GENERALIZATIONS OF COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 37
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

SPINORS IN BRAIDED GEOMETRY

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Abstract. Let V be a $\mathbb{C}$-space, $\sigma \in \operatorname{End}(V^{\otimes 2})$ be a pre-braid operator and let $F \in \operatorname{lin}(V^{\otimes 2}, \mathbb{C})$. This paper offers a sufficient condition on (σ, F) that there exists a Clifford algebra $\mathcal{C}\ell(V, \sigma, F)$ as the Chevalley F-dependent deformation of an exterior algebra $\mathcal{C}\ell(V, \sigma, 0) \equiv V^{\wedge}(\sigma)$. If $\sigma \neq \sigma^{-1}$ and F is non-degenerate then F is not a σ-morphism in σ-braided monoidal category. A spinor representation as a left $\mathcal{C}\ell(V, \sigma, F)$-module is identified with an exterior algebra over F-isotropic $\mathbb{C}$-subspace of V. We give a sufficient condition on braid σ that the spinor representation is faithful and irreducible.

1. Introduction. Clifford and Weyl algebra for a Hecke braid was considered among other in [Hayashi 1990, Oziewicz 1995, Bautista *et al.* 1996]. The aim of this paper is to extend Clifford algebra and spinors to braided geometry and we do not assume that a braid need to be a Hecke braid.

Let V be a finite dimensional $\mathbb{C}$-space, $\sigma \in \operatorname{End}(V^{\otimes 2})$ be a pre-braid operator and let $F \in \operatorname{lin}(V^{\otimes 2}, \mathbb{C})$. This paper offers a sufficient condition on (σ, F) that there exists a Clifford algebra $\mathcal{C}\ell(V, \sigma, F)$ as the Chevalley F-dependent deformation of a braided exterior algebra $\mathcal{C}\ell(V, \sigma, 0) \equiv V^{\wedge}(\sigma)$,

$$(F \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma) = (\mathrm{id}_V \otimes F)(\sigma \otimes \mathrm{id}_V). \tag{1}$$

1991 *Mathematics Subject Classification*: Primary 15A66, 15A75, 16S80; Secondary 18D10.

Z. Oziewicz is a member of Sistema Nacional de Investigadores, México. Partially supported by KBN grant 2 P302 023 07.

Resubmitted May 17, 1996.

The paper is in final form and no version of it will be published elsewhere.

The condition (1) should be contrasted with a condition that a form F is a σ-morphism in a σ-braided monoidal category. The condition (1) is valid also for $\sigma = 0$ in contrast to the σ-morphism which for $F \neq 0$ do not holds for $\sigma = 0$. For involutive braids $\sigma = \sigma^{-1}$ condition (1) coincides with a σ-morphism for F. The condition (1) is invariant with respect to rescaling, for $\lambda \in \mathbb{C}$, $\sigma \to \lambda \cdot \sigma$, whereas a condition for F to be a σ-morphism is not rescaling invariant.

From the point of view of the deformation theory one can ask if exists an F-dependent deformation of exterior Hopf algebra $V^\wedge(\sigma)$?

Let sh be a shuffle comultiplication on a $\mathbb{C}$-space $V^\otimes$. Then, from one point of view, the answer to the above question is positive in the framework of generalized braided quantum groups [Đurđević 1996],

$$V^\wedge(\sigma) \hookrightarrow \{\mathcal{C}\ell(V, \sigma, F), \mathrm{sh}\}.$$

From another point of view the answer is negative because for $F \neq 0$, $\{\mathcal{C}\ell(V, \sigma, F), \mathrm{sh}\}$ is not a braided Hopf algebra as in [Majid 1991-1993].

An open question is whether exists an F-dependent deformation, in the framework of a braided category, of the braided exterior bialgebra $V^\wedge(\sigma)$ which need not to be bi-unital (*i.e.* not unital or not co-unital)?

In the last section the Cartan approach to algebraic spinors is applied in the braided geometry. This last section is a variation and a generalization of a construction given in [Bautista at al. 1996]. A spinor space S is defined as an exterior Hopf $\mathbb{C}$-algebra over F-isotropic $\mathbb{C}$-subspace of V. A spinor space S is a left Clifford module and in contrast to the standard formulation, we shall not require the existence of a hermitian form (a spinor norm), see e. g. [Crumeyrolle 1990]. We found sufficient conditions on a braid σ that the spinor representation $\{\mathcal{C}\ell(V, \sigma, F) \to \mathrm{End}\, S\}$ is irreducible and faithful.

Throughout this paper algebra, cogebra, Hopf algebra means $\mathbb{C}$-algebra, $\mathbb{C}$-cogebra and Hopf $\mathbb{C}$-algebra and $\otimes \equiv \otimes_{\mathbb{C}}$. All maps are $\mathbb{C}$-linear, $\mathrm{lin} \equiv \mathrm{lin}_{\mathbb{C}}$ and $\mathrm{End} \equiv \mathrm{End}_{\mathbb{C}}$.

