \mid t \mid^\alpha \sin \mid t \mid, \quad \alpha < 1; \quad \phi = \mid t \mid^{1/2}$$

then both w and ϕ are weighted functions of the considered type and generate *sharp* estimates

$$\|wf\|_p \geq c\sigma^{-\alpha}\|f\|_p, \quad \|wf\|_p \geq c\sigma^{-1/2}\|f\|_p,$$

for all $f \in L_{p,\sigma}$. But for the composite function $W = w \circ \phi$ the estimate $\|Wf\|_p \geq c\|f\|_p, \quad f \in L_{p,\sigma}$ can be valid only when $c = 0$.

2. Multidimensional case. Consider a bounded convex domain $\Omega \subset \mathbb{R}^n, \quad 0 \in \Omega$ with the support function

$$\mathcal{H}_\Omega(\tau) = \sup_{t\in\Omega} < t, \tau >$$

For an arbitrary unit vector $\tau \in \mathbb{R}^n, \quad |\ \tau\ |= 1$, we denote by $\delta_\tau(\Omega)$ the *width* of the domain Ω in the direction τ, i.e. $\delta_\tau(\Omega) = \mathcal{H}_\Omega(\tau) + \mathcal{H}_\Omega(-\tau)$. In addition to this notation, for an arbitrary measurable set $U \subset \mathbb{R}^n$ and unit vector $\tau \in \mathbb{R}^n$ we denote by $d_\tau(U)$ the *diameter* of the set U in the direction τ. In other words

$$d_\tau(U) = \sup_{\xi\in U} |\ \{t \in \mathbb{R}^1 : \xi + t\tau \in U\}\ | \ .$$

Let $\mathfrak{N}$ be the Stiefel manifold of all orthonormal bases $\mathrm{w} = \{w_1, w_2, \ldots, w_n\}$ in the space $\mathbb{R}^n$. Put

$$\delta_{\mathrm{w}}(\Omega) = \{\delta_{\mathrm{w}_1}(\Omega), \delta_{\mathrm{w}_2}(\Omega), \ldots, \delta_{\mathrm{w}_n}(\Omega)\}$$

and for an arbitrary $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^n$ put

$$\delta_{\mathrm{w}}^{-\sigma}(\Omega) = \delta_{\mathrm{w}_1}^{-\sigma_1}(\Omega)\delta_{\mathrm{w}_2}^{-\sigma_2}(\Omega)\ldots\delta_{\mathrm{w}_n}^{-\sigma_n}(\Omega)$$

We denote by $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$ an arbitrary multi-index of nonnegative integers α_j with the lenght $|\ \alpha\ |= \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

For arbitrary $\xi \in \mathbb{R}^n$ we put

$$\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\ldots\xi_n^{\alpha_n}.$$

Further, for any nonnegative n-tuple $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ we set

$$(\delta/\alpha)^\alpha = (\delta_1/\alpha)^{\alpha_1}(\delta_2/\alpha_2)^{\alpha_2}\ldots(\delta_n/\alpha_n)_n^\alpha$$

where the factor $(\delta_j/\alpha_j)^{\alpha_j}$ is omitted if $\alpha_j = 0$.

An arbitrary polynomial $P(\xi) = P(\xi_1, \xi_2, \ldots, \xi_n)$ of degree $\mathcal{M}$ may be written in the form $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha\xi^\alpha$, where $a_\alpha \neq 0$ for at least one multi-index α with $|\ \alpha\ |:= \alpha_1 + \alpha_2 + \ldots + \alpha_n = m$.

For every unit vector $\tau \in \mathbb{R}^n$ let $\partial_\tau P(\xi)$ denote the derivative of $P(\xi)$ in the direction of τ. If $\mathrm{w} \in \mathfrak{N}$ is one of the orthonormal bases in $\mathbb{R}^n$ we put

$$\partial_{\mathrm{w}}^\alpha P = \partial_{w_1}^{\alpha_1}\partial_{w_2}^{\alpha_2}\ldots\partial_{w_n}^{\alpha_n}P.$$

The following definition plays an important role what follows.

Definition 5. Given a polynomial $P(\xi) = \sum a_\alpha\xi^\alpha$, we call a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ a *leading multi-index* of $P(\xi)$ with respect to a basis $\mathrm{w} \in \mathfrak{N}$ if $\partial_{\mathrm{w}}^\alpha P(\xi) \equiv$ const $\neq 0$ and $\partial_{w_1}^{\alpha_1}\partial_{w_2}^{\alpha_2}\ldots\partial_{w_j}^{\alpha_j+1}P(\xi) \equiv 0$ for all $j = 1, 2, \ldots, n$ such that $\alpha_j \neq 0$.

