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PAIRS OF CLIFFORD ALGEBRAS OF THE HURWITZ TYPE

WIESŁAW KRÓLIKOWSKI

Institute of Mathematics, Polish Academy of Sciences, Łódź Branch
Narutowicza 56, 90-136 Łódź, Poland

Abstract. For a given Hurwitz pair $[S(Q_S), V(Q_V), \circ]$ the existence of a bilinear mapping $\star : C(Q_S) \times C(Q_V) \to C(Q_V)$ (where $C(Q_S)$ and $C(Q_V)$ denote the Clifford algebras of the quadratic forms Q_S and Q_V, respectively) generated by the Hurwitz multiplication "$\circ$" is proved and the counterpart of the Hurwitz condition on the Clifford algebra level is found. Moreover, a necessary and sufficient condition for "$\star$" to be generated by the Hurwitz multiplication is shown.

1. Introduction. The general Hurwitz problem was studied e.g. by Ławrynowicz and Rembieliński [2-4]. They introduced the notions of "Hurwitz pairs" and "pseudo-Hurwitz pairs" and gave their systematic classification according to the relationship with real Clifford algebras. In the present work we show the existence of a bilinear mapping $\star$: $C(Q_S) \times C(Q_V) \to C(Q_V)$, where $(S, V, \circ)$ is a given Hurwitz pair which makes the following diagram:

$$
\text{(1)} \qquad \begin{array}{ccc}
S \times VV & \xrightarrow{\ \circ\ \text{(Hurwitz multiplication)}\ } & V \\
{\scriptstyle \mathrm{i}_S \times \mathrm{i}_V}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{i}_V} \\
C(Q_S) \times C(Q_V) & \xrightarrow{\ \star\ } & C(Q_V)
\end{array}
$$

commutative.

Moreover, we prove that if such a mapping exists and satisfies the following "algebraic Hurwitz condition": $N(x_S \star y_V) = N(x_S)N(y_V)$ for any $x_S \in \Gamma_S$ and $y_V \in \Gamma_V$, where Γ denotes the Clifford group of the Clifford algebra $C(Q)$ and N is a spinor norm, then $\star$ is generated by the Hurwitz multiplication, i.e. $\star_{|S\times V} = \circ$. An example of a mapping $\star$ which does not satisfy the N-norm condition is given. Since in the meantime the detailed proofs have appeared in [1], they are only sketched here.

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2. Product of Clifford algebras generated by the Hurwitz multiplication. Let $(S,\, V,\, \circ)$ be a Hurwitz pair. Suppose that the vector spaces S and V are equipped with non-degenerate quadratic forms Q_S and Q_V, respectively. We will only consider the elliptic and hyperbolic cases (see, e.g. [2-4]). In S and V we choose some bases (ϵ_α) and (e_j) with $\alpha = 1, \ldots, p = \dim S$; $j = 1, \ldots, n = \dim V$. Assume that $p \le n$.

Let $C(Q_S)$ (resp. $C(Q_V)$) denote the Clifford algebra of (S, Q_S) (resp. (V, Q_V)). There are canonical injections $i_S : S \to C(Q_S)$ and $i_V : V \to C(Q_V)$. Then we get the diagram (2). It would be interesting to complete the diagram (2) by the suitable mapping $C(Q_S) \times C(Q_V) \to C(Q_V)$. Define the following mapping $\star : C(Q_S) \times C(Q_V) \to C(Q_V)$ by:

$$\begin{cases} 1_S \star y_V := y_V, \\ (\epsilon_{i_1} \ldots \epsilon_{i_r}) \star (e_{j_1} \ldots e_{j_k}) := \begin{cases} e_{j_k} \ldots e_{j_{r+1}} (\epsilon_{i_r} \circ e_{j_r}) \ldots (\epsilon_{i_1} \circ e_{j_1}), & r < k, \\ (\epsilon_{i_r} \circ e_{j_r}) \ldots (\epsilon_{i_1} \circ e_{j_1}), & r = k, \\ \epsilon_{i_r} \circ [\epsilon_{i_{r-1}} \circ [\ldots \circ [\epsilon_{i_{k+1}} \\ \circ [(\epsilon_{i_k} \circ e_{j_k}) \ldots (\epsilon_{i_1} \circ e_{j_1})] \ldots], & r > k, \end{cases} \\ (\epsilon_{i_1} \ldots \epsilon_{i_r}) \star 1_V := \|\epsilon_{i_1}\| \ldots \|\epsilon_{i_r}\| 1_V \end{cases} \tag{2}$$

for $1 \le r \le p$, $1 \le i_1 < \ldots < i_r \le p$; $1 \le k \le n$, $1 \le j_1 < \ldots < j_k \le n$. Then, the required mapping $\star : C(Q_S) \times C(Q_V) \to C(Q_V)$ is defined by the bilinear extension of (2).

Remark. If (S, Q_S) is a Euclidean vector space then all $\|\epsilon_i\|^2 > 0$. In this case the Clifford algebras $C(Q_S)$ and $C(Q_V)$ are considered to be real. But, if (S, Q_S) is a pseudo-Euclidean vector space then there are some $\epsilon_{i_1}, \ldots, \epsilon_{i_r}$, $1 \le r \le p$, such that $\|\epsilon_{i_s}\|^2 < 0$, $1 \le s \le r$. This time the Clifford algebras have to be treated as complex ones.

PROPOSITION. *$\star$ is a well defined bilinear mapping. Moreover, $\star_{|S\times V} = \circ$, the Hurwitz multiplication, i.e. the diagram* (1) *is commutative.*

LEMMA. *Let $x_S \in \Gamma_S$ and $y_V \in \Gamma_V$, where Γ_S (resp. Γ_V) denotes the Clifford group in $C(Q_S)$ (resp. $C(Q_V)$) and let N_S, N_V be the spinor norms in $C(Q_S)$ and $C(Q_V)$, respectively. Then*

$$N_V(x_S \star y_V) = N_S(x_S) N_V(y_V). \tag{3}$$

THEOREM. *Let S and V be real vector spaces equipped with non-degenerate quadratic forms Q_S and Q_V, respectively. Denote by $C^{\mathbb{C}}(Q_S)$ (resp. $C^{\mathbb{C}}(Q_V)$) the complex Clifford algebras of (S, Q_S) (resp. (V, Q_V)). Suppose that there is a bilinear mapping $\star : C^{\mathbb{C}}(Q_S) \times C^{\mathbb{C}}(Q_V) \to C^{\mathbb{C}}(Q_V)$ satisfying the condition* (3). *Then $\star$ is generated by the Hurwitz multiplication, i.e. $\star_{|S\times V} = \circ$, where $\circ : S \times V \to V$ is a bilinear mapping such that $\|s \circ v\|_V = \|s\|_S \|v\|_V$ for all $s \in S$ and $v \in V$.*

Proof. Let $s \in S \subset \Gamma_S$ and $v \in V \subset \Gamma_V$. By definition of N we have

$$N_V(s \star v) = N_S(s) N_V(v) = \|s\|_S^2 \|v\|_V^2 \in \mathbb{R}. \tag{4}$$

Let $(e_1, \ldots, e_n)$ be an orthogonal base in V. Suppose

$$s \star v = a_0 + \sum_{i=1}^{n} a^i e_i + \sum_{l=2}^{n} \sum_{i_1 < \ldots < i_l} a_l{}^{i_1 \ldots i_l} e_{i_1} \ldots e_{i_l}.$$