2. Deformation of shuffle comutiplication. In what follows V is a $\mathbb{C}$-space, TV is a tensor algebra universal on V with a realization $TV \equiv \{V^\otimes, \otimes\}$, and CV is a tensor cogebra co-universal on V with a shuffle co-universal commultiplication sh, *i.e.* $CV \equiv \{V^\otimes, \mathrm{sh}\}$, see [Sweedler 1969, chapter XII, page 247].

Henceforth we define

$$\begin{gathered} s_{p,q} \in \mathrm{lin}(V^{\otimes(p+q)}, V^{\otimes p} \otimes V^{\otimes q}), \\ s_{p,q}(v_1 \otimes \ldots \otimes v_{p+q}) \equiv (v_1 \otimes \ldots \otimes v_p) \otimes (v_{p+1} \otimes \ldots \otimes v_{p+q}), \\ s_{0,n} \equiv 1 \otimes \mathrm{id}, \qquad s_{n,0} \equiv \mathrm{id} \otimes 1, \\ \mathrm{sh}\colon V^\otimes \to V^\otimes \otimes V^\otimes, \quad \mathrm{sh}|V^{\otimes n} \equiv \sum_{i=0}^{n} s_{i,n-i}. \end{gathered}$$

In this section we consider a pre-braid - dependent coassociative deformation of co-universal shuffle comultiplication sh.

A pre-braid on $\mathbb{C}$-space V is a map $\sigma \in \mathrm{End}(V^{\otimes 2})$ for which the braid equation holds

$$(\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma)(\sigma \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes \sigma)(\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma). \tag{2}$$

A pre-braid σ extends to pre-braiding $\sigma^{\otimes} \in \text{End}(V^{\otimes} \otimes V^{\otimes})$ on $V^{\otimes}$, e. g.

$$(\sigma^{\otimes})_{1,2} \equiv \sigma^{\otimes}|\{V \otimes V^{\otimes 2}\} = (\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V),$$
$$(\sigma^{\otimes})_{2,1} \equiv \sigma^{\otimes}|\{V^{\otimes 2} \otimes V\} = (\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma).$$

Let $V^* \equiv \text{lin}(V, \mathbb{C})$ be a dual $\mathbb{C}$-space. The evaluation $V^* \otimes V \to \mathbb{C}$ extends to a pairing $TV^* \otimes TV \to \mathbb{C}$ according to the following convention

$$\text{for } \alpha, \beta \in TV^*,\ t, u \in TV \text{ and } |\alpha| = |u|,\ |\beta| = |t|,$$
$$(\alpha \otimes \beta)(t \otimes u) \equiv (\beta t)(\alpha u) \quad \in \mathbb{C}.$$

The dual space $V^* \equiv \text{lin}(V, \mathbb{C})$ possess the transposed pre-braid $\sigma^* \in \text{End}(V^{*\otimes 2})$ with respect to the above pairing.

Let $V\sigma V$ denote an associative unital algebra and a coassociative counital cogebra with a $\mathbb{C}$-space $V^{\otimes} \otimes V^{\otimes}$ and with σ-dependent structures,

$$\begin{aligned} \text{a multiplication:} \quad & [(\otimes) \otimes (\otimes)] \circ (\text{id}_{V^{\otimes}} \otimes \sigma^{\otimes} \otimes \text{id}_{V^{\otimes}}), \\ \text{a unit:} \quad & 1 \otimes 1, \quad 1 \in \text{lin}(\mathbb{C}, V^{\otimes}), \\ \text{a comultiplication:} \quad & (\text{id}_{V^{\otimes}} \otimes \sigma^{\otimes} \otimes \text{id}_{V^{\otimes}}) \circ [\text{sh} \otimes \text{sh}], \\ \text{a counit:} \quad & \varepsilon \otimes \varepsilon, \quad \varepsilon \in \text{lin}(V^{\otimes}, \mathbb{C}). \end{aligned}$$

It is an interesting question whether the above structure can be extended to some braided bialgebra. This question is not investigated in this paper.

For a pre-braid $\sigma \in \text{End}(V^{\otimes 2})$ we shall define a σ-dependent coassociative comultiplication $C(\sigma)$ and associative multiplication $Q(\sigma)$ in a $\mathbb{C}$-space $V^{\otimes}$,

$$V^{\otimes} \xrightarrow{C(\sigma)} V^{\otimes} \otimes V^{\otimes},$$
$$V^{\otimes} \xleftarrow{Q(\sigma)} V^{\otimes} \otimes V^{\otimes},$$
$$Q(\sigma) \equiv [C(\sigma^*)]^g, \quad \text{the graded dual.} \tag{3}$$

Definition 1 (Comultiplication). Let $1 \in TV$. An algebra map $C(\sigma) \in \text{alg}(TV, V\sigma V)$ is determined by value on generating $\mathbb{C}$-space V,

$$C(\sigma)1 \equiv 1 \otimes 1, \quad C(\sigma)|V \equiv 1 \otimes \text{id}_V + \text{id}_V \otimes 1.$$

Let S_n be the permutation group on n elements, $\pi \in S_n$ and let $\sigma_\pi \in \text{End}(V^{\otimes n})$ be a map obtained by replacing transpositions in a minimal decomposition of π by a pre-braid $\sigma \in \text{End}(V^{\otimes 2})$.