The set of all leading multi-indeces of $P(\xi)$ with respect to a basis w will be denoted by $\mathfrak{A}_{\mathrm{w}}(P)$. Let us introduce the constant

$$\mathrm{K}_P(\Omega) = \sup_{\substack{\mathrm{w}\in\mathfrak{N} \\ \alpha\in\mathfrak{A}_{\mathrm{w}}(P)}} \delta_{\mathrm{w}}^{-\alpha}(\Omega) \mid \partial_{\mathrm{w}}^{\alpha} P \mid$$

The following theorem contains the main multi-dimensional result of the paper.

THEOREM 3. *Let* $\Phi : [0,\infty) \to [0,\infty)$ *be an arbitrary nondecreasing function and let* $P(\xi)$ *be an arbitrary complex valued polynomial of degree* m. *Then for every* $p \geq 1$ *there exists a constant* $c = c(p,n,m)$ *such that for all function* $u \in \mathrm{L}_{p,\Omega}$ *the inequality*

$$\|\Phi\big(\mid P \mid \big) f\|_p \geq c\Phi\big(\mathrm{K}_P(\Omega)\big)\|f\|_p. \tag{3}$$

holds for all functions $f \in L_{p,\Omega}$. *If* $\Phi(\infty) = \infty$ *and* $\partial_\tau P \not\equiv 0$ *for every vector* $\tau \neq 0$, *then* $\Phi\big(\mathrm{K}_P(\Omega)\big) \to \infty$ *as the domain* Ω *shrinks.*

Remarks. The condition $\partial_\tau P \not\equiv 0$ for all $\tau \neq 0$ means that the polynomial P really depends on all variables.

The domain Ω shrinks if there exists a system of convex domains Ω_s and unit vectors $\tau(s)$, $s \geq 0$ such that $\Omega_0 = \Omega$, $\Omega_s \supset \Omega_r$ for $s < r$ and $\delta_{\tau(s)}(\Omega_s) \to 0$ as $s \to \infty$. Let us consider some particular cases of this result.

Take $p = 2$, $\Phi(z) = z$. Than the inequality of the Theorem and Parsevals' equality give us *support dependent form* of the famous Hörmander's inequality for an arbitrary PDO,

$$\|P(\mathcal{D})F\|_{\mathrm{L}_2} \geq c\mathrm{K}_P(\Omega)\|F\|_{\mathrm{L}_2}, \quad F \in \mathrm{L}_2(\Omega).$$

Take $p = 2$, $\Phi(z) = \sqrt{z}$ and $P(\xi) \geq 0$. Then (3) coincides with *a support dependent form* of Gårding's inequality

$$\mathrm{Re}\,\big(P(\mathcal{D})F, F\big) \geq c\mathrm{K}_P(\Omega)\|F\|^2_{\mathrm{L}_2}, \quad F \in \mathrm{L}_2(\Omega).$$

Take $p = 1$. Then (3) gives us a good estimate of another kind, namely,

$$\|\Phi\big(\mid P \mid \big)F\|_1 \geq c\Phi\big(\mathrm{K}_P(\Omega)\big)\|F\|_1 \geq c\Phi\big(\mathrm{K}_P(\Omega)\big) \sup_\Omega \mid F \mid$$

(We remind that F is the Fourier transform of f).

It turns out that function Φ does not have to be nondecreasing for some estimate of the form (3) to be valid. For instance, if $\beta\big(M_\tau^\Phi\big) \geq \beta_0 > 0$, then for some constant $c > 0$

$$\|\Phi\big(\mid P \mid \big) f\|_p \geq c\|f\|_p, \quad f \in \mathrm{L}_{p,\sigma} \tag{4}$$

(E. Tel, Thesis, Technion, 1994).

It will be interesting to generalize the result of Theorem 2 and to find dependence of the constant c in (4) on Φ and P. Nothing is known about the estimate $\|wf\| \geq \mathrm{K}(\Omega)\|f\|_p$ for the general weighted function w in the multidimensional case.

In conclusion let us point out that the first part of this paper has some intersections with my paper [1] "On sharp support-dependent wieghted norm estimates for Fourier transforms", *International Mathematical Research Notices* (*IMNR*) 11 (1993), 289-294. The proof of Theorem 3 will be published in [2] "Support dependent weighted norm estimates for Fourier transforms", *J. of Math. Anal. and Appl.*, to appear.