Then

$$N(s\star v) = a_0^2 + \sum_{i=1}^{n} (a^i)^2 Q_V(e_i) + \sum_{l=2}^{n} \sum_{i_1<..<i_l} (a_l^{i_1..i_l})^2 Q_V(e_{i_1})\dots Q_V(e_{i_l}) + R(e_1,\dots,e_n),$$

where

$$R(e_1,\dots,e_n) = \sum b^i e_i + \sum_{i<j} b^{ij} e_i e_j + \dots + \sum_{i_1<\dots<i_m} b^{i_1\dots i_m} e_{i_1}\dots e_{i_m} + b e_1 \dots e_n.$$

Since $N(s\star v)$ is a scalar then $R(e_1,\dots,e_n)$ must vanish. The multiplication $\star$ is bilinear so the coefficients a_0, a^i and $a_l^{i_1\dots i_l}$ are bilinear functions in s and v. Thus $N(s\star v)$ should be separated into two parts, first depending only on s and second only on v. Then we can write

$$\begin{aligned} a_0^2 + \sum_{i=1}^{n} (a^i)^2 Q_V(e_i) &+ \sum_{l=2}^{n} \sum_{i_1<\dots<i_l} (a_l^{i_1\dots i_l})^2 Q_V(e_{i_1})\dots Q_V(e_{i_l}) \\ &= \|s\|_S^2 \|v\|_V^2 \\ &= \|s\|_S^2 [c_0^2 + \sum_{i=1}^{n} (c^i)^2 Q_V(e_i) + \sum_{l=2}^{n} \sum_{i_1<\dots<i_l} (c_l^{i_1\dots i_l 2} Q_V(e_{i_1})\dots Q_V(e_{i_l})] \end{aligned}$$

Thus, the following equality has to be satisfied:

$$c_0^2 + \sum_{i=1}^{n} (c^i)^2 Q_V(e_i) + \sum_{l=2}^{n} \sum_{i_1<\dots<i_l} (c_l^{i_1\dots i_l})^2 Q_V(e_{i_1})\dots Q_V(e_{i_l}) = \sum_{i=1}^{n} (v^i)^2 Q_V(e_i).$$

The coefficients $c_0, c^i, c_l^{i_1\dots i_l}$ are linear in v so, by continuity, we can write

$$c_0(v) = c_{0j} v^j, \quad c^i(v) = c_j^i v^j, \quad c_l^{i_1\dots i_l}(v) = c_{lj}^{i_1\dots i_l} v^j.$$

Thus, for any $1 \le j,\, k \le n$ we get the identity

$$c_{0j}c_{0k} + \sum_{i=1}^{n} (c_j^i c_k^i - \delta_j^i \delta_k^i) Q_V(e_i) + \sum_{l=2}^{n} \sum_{i_1<\dots<i_l} c_{lj}^{i_1\dots i_l} c_{lk}^{i_1\dots i_l} Q_V(e_{i_1})\dots Q_V(e_{i_l}) \equiv 0.$$

Take an orthogonal transformation $R \in O(Q_V)$. In a new base $e' = Re$ we have

$$c_{0j}c_{0k} + \sum_{i=1}^{n} (\widetilde{c}_j^i \widetilde{c}_k^i - \delta_j^i \delta_k^i) Q_V(Re_i) + \sum_{l=2}^{n} \sum_{i_1<\dots<i_l} \widetilde{c}_{lj}^{i_1\dots i_l} \widetilde{c}_{lk}^{i_1\dots i_l} Q_V(Re_{i_1})\dots Q_V(Re_{i_l}) \equiv 0.$$

But $Q_V(Re_i) = Q_V(e_i)$. Then the new coefficients $\widetilde{c}_j$ and $\widetilde{c}_{lj}^{i_1\dots i_l}$, obtained by the changing of the base, satisfy the same identity as the previous ones. This is possible if and only if

$$\begin{aligned} &c_{0j} \equiv 0 \quad \text{for } j = 1,\dots,n, \\ &c_j^i c_k^i - \delta_j^i \delta_k^i \equiv 0 \quad \text{for } 1 \le i,j,k \le n, \\ &c_{lj}^{i_1\dots i_l} \equiv 0 \quad \text{for } l = 2,\dots,n;\ 1 \le i_1 < \dots < i_l \le n;\ j = 1,\dots,n. \end{aligned}$$