Let $\text{sh}_{n,k} \subseteq S_{n+k}$ be a set of riffle shuffles with a cut n from S_{n+k} [Sweedler 1969, chapter XII; Sternberg 1993, p. 43], *i.e.* a set of permutations preserving an order of sub-sets $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+k\}$.

Proposition 2 (Deformation of shuffle comultiplication). *An algebra map $C(\sigma)$ is a coassociative σ-deformation of co-universal tensor shuffle comultiplication*,

$$C(\sigma)|V^{\otimes n} = \sum_{i=0}^{n} \{ s_{i,n-i} \circ C_{i,n-i}(\sigma) \},$$
$$C_{0,n}(\sigma) = C_{n,0}(\sigma) = \text{id}_{V^{\otimes n}},$$

$$C_{n,k}(\sigma) = \sum_{\pi \in \mathrm{sh}_{n,k}} \sigma_\pi \quad \in \quad \mathrm{End}(V^{\otimes(n+k)}),$$
$$C_{n,k}(0) = \mathrm{id}_{V^{\otimes(n+k)}}, \qquad C(0) = \mathrm{sh}.$$

P r o o f. The above formulas are proved by induction on n. The coassociativity of the coproduct $C(\sigma)$ is due to braid equation (2). ∎

The operators $C_{n,k}(\sigma)$ for $n = 1$ or $k = 1$ are the same as *braided integers* in [Majid 1993] or in [Majid 1995, Definition 10.4.8],

$$\begin{bmatrix} 1+n \\ n \end{bmatrix} ; \sigma \Bigg] = C_{n,1}(\sigma), \quad [1+k;\sigma] = C_{1,k}(\sigma).$$

In particular

$$\begin{aligned} C_{1,1}(\sigma) &= \mathrm{id}_{V^{\otimes 2}} + \sigma, \\ C_{1,2}(\sigma) &= \mathrm{id}_{V^{\otimes 3}} + \sigma \otimes \mathrm{id}_V + (\mathrm{id}_V \otimes \sigma)(\sigma \otimes \mathrm{id}_V), \\ C_{2,1}(\sigma) &= \mathrm{id}_{V^{\otimes 3}} + \mathrm{id}_V \otimes \sigma + (\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma). \end{aligned}$$

3. Braided exterior Hopf algebra. Hopf algebras in a braided monoidal category, *i.e.* a braided Hopf algebras, has been introduced by Majid in a series of papers in years 1991-1993, we refer to monograph by Majid [1995]. In this section we generalize this to pre-braided case when a braid needs not to be invertible and we consider two important examples of pre-braid - dependent families: pre-braided universal Hopf algebra and pre-braided co-universal Hopf algebra as the deformation of bi-universal (*i.e.* universal and couniversal) Hopf algebra.

A bi-universal σ-braided Hopf algebra exists if and only if $\sigma|V^{\otimes 2} = 0$ [Oziewicz, Paal and Różański 1995, section 8]. A realization of the bi-universal 0-braided Hopf algebra is given by $\{V^{\otimes}, \otimes, \mathrm{sh}\}$ with the antipode

$$S|\mathbb{C} = \mathrm{id}, \qquad S|V = -\mathrm{id}, \qquad S|V^{\otimes n>1} = 0.$$

Proposition 3 (Braided Hopf algebras).

(i) $Q(0) = \otimes$, *i.e.* $Q(\sigma)$ *is an associative* σ*-deformation of a tensor multiplication* (*concatenation*) *and* $Q(\sigma) \in \mathrm{cog}(V\sigma V, CV)$.

(ii) $CV(\sigma) \equiv \{V^{\otimes}, \otimes, C(\sigma)\}$ *is a* σ*-braided universal biassociative and biunital Hopf algebra which is a* σ*-deformation of bi-universal Hopf algebra* $CV(0)$.

(iii) $QV(\sigma) \equiv \{V^{\otimes}, Q(\sigma), \mathrm{sh}\}$ *is a* σ*-braided co-universal biassociative and biunital Hopf algebra which is* σ*-deformation of bi-universal Hopf algebra* $QV(0) = CV(0)$.