Thus, we get $s\star v = \|s\|_S \sum_{i=1,j=1}^{n} c_j^i v^j e_i \in V$ and $\|s\|_S^2 \|v\|_V^2 = N_V(s\star v) = \|s\star v\|_V^2$, so $\star_{|S\times V}$ satisfies the Hurwitz condition, as required. ∎

EXAMPLE. We now construct a bilinear map $\square : C^{\mathbb{C}}(Q_S) \times C^{\mathbb{C}}(Q_V) \to C^{\mathbb{C}}(Q_V)$ which does not satisfy the condition (4). Choose some bases (ϵ_α) and (e_j) in S and V, respectively. Define

$$(5)\qquad \begin{cases} 1_S \,\square\, 1_V := e_1 \dots e_n, \\ 1_S \,\square\, (e_{i_1} \dots e_{i_k}) := e_{i_1} \widehat{\dots} e_{i_k}, \\ 1_S \,\square\, (e_1 \dots e_n) := 1_V, \\ (\epsilon_{j_1} \dots \epsilon_{j_r}) \,\square\, (e_{i_1} \dots e_{i_k}) := \|\epsilon_{j_1}\| \dots \|\epsilon_{j_r}\| e_{i_1} \widehat{\dots} e_{i_k}, \\ (\epsilon_{j_1} \dots \epsilon_{j_r}) \,\square\, 1_V := \|\epsilon_{j_1}\| \dots \|\epsilon_{j_r}\| e_1 \dots e_n, \\ (\epsilon_{j_1} \dots \epsilon_{j_r} \,\square\, (e_1 \dots e_n) := \|\epsilon_{j_1}\| \dots \|\epsilon_{j_r}\| 1_V, \end{cases}$$

where "$\widehat{\cdot}$" is defined by

$$e_{i_1} \widehat{\dots} e_{i_r} := e_{j_1} \dots e_{j_s} \text{ with } j_1 < \dots < j_s \text{ and } (i_1, \dots, i_r, j_1, \dots, j_s) = (1, \dots, n).$$

The map $\square : C^{\mathbb{C}}(Q_S) \times C^{\mathbb{C}}(Q_V) \to C^{\mathbb{C}}(Q_V)$ is defined by the bilinear extension of (5). It is easy to see that $\square$ does not satisfy the condition (4). Indeed, take $s \in S$ and $v \in V$. We have

$$s \,\square\, v = s^\alpha v^i \epsilon_\alpha \,\square\, e_i = s^\alpha v^i \|\epsilon_\alpha\|_S e_1 \dots \widehat{e_i} \dots e_n \notin V.$$

and

$$N_V(s \,\square\, v) = s^\alpha s^\beta \|\epsilon_\alpha\|_S \|\epsilon_\beta\|_S (v^i)^2 Q_V(e_1) \dots \widehat{Q_V(e_i)} \dots O_V(e_n).$$

Suppose that $N_V(s \,\square\, v) = N_S(s) N_V(v)$. Then we get

$$\sum_{\alpha,\beta} s^\alpha s^\beta \|\epsilon_\alpha\|_S \|\epsilon_\beta\|_S = \sum_\alpha (s^\alpha)^2 \|\epsilon_\alpha\|_S^2,$$

$$\sum_i (v^i)^2 Q_V(e_1) \dots \widehat{Q_V(e_i)} \dots Q_V(e_n) = \sum_i (v^i)^2 Q_V(e_i).$$

The above condition is equivalent to

$$\|\epsilon_\alpha\|_S = 0 \text{ and } \|e_1\|_V^2 \dots \|\widehat{e_i}\|_V^2 \dots \|e_n\|_V^2 = \|e_i\|_V^2,$$

but this is impossible. ■

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