(iv) *There exists a unique homomorphism* $W(\sigma) \in \mathrm{hopf}[CV(\sigma), QV(\sigma)]$ *of* σ*-deformed universal Hopf algebra into* σ*-deformed co-universal Hopf algebra such that* $W(\sigma)|(\mathbb{C} \oplus V) = \mathrm{id}$. *An operator* $W(\sigma)$ *commutes with antipod and* $W(\sigma)$ *is* σ*-deformation of identity*, *i.e.* $W(0) = \mathrm{id}_{V^{\otimes}}$.

P r o o f. See [Oziewicz, Paal and Różański 1995, section 8]. In the proof we need to use braid equation (1).

From the graded duality in definition (3) and from Proposition 2 we get

$$Q_{n,k}(\sigma) \equiv Q(\sigma)|(V^{\otimes n} \otimes V^{\otimes k}) = \sum_{\pi \in \mathrm{sh}_{n,k}} [(\sigma^*)_\pi]^* \quad : \quad V^{\otimes n} \otimes V^{\otimes k} \longrightarrow V^{\otimes(n+k)}.$$

Moreover

$$[(\sigma^*)_\pi]^* = \sigma_{\pi^{-1}}$$

and this follows from the equality of the tangles corresponding to the left and the right sides of the above equation.

By definition an operator $W(\sigma)$ must be an algebra and cogebra map,

$$(4)\qquad W(\sigma) \in \mathrm{alg}\,[TV, (V^{\otimes}, Q(\sigma))], \qquad \begin{array}{ccc} V^{\otimes}\otimes V^{\otimes} & \stackrel{\otimes}{\longrightarrow} & V^{\otimes} \\ \downarrow W\otimes W & & \downarrow W \\ V^{\otimes}\otimes V^{\otimes} & \stackrel{Q(\sigma)}{\longrightarrow} & V^{\otimes} \end{array}$$

$$(5)\qquad W(\sigma) \in \mathrm{cog}\,[(V^{\otimes}, C(\sigma)), CV], \qquad \begin{array}{ccc} V^{\otimes} & \stackrel{C(\sigma)}{\longrightarrow} & V^{\otimes}\otimes V^{\otimes} \\ \downarrow W & & \downarrow W\otimes W \\ V^{\otimes} & \stackrel{\mathrm{sh}}{\longrightarrow} & V^{\otimes}\otimes V^{\otimes} \end{array}$$

An image $\mathrm{im}\,W(\sigma)$ is a Hopf sub-algebra of $QV(\sigma)$.

In particular a Hopf algebra map $W(\sigma)$ coincide with a braided symmetrizer/alternator introduced by Woronowicz [1989, pp. 153-155]

$$\mathrm{End}(V^{\otimes n}) \ \ni\ W_n(\sigma) \equiv \sum_{\pi\in S_n} \sigma_\pi.$$

Moreover

$$\text{for} \quad \lambda\in\mathbb{C}, \quad (\lambda\cdot\sigma)_\pi = \lambda^{|\pi|}\cdot\sigma_\pi. \ \blacksquare$$

A subspace $\ker W(\sigma) < CV(\sigma)$ is a two-sided biideal in a universal tensor Hopf algebra. A factor Hopf algebra $V^\wedge(\sigma) \equiv CV(\sigma)/\ker W(\sigma)$ is said to be an exterior Hopf algebra for a pre-braid σ. Let $\pi_\sigma \in \mathrm{hopf}\,[CV(\sigma), V^\wedge(\sigma)]$ be an epimorphism of Hopf algebras and $\ker \pi_\sigma \equiv \ker W(\sigma)$,

$$\wedge \equiv \wedge_\sigma \equiv \otimes \quad \mathrm{mod} \quad \ker W(\sigma) : V^\wedge \otimes V^\wedge \to V^\wedge.$$

One can show that a pre-braiding $\sigma^{\otimes}$ factors to a pre-braiding $\sigma^\wedge$ on a factor algebra $V^\wedge\otimes V^\wedge$ and that an exterior Hopf algebra $V^\wedge(\sigma)$ is $\sigma^\wedge$-braided, i. e. all structure maps are $\sigma^\wedge$-morphisms and in particular a factor multiplication $\wedge_\sigma$ is a $\sigma^\wedge$-morphism,

$$\begin{aligned} \sigma^\wedge(\wedge\otimes\mathrm{id}) &= (\mathrm{id}\otimes\wedge)(\sigma^\wedge\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^\wedge), \\ \sigma^\wedge(\mathrm{id}\otimes\wedge) &= (\wedge\otimes\mathrm{id})(\mathrm{id}\otimes\sigma^\wedge)(\sigma^\wedge\otimes\mathrm{id}). \end{aligned}$$

A Hopf algebra $\mathrm{im}\ W(\sigma)$ is isomorphic as Hopf algebra to the exterior algebra. The following algebra map is invertible,

$$\pi_\sigma|\mathrm{im}\ W(\sigma) \in \mathrm{alg}\,[(\mathrm{im}\ W(\sigma), Q(\sigma)), (V^\wedge(\sigma), \wedge_\sigma)],$$

$$V^\wedge(\sigma) \ni \pi_\sigma\psi \equiv \psi^\wedge \equiv [\psi \bmod \ker W(\sigma)] \quad \longleftrightarrow \quad W(\sigma)\psi \in \mathrm{im}\,W(\sigma) < V^{\otimes}.$$

We have a pairings

$$\text{for } \alpha,\beta\in TV^*,\ t,u\in TV \text{ and } |\alpha| = |u|,\ |\beta| = |t|,$$

$$(\alpha\otimes\beta)(t\otimes u) \equiv (\beta t)(\alpha u) \quad \in\mathbb{C},$$

$$V^{*\wedge}(\sigma)\otimes V^\wedge(\sigma) \quad \to \quad \mathbb{C},$$

$$(6)\qquad \mathbb{C}\ni \quad \alpha^\wedge t^\wedge \equiv \alpha W(\sigma) t \quad \text{or} \quad 0 \quad \text{if } |\alpha| \neq |t|.$$

4. Inner product. An inner product is a graded derivation of degree -1. In this section an inner product is generalized to a braided geometry. A general theory of derivations of arbitrary degree in braided geometry was presented in [Oziewicz, Paal and Różański 1995].

For $f \in V^*$ and $c_f \equiv f \otimes \mathrm{id}$ let $c_f | V^{\otimes n} \in \mathrm{lin}(V^{\otimes n}, V^{\otimes(n-1)})$ be contraction in a tensor algebra $V^{\otimes}$. Then exists $k_f(\sigma) \in \mathrm{End}(V^{\otimes})$ such that

$$c_f \circ W(\sigma) = W(\sigma) \circ k_f(\sigma).$$

Therefore $k_f(\sigma)$ factors to a map $k(\sigma) \in \mathrm{lin}(V^* \otimes V^\wedge, V^\wedge)$.

LEMMA 4 (Braided derivation). *The Leibniz rule holds*,

$$k_f(\sigma) \circ \wedge = \wedge \circ \{k_f(\sigma) \otimes \mathrm{id} + \sigma^{\wedge -1} \circ [k_f(\sigma) \otimes \mathrm{id}] \circ \sigma^\wedge\}.$$

Proof. The Leibniz rule for $k_f(\sigma)$ in a tensor algebra factors to the above Leibniz rule in a factor algebra. ∎

An operator $k(\sigma)$ extends to an algebra map

$$\sqcup \in \mathrm{alg}\,[TV^*, \mathrm{End}(V^\wedge)], \qquad \sqcup | V^* \equiv k(\sigma).$$

LEMMA 5. *If* $f \in \ker W^*(\sigma)$ *then* $\sqcup_f = 0$.

Proof. The statement is the consequence of (3) because $W(\sigma) \in \mathrm{alg}\,[TV, QV(\sigma)]$. ∎

Hence, we can pass from $V^{*\otimes}$ to $V^{*\wedge}$ in the first argument of $\sqcup$ and we obtain a map $\sqcup\colon V^{*\wedge} \otimes V^\wedge \to V^\wedge$.

DEFINITION 6. The map $\sqcup\colon V^{*\wedge} \otimes V^\wedge \to V^\wedge$ is said to be *the inner product*.

The inner product $\sqcup$ can be defined equivalently as the transposed exterior multiplication. For $f \in V^{*\wedge}$, let $\wedge_f \in \mathrm{End}(V^{*\wedge})$ be a linear map given by $\wedge_f \phi \equiv \phi \wedge f$.

PROPOSITION 7. *Let* $f, g \in TV^*$, $\psi \in TV$ *and* $|f| + |g| = |\psi|$. *Then the inner product* $\sqcup_f$ *and an exterior product* $\wedge_f$ *are mutually transposed*,

$$(\pi_\sigma g)(\sqcup_{\pi_\sigma f} \pi_\sigma \psi) = (\wedge_{\pi_\sigma f} \pi_\sigma g)(\pi_\sigma \psi).$$

Proof. We have

$$\begin{aligned}(g^\wedge \wedge f^\wedge)\psi^\wedge &= (\wedge_{f^\wedge} g^\wedge)\psi^\wedge = [W^*(g \otimes f)]\psi \\ &= (g \otimes f) W\psi = g^\wedge(f^\wedge \sqcup \psi^\wedge) = g^\wedge(\sqcup_{f^\wedge} \psi^\wedge). \ \blacksquare\end{aligned}$$

5. Clifford Algebra. Let $F \in \mathrm{lin}(V^{\otimes 2}, \mathbb{C})$ be a scalar product and $\ell_F \in \mathrm{lin}(V, V^*)$ be an associated correlation,

$$\mathrm{ev}(\ell_F \otimes \mathrm{id}_V) \equiv F \quad \text{and} \quad T\ell_F \in \mathrm{alg}\,(TV, TV^*).$$

Clifford and Weyl algebras for Hecke braids were considered among others in [Hayashi 1990, Oziewicz 1995, Bautista *et al.* 1996]. In this section a braid $\sigma \in \mathrm{End}(V^{\otimes 2})$ needs not to be a Hecke braid.

In what follows we shall assume

$$T\ell_F \circ W(\sigma) = W^*(\sigma) \circ T\ell_F, \quad W^*(\sigma) = W(\sigma^*). \tag{7}$$

This is equivalent that

$$(\ell_F \otimes \ell_F) \circ \sigma = \sigma^* \circ (\ell_F \otimes \ell_F).$$

Factorizing $T\ell_F$ through ideals $\ker\{W, W^*\}$ we obtain an algebra homomorphism $\wedge\ell_F \in \operatorname{alg}(V^\wedge, V^{*\wedge})$.

A sufficient condition that (7) holds is

$$(F \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma) = (\mathrm{id}_V \otimes F)(\sigma \otimes \mathrm{id}_V). \tag{8}$$

Let ι_F be a contraction map multiplicative on the first factor with values in braided derivations (Lemma 4),

$$\iota_F : V^\wedge \otimes V^\wedge \to V^\wedge, \qquad \iota_F \equiv \sqcup[(\wedge\ell_F) \otimes \mathrm{id}], \tag{9}$$

$$\text{for } x \in V \text{ and } \quad \sigma(x \otimes \vartheta) = \sum\nolimits_k \vartheta_k \otimes x_k,$$

$$\iota_{Fx} \circ \wedge_\vartheta = \wedge_{\iota_{Fx}\vartheta} + \sum \wedge_{\vartheta_k} \circ \iota_{Fx_k}.$$

We shall introduce contraction operators $\langle\,,\rangle_k$ in $V^\wedge$ which we need for construction of a Clifford algebra $\mathcal{C}\ell(V, \sigma, F)$ as the Chevalley F-deformation of a σ-braided exterior algebra $\mathcal{C}\ell(V, \sigma, 0) \equiv V^\wedge(\sigma)$. We define

$$\text{for} \quad \psi \in V^{\otimes n}, \quad \psi^j \in V^{\wedge(n-k)}, \quad \psi_j \in V^{\wedge k},$$

$$[\pi_\sigma \circ C_{n-k,k}(\sigma)]\psi = \sum \psi^j \wedge \psi_j,$$

$$\langle\,,\rangle_k : V^\wedge \otimes V^\wedge \to V^\wedge,$$

$$\langle \psi, \cdot \rangle_k \equiv \sum \wedge_{\psi^j} \circ \iota_{F\psi_j}, \quad \text{if } n < k \text{ then } \langle\,,\rangle_k \equiv 0.$$

Consistency of this definition follows from bialgebra map (4-5).

The Chevalley F-dependent deformed product $\vee \equiv \vee_{\sigma,F}$ on $V^\wedge(\sigma)$ is defined as follows,

$$\vee \equiv \wedge + \sum_{k\geq 1} \langle \cdot, \cdot \rangle_k,$$

$$\text{in particular for } x \in V, \quad \vee_x = \wedge_x + \iota_{Fx}.$$

Proposition 8. *An F-deformed algebra $\{V^\wedge(\sigma), \vee_{\sigma,F}\}$ is an associative algebra with the unity* $1 \in V^\wedge(\sigma)$.

Proof. The proof can be performed diagramatically, using tangle and braid diagrams, and using condition $(1-8)$. ∎

Definition 9 (Clifford algebra as a deformation). An algebra

$$\mathcal{C}\ell(V, \sigma, F) \equiv \{V^\wedge(\sigma), \vee_{\sigma,F}\}$$

is said to be *a Clifford algebra* as the Chevalley F-deformation of an exterior algebra $V^\wedge(\sigma)$.

The graded algebra associated to a filtered algebra $\mathcal{C}\ell(V, \sigma, F)$ is isomorphic to $V^\wedge(\sigma)$.

Let $\pi_{\sigma,F} \in \operatorname{alg}[TV, \mathcal{C}\ell(V, \sigma, F)]$ be an algebra epimorphism extending the identity map on V and let $I_{\sigma,F}$ be two-sided ideal in a tensor algebra,

$$I_{\sigma,F} \equiv \ker \pi_{\sigma,F} \quad \triangleleft \quad TV.$$

The Clifford algebra $\mathcal{C}\ell(V,\sigma,F)$ can be presented as a factor algebra $\mathcal{C}\ell(V,\sigma,F) = TV/I_{\sigma,F}$.

6. The Bourbaki bijection in a tensor algebra. The ideal $I_{\sigma,F} \triangleleft TV$ can be described in another way, using a bijection $\lambda_F \in \operatorname{End} V^{\otimes}$ introduced by Bourbaki [1959].

Let for $x \in V$, $\iota_{Fx} \in \sigma\mathrm{der}\, TV$ be a braided derivation on TV, Lemma 4. Bourbaki [1959] introduced the following map $\lambda_F \in \mathrm{End} V^{\otimes}$,

$$\text{for } x \in V \text{ and } \psi \in V^{\otimes},$$
$$\lambda_F|\mathbb{C} = \mathrm{id}, \qquad \lambda_F(x \otimes \psi) = x \otimes (\lambda_F \psi) + (\iota_{Fx} \circ \lambda_F)\psi.$$

Then $\lambda_F|V = \mathrm{id}$ and the Bourbaki map λ_F is bijective.

Lemma 10. *We have*

$$\ker W(\sigma) = \lambda_F(I_{\sigma,F}). \tag{10}$$

Proof. The statement follows from $\pi_\sigma \circ \lambda_F = \pi_{\sigma,F}$. ∎

The Bourbaki bijection λ_F allows to define a new product $\vee_F$ in $\mathbb{C}$-space $V^{\otimes}$,

$$\vee_F \equiv \lambda_F \circ \left[\lambda_F^{-1} \otimes \lambda_F^{-1}\right]. \tag{11}$$

With respect to product $\vee_F$ (11) the space ker $W(\sigma)$ is a left ideal in $\{TV, \vee_F\}$. The condition $(1-8)$ ensures that ker $W(\sigma)$ is also a right $\vee_F$-ideal.

An algebra epimorphism $\pi_\sigma \in \mathrm{alg}\,[TV, V^\wedge(\sigma)]$ by construction is also an algebra epimorphism of F-deformed algebras.

If the braid operator σ is such that ker $W(\sigma)$ is quadratic, then the ideal $I_{\sigma,F}$ is generated by elements of the form

$$\psi - F(\psi)1 \otimes 1, \quad \text{where } \psi \in V^{\otimes 2} \text{ is } \sigma\text{-invariant}, \quad \sigma\psi = \psi.$$

This covers Clifford and Weyl algebras for a Hecke braid [Hayashi 1990, Oziewicz 1995].

To define a Clifford algebra as the Chevalley F-deformation of braided exterior algebra $V^\wedge(\sigma)$, it is neccessary and sufficient that ker $W(\sigma)$ is also a right-ideal in $(V^{\otimes}, \vee_F)$. This assumption is weaker then $(1-8)$. If (8) does not hold, then the symmetry between left and right is broken.

If $\sigma = \sigma^{-1}$ then the braided Hopf algebra $V^\wedge(\sigma)$ can be deformed to a generalized braided quantum group $\{\mathcal{C}\ell(V,\sigma,F), \mathrm{sh}\}$ [Đurđević 1994, 1996]. A generalized braided quantum group $\{\mathcal{C}\ell(V,\sigma,F), \mathrm{sh}\}$ is not a braided Hopf algebra as in [Majid 1991-1993]. However, the axiom for the antipode is the same as for Hopf algebra. The antipode is F-dependent. The shuffle coproduct sh in $\{\mathcal{C}\ell(V,\sigma,F), \mathrm{sh}\}$ is the same as in $V^\wedge(\sigma)$, however the intrinsic braid determined by the Clifford product, shuffle coproduct and the antipode will be F-dependent. The generalized braided quantum group is not includable in the framework of braided categories, because the coproduct map does not obey the functoriality properties relative to the mentioned braiding.

7. Spinors. This section is devoted to braided generalization of the Cartan theory of spinors [3], [1]. Consider the Witt F-isotropic splitting

$$V = V_1 \oplus V_2, \qquad V_i \text{ are } F\text{-isotropic}. \tag{12}$$

The Witt splitting is said to be compatible with the braid $\sigma \in \mathrm{End}(V^{\otimes 2})$ if

$$\text{for } i \neq j, \quad \sigma(V_i \otimes V_j) = V_j \otimes V_i,$$
$$\sigma^2 | \{(V_1 \otimes V_2) \oplus (V_2 \otimes V_1)\} = \mathrm{id}.$$

Let $F|\big(V_2 \otimes V_1\big)$ be nondegenerate. In this case $V_2 \simeq V_1^*$. Let for $f \in V_1^*$ and $x \in V_1$ a form F be given by $F(f \otimes x) \equiv f(x)$.

Exterior algebras $(V_1)^\wedge$ and $(V_2)^\wedge$ are subalgebras of $\mathcal{C}\ell(V, \sigma, F)$.

Lemma 11 (The Cartan map). *The following Cartan map μ is bijective*,

$$\mu\colon (V_1)^\wedge \otimes (V_2)^\wedge \longrightarrow \mathcal{C}\ell(V, \sigma, F), \qquad \mu \equiv \vee_{\sigma,F}. \tag{13}$$

Proof. Let $u \in V_1$, $v \in V_2$ and $\sigma(v \otimes u) \equiv \sum_k u_k \otimes v_k$. Then

$$vu + \sum_k u_k v_k - F(v \otimes u) \cdot 1 = 0.$$

This implies that μ is surjective.

Let $p_{kl}\colon V^{\otimes(k+l)} \to (V_1)^{\otimes k} \otimes (V_2)^{\otimes l}$ be a projection. We shall prove that the map $\wedge\colon (V_1)^\wedge \otimes (V_2)^\wedge \to V^\wedge$, which is the grade-preserving component of μ, is injective. Indeed, we have $p_{kl} \circ \wedge = \mathrm{id}$ and hence $\wedge$ is injective. ∎

Denote $\mathcal{K} = (V_2)^\wedge$ and let $\kappa \in \mathrm{alg}\,(\mathcal{K}, \mathbb{C})$ be a character, $\kappa(1) = 1$ such that $\kappa(V_2) = 0$. This gives a left $\mathcal{K}$-module structure on $\mathbb{C}$. On the other hand $\mathcal{C}\ell(V, \sigma, F)$ is a right $\mathcal{K}$-module.

Definition 12. A left $\mathcal{C}\ell(V, \sigma, F)$-module $\mathcal{S} \equiv \mathcal{C}\ell(V, \sigma, F) \otimes_{\mathcal{K}} \mathbb{C}$ is called *the spinor module* associated to the F-isotropic splitting (12).

According to Lemma 11, the space $\mathcal{S}$ is isomorphic to the exterior algebra $(V_1)^\wedge$ and

$$\text{for } x = x_1 + x_2 \in V,\ x_i \in V_i, \qquad x\xi = x_1 \wedge \xi + x_2 \sqcup \xi.$$

Proposition 13. *A $\mathcal{C}\ell(V, \sigma, F)$-module $\mathcal{S}$ is faithful and irreducible.*

Proof. We prove that each vector $\psi \in \mathcal{S} \setminus \{0\}$ is cyclic which implies that the module is simple. The unit element $1_{\mathcal{S}} \in \mathcal{S}$ is cyclic, by construction. The duality between spaces V_1 and V_2 extends to the duality between exterior algebras $(V_1)^\wedge$ and $(V_2)^\wedge$, as explained in Section 4. In terms of this duality, the contraction between elements of the same degree becomes the pairing map. It follows that there exists $\varphi \in \mathcal{K}$ such that $\varphi \sqcup \psi = 1$. Hence, ψ is cyclic.

Let $\sum u_k \otimes v_k \neq 0$ be a component consisting of summands having the minimal second degree n. We can assume that v_k are linearly independent vectors. Let $x \in \mathcal{C}\ell(V, \sigma, F) \setminus \{0\}$. We have

$$\mu^{-1}(x) = \sum u_k \otimes v_k + \psi.$$

There exist spinors $\xi_j \in \mathcal{S}$ satisfying $\psi\xi_j = 0$ and $v_k \xi_j = \delta_{kj}$. This gives $x\xi_j = u_j$ which implies that the representation is faithful. ∎

A left $\mathcal{C}\ell(V, \sigma, F)$-module $\mathcal{S}$ is completely characterized by the existence of a cyclic vector $1_{\mathcal{S}} \in \mathcal{S}$ anihilated by the subspace V_2.

In other words let $\mathcal{V}$ be an arbitrary left $\mathcal{C}\ell(V,\sigma,F)$-module, possesing a vector v satisfying $\{V_2\}v=0$. Then exists the unique module map $\varrho\colon\mathcal{S}\to\mathcal{V}$ satisfying $\varrho(1_{\mathcal{S}})=v$. The map ϱ is injective because of the simplicity of $\mathcal{S}$. In particular, if v is cyclic then ϱ is a module isomorphism.

Clifford and Weyl algebra for a Hecke braid considered by Hayashi [1990] and by Oziewicz [1995] and spinors for a Hecke braid introduced in [Bautista et al. 1996] are included in the theory presented here. Clifford algebra of [1] is defined for Hecke braid $\sigma\in\mathrm{End}(V^{\otimes 2})$ (where $V\equiv W\oplus W^*$ is a finite-dimensional space) admitting extensions to braidings between W and W^*, so that the contraction map is a σ-morphism. If $V\equiv W\oplus W^*$ then a form F and a braid σ on V are expressible in terms of the extended braiding τ and the contraction map.

In the classical theory spinors can be equivalently viewed as elements of the left $\mathcal{C}\ell(V,\sigma,F)$-ideal, generated by a volume element of $V_2<V\equiv V_1\oplus V_2$. A similar description is possible in the braided geometry if an external algebra $(V_2)^\wedge$ admits a volume element. A form $\omega\in V^\wedge(\sigma)$ is said to be a volume on $V^\wedge(\sigma)$ if $V^\wedge\omega=0$. Let ω be a volume on V_2. Then a left $\mathcal{C}\ell(V,\sigma,F)$-ideal $\mathrm{gen}\,\omega$ is isomorphic to $\mathcal{S}$ as a left $\mathcal{C}\ell(V,\sigma,F)$-module.

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